Lecture 07
Multiattribute Utility Theory (I)

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ME 597: Decision Making for Engineering Systems Design

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Lecture Outline

1. Approaches for Multiattribute Assessment
   - Direct Utility Assessment
   - Conditional Assessments
   - Assessing Utility Functions over “Value” Functions
   - Qualitative Structuring of Preferences

2. Utility Independence
   - Conditional Utility Independence
   - Mutual Utility Independence
   - Additive Independence and Additive Utility Function
   - General Case – No Utility Independence

What have we covered so far?
(a) Deterministic scenario

Tradeoff problem (Lecture 3)
How much achievement on Objective 1 is the decision maker willing to give up in order to improve achievement on Objective 2 by some fixed amount?
What have we covered so far?
(b) Probabilistic scenario

Lectures 5 and 6

How much of an attribute is a (risk averse) decision maker willing to “give up” from the average to avoid the risks associated with a lottery?

**Lottery questions** such as

\[ \langle x_n, \pi_i, x_1 \rangle \sim x_i \]

Positive Linear Transformation:

\[ u_i = a + b\pi_i, \quad b > 0, \quad i = 1, \ldots, n \]

**Qualitative Characteristics** of Utility:

- Monotonicity
- Risk aversion
- Increasing, decreasing, and constant risk aversion
Assume that the attributes for the problem are $X, Y, Z, \ldots$

$x$ denotes a specific level of $X$; $y$ denotes a specific level of $Y$, etc.

**Goal:** to find a utility function over the attribute (consequence) space

$$u() = u(x, y, z, \ldots)$$

**Question**

How can we adapt the procedure for single attribute utility functions to multiple attributes?
Potential Approaches

1. Direct utility assessment
2. Fixing all attributes except for one
3. Using value functions
4. Qualitative structuring of preferences
Assign utilities to possible consequences \((y_i, z_j)\) directly.
Arbitrarily set \(u(y^0, z^0) = 0\) and \(u(y^*, z^*) = 1\) where \(o\) is the least preferred and \(\ast\) is the most preferred.

If lottery \(\langle (y^*, z^*), \pi, (y^0, z^0) \rangle \sim (y_i, z_j)\) then \(u(y_i, z_j) = \pi\)
1. Direct Utility Assessment

Limitations

Shortcomings of the procedure:

1. it fails to exploit the basic preference structure of the decision maker,
2. the requisite information is difficult to assess, and
3. the result is difficult to work with in expected utility calculations and sensitivity analysis.
2. Fixing all Attributes Except One

Consider a two attribute scenario with consequence space defined by $Y, Z$. Fix $Z = z_f$ and carry out utility assessments for single attribute $Y$.

$$y_i \sim \langle y_n, \pi_i, y_1 \rangle$$

$\pi_i(y)$ is the conditional utility function for $y$ values, conditioned on $Z = z_f$, and normalized by $\pi_i(y_1) = 0$ and $\pi_i(y_n) = 1$. 

![Graph showing a constant utility function](image)
A utility function is a value function, but a value function is not necessarily a utility function!

1. Assign a value function for each point \((y, z)\) in the consequence space.
2. The utility function is monotonically increasing in \(v\).
3. Assess unidimensional utility functions for \(u[v(y, z)]\).
3. Using Value Functions

Example

Figure: 5.1 on page 221 (Keeney and Raiffa) [adapted]
4. Qualitative Structuring of Preferences

Basic approach:

1. Postulate various assumptions about the preference attitudes of the decision maker.

2. Derive functional forms of the multiattribute utility function consistent with these assumptions.
Utility Independence

Important property: Independence

Ideal scenario: Utility function such that

\[ u(x, y, z, \ldots) = f[f_1(x), f_2(y), f_3(z), \ldots] \]
Definition (Utility Independence)

Y is utility independent of Z when conditional preferences for lotteries on Y given z do not depend on the particular level of z.

