AC-3
August 2015 QE

1. **(20 pts)** Solve the following linear program,

$$\begin{aligned} \text{maximize} \quad & -x_1 - 2x_2 + 4x_3 \\ \text{subject to} \quad & x_1 + 2x_2 - x_3 = 5 \\ & 2x_1 + 3x_2 - x_3 = 6 \\ & x_1 \text{ free},\ x_2 \geq 0,\ x_3 \leq 0. \end{aligned}$$

2. **(20 pts)** Formulate the first-order necessary conditions for the quadratic program,

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} - \boldsymbol{b}^\top \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{c}, \end{aligned}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{c} \in \mathbb{R}^m$, $m \leq n$, and $\boldsymbol{Q} = \boldsymbol{Q}^\top > 0$.

- **(15 pts)** Represent the obtained conditions as a system of linear equations and **write down the solution to the problem.**
- **(5 pts)** What is the condition, involving $\boldsymbol{A}$ and $\boldsymbol{Q}$, that must be satisfied for the solution to be unique?

3. **(20 pts)** Consider the optimization problem,

$$\begin{aligned} \text{maximize} \quad & -x_1^2 + x_1 - x_2 - x_1 x_2 \\ \text{subject to} \quad & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

(i) **(10 pts)** Characterize feasible directions at the point

$$x^* = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}.$$

(ii) **(10 pts)** Write down the second-order necessary condition for x^*. Does the point x^* satisfy this condition?

4. (20 pts) Consider the following primal problem:

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & -2x_1 + x_2 + x_3 = 2 \\ & -x_1 + 2x_2 + x_4 = 7 \\ & x_1 + x_5 = 3 \\ & x_i \geq 0, \quad i = 1, 2, 3, 4, 5. \end{aligned}$$

(i) (5 pts) Construct the dual problem corresponding to the above primal problem.

(ii) (15 pts) It is known that the solution to the above primal is $x^* = \begin{bmatrix} 3 & 5 & 3 & 0 & 0 \end{bmatrix}^\top$. Find the solution to the dual.

5. (20 pts) Find the minimizer of

$$f(x_1, x_2) = \frac{1}{2}x_1^2 + x_2^2 + x_1 + \frac{1}{2}x_2 + 3$$

using the conjugate gradient algorithm. The starting point is $x^{(0)} = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$.