Optimization

1. (20 pts) Consider the following linear program,

$$\begin{aligned} \text{minimize} \quad & 2x_1 + x_2 \\ \text{subject to} \quad & x_1 + 3x_2 \geq 6 \\ & 2x_1 + x_2 \geq 4 \\ & x_1 + x_2 \leq 3 \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{aligned}$$

Convert the above linear program into standard form and find an initial basic feasible solution for the program in standard form.

2. (20 pts)

- **(15 pts)** Find the largest range of the step-size, α, for which the fixed step gradient descent algorithm is guaranteed to converge to the minimizer of the quadratic function

$$f = \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} - \boldsymbol{b}^\top \boldsymbol{x}$$

  starting from an arbitrary initial condition $\boldsymbol{x}^{(0)} \in \mathbb{R}^n$, where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^n$, and $\boldsymbol{Q} = \boldsymbol{Q}^\top > 0$.

- **(5 pts)** Find the largest range of the step size, α, for which the fixed step gradient descent algorithm is guaranteed to converge to the minimizer of the quadratic function

$$f = 6x_1^2 + 2x_2^2 - 5,$$

starting from an arbitrary initial condition $\boldsymbol{x}^{(0)} \in \mathbb{R}^2$.

3. (20 pts) Is the function

$$f(x_1, x_2) = \frac{1}{(x_1 - 2)^2 + (x_2 + 1)^2 + 3}$$

locally convex, concave, or neither in the neighborhood of the point $\begin{bmatrix} 2 & -1 \end{bmatrix}^\top$? Justify your answer by giving all the details of your argument.

4. (20 pts) Solve the following optimization problem:

$$\begin{aligned} \text{optimize} \quad & x_1 x_2 \\ \text{subject to} \quad & x_1 + x_2 + x_3 = 1 \\ & x_1 + x_2 - x_3 = 0. \end{aligned}$$

5. (20 pts) Solve the following optimization problem:

$$\begin{aligned} \text{maximize} \quad & 14x_1 - x_1^2 + 6x_2 - x_2^2 + 7 \\ \text{subject to} \quad & x_1 + x_2 \leq 2 \\ & x_1 + 2x_2 \leq 3. \end{aligned}$$

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