AC-2
August 2011 QE

Linear Time-Invariant and Time-Varying Systems: A State Space Approach

Problem 1. (30 points) Consider the following linear system

$$\dot{x} = Ax + bu = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ -3 \end{bmatrix} u$$
$$y = cx = \begin{bmatrix} 3 & 1 \end{bmatrix} x.$$

(a) **(4 pts)** Is the system controllable? Find its controllable subspace.

(b) **(4 pts)** Is the system observable? Find its unobservable subspace.

(c) **(4 pts)** Is the system stabilizable? Justify your answer.

(d) **(4 pts)** Does there exist a feedback gain $k \in \mathbb{R}^{1\times 2}$ such that $A - bk$ has eigenvalues $\{-1, -1\}$?

(e) **(4 pts)** Is the system detectable? Justify your answer.

(f) **(4 pts)** Does there exist $l \in \mathbb{R}^{2\times 1}$ such that $A - lc$ has eigenvalues $\{-1, -1\}$?

(g) **(6 pts)** Suppose $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $u(t) \equiv 0$. Find the output $y(t)$ for all $t \geq 0$.

Problem 2. (10 points) Given two linear systems (A, B, C) and $(A, \tilde{B}, C)$ with $A \in \mathbb{R}^{n\times n}$, $B, \tilde{B} \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{r\times n}$, suppose (C, A) is observable and both systems have the same transfer function $C(zI - A)^{-1}B = C(zI - A)^{-1}\tilde{B}$. Show that we must have $B = \tilde{B}$.

Problem 3. (25 points) Consider a linear system with state $x[k] = \begin{bmatrix} x_1[k] & x_2[k] & x_3[k] \end{bmatrix}^T \in \mathbb{R}^3$:

$$x[k+1] = A\,x[k] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} x[k],$$

with an initial condition $x[0]$. It is known that A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = \frac{1}{2}$, and $\lambda_3 = -\frac{1}{6}$.

(a) **(5 pts)** Show that $\begin{bmatrix} \frac{2}{7} & \frac{3}{7} & \frac{2}{7} \end{bmatrix}$ is a left eigenvector of A, and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a right eigenvector of A, both corresponding to the eigenvalue 1.

(b) **(10 pts)** Prove that $\frac{2}{7}x_1[k] + \frac{3}{7}x_2[k] + \frac{2}{7}x_3[k]$ for all k is a constant $\bar{x} = \frac{2}{7}x_1[0] + \frac{3}{7}x_2[0] + \frac{2}{7}x_3[0]$.

(c) **(10 pts)** Show that $x[k]$ converges to $\begin{bmatrix} \bar{x} & \bar{x} & \bar{x} \end{bmatrix}^T$ as $k \to \infty$.

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Problem 4. (15 points) Consider a single-input single-output system:

$$\begin{cases} \dot{x} &= Ax + bu \\ y &= cx. \end{cases}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n\times 1}$, $c \in \mathbb{R}^{1\times n}$ $(n \geq 2)$. Suppose its transfer function is known:

$$H(s) = c(sI - A)^{-1}b = \frac{1}{(s+1)^2}.$$

Determine **true or false** for the following statements. You do **not** need to justify your answers.

(a) **(3 pts)** If (A, b) is controllable, then A is asymptotically stable.

(b) **(3 pts)** If (c, A) is observable, then A is asymptotically stable.

(c) **(3 pts)** If (A, b) is controllable and (c, A) is observable, then A is asymptotically stable.

(d) **(3 pts)** If (c, A) is observable and $x(0) = 0$, then $y(t)$ is bounded under all bounded $u(t)$.

(e) **(3 pts)** If (c, A) is observable and $u(t) \equiv 0$, then $y(t) \to 0$ as $t \to \infty$ for all $x(0)$.

Problem 5. (20 points) Consider the following linear time-varying system

$$\dot{x}(t) = \begin{bmatrix} -\frac{1}{t+1} & 0 \\ -\frac{1}{t+1} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \quad t \geq 0.$$

(a) **(15 pts)** Assume $u(t) \equiv 0$. Find the state transition matrix $\Phi(t, \tau)$ of the system.

(b) **(5 pts)** Is the system asymptotically stable under $u(t) \equiv 0$?

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