What is the certainty equivalent ($\hat{y}, z^o$) for the lottery $\langle (y^*, z^o), 0.5, (y^o, z^o) \rangle$?
(Note: z is fixed at $z^o$)

Does $\hat{y}$ change if z is changed to another value (say $z'$)?

**Figure**: 5.2 on page 225 (Keeney and Raiffa)
Conditional Utility Independence

Question
If $Y$ is utility independent of $Z$, does that also mean that $Z$ is utility independent of $Y$?
Conditional Utility Independence

If $Y$ is utility independent of $Z$, all conditional utility functions along horizontal cuts would be positive linear transformations of each other.

Therefore,

$$u(y, z) = g(z) + h(z)u(y, z')$$

for an arbitrarily chosen $z'$. In other words, the conditional utility function over $Y$ given $z$ does not strategically depend on $z$.

Figure: 5.2 on page 225 (Keeney and Raiffa)
Some Examples

Which utility functions satisfy the conditional utility independence conditions? If so, for which attribute?

1. \( u(y, z) = \frac{y^\alpha z^\beta}{y + z} \)

2. \( u(y, z) = g(z) + h(z)u_Y(y) \)

3. \( u(y, z) = k(y) + m(y)u_Z(z) \)

4. \( u(y, z) = k_1 u_Y(y) + k_2 u_Z(z) + k_3 u_Y(y)u_Z(z) \)

5. \( u(y, z) = [\alpha + \beta u_Y(y)][\gamma + \delta u_Z(z)] \)

6. \( u(y, z) = k_Y u_Y(y) + k_Z u_Z(z) \)
Some Examples

Which utility functions satisfy the conditional utility independence conditions? If so, for which attribute?

1. \( u(y, z) = \frac{y^\alpha z^\beta}{y + z} \)
   Neither attribute is utility independent

2. \( u(y, z) = g(z) + h(z)u_Y(y) \)
   Here, \( Y \) is utility independent of \( Z \), but not vice versa

3. \( u(y, z) = k(y) + m(y)u_Z(z) \)
   Here, \( Z \) is utility independent of \( Y \), but not vice versa

4. \( u(y, z) = k_1u_Y(y) + k_2u_Z(z) + k_3u_Y(y)u_Z(z) \)
   Both parameters are utility independent of each other

5. \( u(y, z) = [\alpha + \beta u_Y(y)][\gamma + \delta u_Z(z)] \)
   Both parameters are utility independent of each other

6. \( u(y, z) = k_Yu_Y(y) + k_Zu_Z(z) \)
   Both parameters are utility independent of each other
Assuming that $Z$ is utility independent of $Y$, for any arbitrary $y_0$,

$$u(y, z) = c_1(y) + c_2(y)u(y_0, z), \quad c_2(y) > 0, \quad \forall y$$

The two-attribute utility function can be specified by:

1. three conditional utility functions, or
2. two conditional utility functions and an isopreference curve, or
3. one conditional utility function and two isopreference curves
1. Assessment using Three Conditional Utility Functions

Theorem

If Z is utility independent of Y, then

\[ u(y, z) = u(y, z_0)[1 - u(y_0, z)] + u(y, z_1)u(y_0, z) \]

where \( u(y, z) \) is normalized by

\( u(y_0, z_0) = 0 \) and \( u(y_0, z_1) = 1 \)

**Figure:** 5.7 on page 244 (Keeney and Raiffa)
2. Assessment using Two Conditional Utility Functions and One Isopreference Curve

**Theorem**

If $Z$ is utility independent of $Y$, then

$$u(y, z) = u(y, z_0) + \left[ \frac{u(y_0, z_1) - u(y, z_0)}{u(y_0, z_n(y))} \right] u(y_0, z)$$

where $u(y_0, z_0) = 0$, and $z_n(y)$ is defined such that $(y, z_n(y)) \sim (y_0, z_1)$ for an arbitrary $z_1$.

