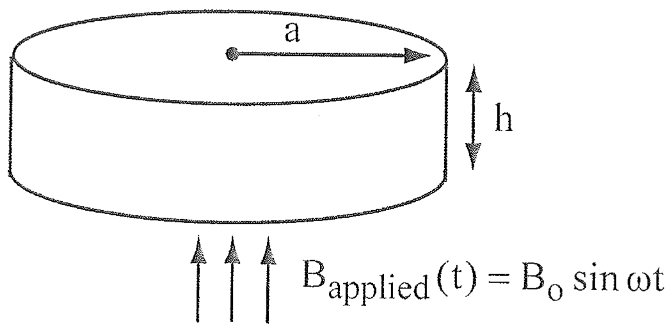
FO-3
August 2011 QE

1) **40 points** A weakly conducting cylindrical disk of radius a and height h is made of a material with permittivity $\varepsilon = \varepsilon_o$, permeability $\mu = 5\mu_o$, and conductivity σ. The disk is exposed to a spatially uniform, time-varying magnetic field $B_{applied}(t) = B_o \sin \omega t$, which is oriented parallel to the axis of the cylindrical shell, as shown. This applied B field fills all space.



Derive an expression for the time-average power dissipated in the weakly conducting disk.

In this problem you may assume that the frequency is sufficiently low that we may set $\partial \mathbf{D} / \partial t = 0$ in Ampere's Law (this is called the magneto-quasi-static approximation). You may also assume that the induced currents are sufficiently small that we may neglect any effect they may have on the **B** field. As a result, for your calculation you may use exactly the **B** field specified.

Some possibly useful formulas are attached at the end of this problem.

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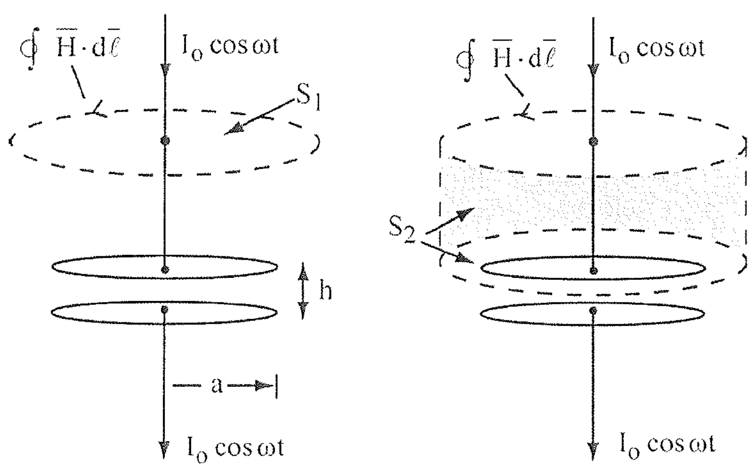
2) **60 points** Consider an infinitely long straight wire carrying current $I = I_o \cos\omega t$ to and from a pair of circular capacitor plates of radius a. The plates are separated by a distance $h << a$ of free-space $(\varepsilon = \varepsilon_o, \mu = \mu_o)$. In this problem you may assume the frequency is low enough that we may set $\partial \mathbf{B}/\partial t = 0$ in Faraday's Law (this is called the electro-quasi-static approximation). We will also neglect fringing fields at the edge of the capacitor. Hence the electric field may be taken as having the same simple form as for a parallel plate capacitor at zero frequency.

Ampere's Law in integral form is given by

$$\oint \mathbf{H} \cdot d\ell = \int \mathbf{J} \cdot ds + \int \frac{\partial \mathbf{D}}{\partial t} \cdot ds$$

As long as the closed loop for the line integral remains the same, the surface integral may be performed over various surfaces. For our problem consider surface S_1, which is a circular surface above the top capacitor plate (left figure) and surface S_2, which consists of a circular surface between the capacitor plates along with cylindrical side walls concentric with the wire (right figure).



(a) Explicitly show that the surface integrals over surfaces S_1 and S_2 give the same result.

(b) Suppose that instead of free-space, the capacitor plates are now separated by an insulating spacer material with $\varepsilon = 9\varepsilon_o, \mu = \mu_o$. Briefly discuss how this affects the **E** and **H** fields in this problem and how it affects the surface integrals from part (a).

(c) Work out expressions for the time-average energy stored in the volume between the capacitor plates (i) in the electric field and (ii) in the magnetic field. What inequality must the frequency (ω) satisfy in order that the energy stored in the magnetic field remains much less than the energy stored in the electric field? Explain how this inequality governing ω relates to the time required for wave propagation across the capacitor.

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Some or all of the following may be useful.

Maxwell's Equations, in differential and integral form, and other potentially useful relationships

$\nabla \cdot \mathbf{D} = \rho$	$\oint \mathbf{D} \cdot d\mathbf{S} = \int \rho dV$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint \mathbf{E} \cdot d\bar{\ell} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S}$
$\nabla \cdot \mathbf{B} = 0$	$\oint \mathbf{B} \cdot d\mathbf{S} = 0$
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint \mathbf{H} \cdot d\bar{\ell} = \int \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$

$$\mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P} = \varepsilon \mathbf{E}$$

$$\mathbf{B} = \mu_o (\mathbf{H} + \mathbf{M}) = \mu \mathbf{H}$$

Vector Differential Relationships in Cylindrical Coordinates (r, ϕ, z)

$$\nabla \Phi = \hat{r} \frac{\partial \Phi}{\partial r} + \hat{\Phi} \frac{1}{r} \frac{\partial \Phi}{\partial \phi} + \hat{z} \frac{\partial \Phi}{\partial z}$$

$$\nabla \cdot \mathbf{D} = \frac{1}{r} \frac{\partial}{\partial r}(r D_r) + \frac{1}{r} \frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

$$\nabla \times \mathbf{H} = \hat{r} \left[\frac{1}{r} \frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} \right] + \hat{\Phi} \left[\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right] + \hat{z} \left[\frac{1}{r} \frac{\partial (r H_\phi)}{\partial r} - \frac{1}{r} \frac{\partial H_r}{\partial \phi} \right]$$

$$\nabla^2 \Phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

$$\nabla^2 \mathbf{A} = \hat{r} \left(\nabla^2 A_r - \frac{2}{r^2} \frac{\partial A_\phi}{\partial \phi} - \frac{A_r}{r^2} \right) + \hat{\Phi} \left(\nabla^2 A_\phi + \frac{2}{r^2} \frac{\partial A_r}{\partial \phi} - \frac{A_\phi}{r^2} \right) + \hat{z} \left(\nabla^2 A_z \right)$$

Divergence Thm: $\int_V \nabla \cdot \mathbf{A} dV = \oint_S \mathbf{A} \cdot d\mathbf{S}$

Stokes Thm: $\int_V \nabla \times \mathbf{A} \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\bar{\ell}$

Differential volume elements

Cartesian: $dV = dx\ dy\ dz$

Cylindrical: $dV = r\ dr\ d\phi\ dz$

Spherical: $dV = r^2 \sin\theta\ dr\ d\phi\ d\theta$