**Figure**: 5.9 on page 247 (Keeney and Raiffa)
3. Assessment using One Conditional Utility Functions and Two Isopreference Curves

**Theorem**

If $Z$ is utility independent of $Y$, then

$$u(y, z) = \frac{u(y_0, z) - u(y_0, z_m(y))}{u(y_0, z_n(y)) - u(y_0, z_m(y))}$$

where

1. $u(y, z)$ is normalized by $u(y_0, z_0) = 0$ and $u(y_0, z_1) = 1$
2. $z_m(y)$ is defined such that $(y, z_m(y)) \sim (y_0, z_0)$
3. $z_n(y)$ is defined such that $(y, z_n(y)) \sim (y_0, z_1)$

Figure: 5.11 on page 250 (Keeney and Raiffa)
For mutual utility independence of $Y$ and $Z$,

1. $Y$ must be utility independent of $Z$, i.e.,

\[ u(y, z) = c_1(z) + c_2(z)u(y, z') \quad \forall y, z \]

for an arbitrarily chosen $z'$, and

2. $Z$ must be utility independent of $Y$, i.e.,

\[ u(y, z) = d_1(y) + d_2(y)u(y', z) \quad \forall y, z \]

for an arbitrarily chosen $y'$. 

When $Y$ and $Z$ are mutually utility independent, then $u(y, z)$ can be expressed by the **multilinear representation**:

$$u(y, z) = k_Y u_Y(y) + k_Z u_Z(z) + k_{YZ} u_Y(y)u_Z(z)$$

where $k_Y > 0$, and $k_Z > 0$.

**Figure**: 5.4 on page 233 (Keeney and Raiffa)
Theorem

If Y and Z are mutually utility independent, then the two-attribute utility function is multilinear. In particular, u can be written in the form

\[ u(y, z) = k_Y u_Y(y) + k_Z u_Z(z) + k_{YZ} u_Y(y) u_Z(z) \]

or

\[ u(y, z) = u(y, z_0) + u(y_0, z) + ku(y, z_0)u(y_0, z), \]

where

1. \( u(y, z) \) is normalized by \( u(y_0, z_0) = 0 \) and \( u(y_1, z_1) = 1 \) for arbitrary \( y_1 \) and \( z_1 \) such that \( (y_1, z_0) \succ (y_0, z_0) \) and \( (y_0, z_1) \succ (y_0, z_0) \)

2. \( u_Y(y) \) is conditional utility on Y normalized by \( u_Y(y_0) = 0 \) and \( u_Y(y_1) = 1 \)

3. \( u_Z(z) \) is conditional utility on Z normalized by \( u_Z(z_0) = 0 \) and \( u_Z(z_1) = 1 \)

4. \( k_Y = u(y_1, z_0) \)

5. \( k_Z = u(y_0, z_1) \)

6. \( k_{YZ} = 1 - k_Y - k_Z \) and \( k = \frac{k_{YZ}}{k_Y k_Z} \)
Use of Isopreference Curves

**Theorem**

If $Y$ and $Z$ are mutually utility independent, then

$$u(y, z) = \frac{u(y_0, z) - u(y_0, z_n(y))}{1 + ku(y_0, z_n(y))}$$

where

1. $u(y_0, z_0) = 0$
2. $z_n(y)$ is defined such that $(y, z_n(y)) \sim (y_0, z_0)$
3. $k = \frac{u(y_0, z_1) - u(y_1, z_1) - u(y_0, z_n(y_1))}{u(y_1, z_1)u(y_0, z_n(y_1))}$ where $(y_1, z_1)$ is arbitrarily chosen such that $(y_0, z_0)$ and $(y_1, z_1)$ are not indifferent.
If two attributes are mutually utility independent, their utility function can be represented by either a \textit{product form}, when $k \neq 0$, or an \textit{additive form}, when $k = 0$.

The multilinear form

$$u(y, z) = u(y, z_0) + u(y_0, z) + ku(y, z_0)u(y_0, z)$$

Let

$$u'(y, z) = ku(y, z) + 1$$

$$= ku(y_0, z) + ku(y, z_0) + k^2 u(y_0, z)u(y, z_0) + 1$$

$$= [ku(y, z_0) + 1][ku(y_0, z) + 1]$$

$$= u'(y, z_0)u'(y_0, z)$$

i.e., the product form!
For $k = 0$, the utility function reduces to an additive function.

**Additive utility function:**

$$u(y, z) = k_Y u_Y(y) + k_Z u_Z(z)$$

where $k_Y$ and $k_Z$ are positive scaling constants.

Additive utility function implies that $Y$ and $Z$ are **mutually utility independent**. But the converse is *not true*.

For example, $u(y, z) = y^\alpha z^\beta, \ 1 \leq y \leq 10, \ 1 \leq z \leq 10$
Checking for Additive Independence

For additive independence, the following two lotteries must be equally preferable:

\[ \langle (y, z), 0.5, (y', z') \rangle \sim \langle (y, z'), 0.5, (y', z) \rangle \]

for all \((y, z)\) given arbitrarily chosen \(y'\) and \(z'\).

Note: Additive independence is reflexive.
Fundamental Result of Additive Utility Function

Theorem

Attributes Y and Z are additive independent if and only if the two-attribute utility function is additive. The additive form may be either written as

\[ u(y, z) = k_Y u_Y(y) + k_Z u_Z(z) \]

or as

\[ u(y, z) = u(y, z^o) + u(y^o, z) \]

where

1. \( u(y, z) \) is normalized by \( u(y^o, z^o) = 0 \) and \( u(y^1, z^1) = 1 \) for arbitrary \( y^1 \) and \( z^1 \) such that \( (y^1, z^o) \succ (y^o, z^o) \) and \( (y^o, z^1) \succ (y^o, z^o) \)
2. \( u_Y(y) \) is a conditional utility function on Y normalized by \( u_Y(y^o) = 0 \) and \( u_Y(y^1) = 1 \)
3. \( u_Z(z) \) is a conditional utility function on Z normalized by \( u_Z(z^o) = 0 \) and \( u_Z(z^1) = 1 \)
4. \( k_Y = u(y^1, z^o) \) and \( k_Z = u(y^o, z^1) \)
Interpretation of Parameter $k$

$$\begin{bmatrix} \langle A, 0.5, C \rangle \sim \langle B, 0.5, D \rangle \end{bmatrix} \iff k \begin{bmatrix} > \sim < \end{bmatrix} 0 \iff \begin{cases} Y \text{ and } Z \text{ are complements} \\ \text{no interaction of preference} \\ Y \text{ and } Z \text{ are substitutes} \end{cases}$$

Figure: 5.6 on page 240 (Keeney and Raiffa)
Benefits of Utility Independence in Utility Assessment

(a) (b) (c)
(d) (e) (f)

Figure: 5.3 on page 228 (Keeney and Raiffa) (a) No independence condition holds, (b) $Y$ is utility independent of $Z$, (c) $Z$ is utility independent of $Y$, (d, e) $Y, Z$ are mutually utility independent, (f) Additivity assumption holds.
Approaches for Multiattribute Assessment

Utility Independence

Conditional Utility Independence
Mutual Utility Independence
Additive Independence and Additive Utility Function
General Case – No Utility Independence

Degrees of Freedom for Assigning Utility Functions

(a) Additive

(b) Multilinear

(c) Three conditional utility functions

(d) General utility independence

Figure: 5.12 on page 253 (Keeney and Raiffa)
Possibilities:

1. Transformation of $Y$ and $Z$ to new attributes that might allow exploitation of utility independence properties.
2. Direct assessment of $u(y, z)$ for discrete points and then interpolation/curve fitting.
3. Application of independence results in subsets of the $Y \times Z$ space.
4. Development of more complicated assumptions about the preference structure that imply more general utility functions.
Summary

1. Approaches for Multiattribute Assessment
   - Direct Utility Assessment
   - Conditional Assessments
   - Assessing Utility Functions over “Value” Functions
   - Qualitative Structuring of Preferences

2. Utility Independence
   - Conditional Utility Independence
   - Mutual Utility Independence
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References
