Sliding-Mode Observers for Systems With Unknown Inputs

Karanjit Kalsi†, Jianming Lian†, Stefen Hui‡, Stanislaw H. Żak†

Abstract

The problem of designing sliding-mode observers for systems with unknown inputs is considered when the so-called observer matching condition is not satisfied. This condition severely restricts the applicability of sliding-mode observers. To circumvent the observer matching condition, thereby broadening the class of systems for which sliding-mode observers can be constructed, a possible method is to generate auxiliary outputs that are then used to construct the sliding-mode observer. High-order sliding-mode exact differentiators can be used to obtain auxiliary outputs. A proof of the asymptotic stability of the state estimation error is provided. However, the resulting overall observer architecture is complex. In this paper, high-gain approximate differentiators are proposed to generate the estimates of auxiliary outputs instead. The resulting architecture is much simpler than the architecture involving high-order sliding-mode exact differentiators. It is shown that the state estimation error is uniformly ultimately bounded with respect to a ball whose radius can be controlled by design parameters. The performance of the high-gain approximate differentiator based sliding-mode observer is demonstrated to be comparable to that of the observer that uses high-order sliding-mode exact differentiators. The use of the presented observers to reconstruct the unknown inputs is also analyzed and then illustrated by numerical examples.

Keywords: Sliding-mode observer, high-order sliding-mode exact differentiator, high-gain approximate differentiator, unknown input reconstruction.

I. INTRODUCTION

Observers are dynamical systems that can be used to estimate the state of a plant using its input-output measurements; they were first proposed by Luenberger [1]. In some cases, the

† School of Electrical and Computer Engineering, Purdue University, West Lafayette, IN 47907.
‡ Department of Mathematical Sciences, San Diego State University, San Diego, CA 92182.
inputs to the plant are unknown or partially known, which led to the development of the so-called unknown input observer (UIO). Examples of linear UIO architectures that have been developed for linear system are analyzed in [2]–[8]. For non-linear system with unknown inputs, some of the UIO architectures can be found in [9]–[11]. Motivated by the design of sliding-mode controllers, first-order sliding mode based UIOs have been developed, see, for example, [12]–[17]. The main advantage of using sliding-mode observers over their linear counterparts is that while in sliding, they are insensitive to the unknown inputs and, moreover, they can be used to reconstruct unknown inputs which could be a combination of system disturbances, faults or non-linearities. The reconstruction of unknown inputs has found useful applications in fault-detection and isolation [8], [15], [16].

In most of the linear and non-linear unknown input observers proposed thus far, the necessary and sufficient conditions for the construction of such observers is that the invariant zeros of the system must lie in the open left half complex plane, and the transfer function matrix between unknown inputs and measurable outputs satisfies the observer matching condition. However, the second condition seriously limits the applicability of this technique. Recently, high-order sliding mode based unknown input observers [11], [18]–[21] have been developed for systems that do not satisfy the observer matching condition. In [20], a suitable change of coordinates is first provided via a constructive algorithm to transform the system into a quasi-block triangular observable form. Then a step-by-step second order sliding-mode observer is constructed for the transformed system. In [21], auxiliary outputs are defined such that the conventional unknown input sliding-mode observer proposed in [15] can be developed for systems without the observer matching condition. In order to obtain those auxiliary outputs, high-order sliding-mode observers which act as exact differentiators [22] are constructed based on the super-twisting algorithm proposed in [23].

In this paper, we adopt the idea of generating auxiliary outputs from [21]. We first incorporate the high-order sliding-mode observers proposed in [21] into the construction of the sliding-mode observer which was first introduced in [12] and later modified for a more general class of systems in [17]. Then we propose a new method of using high-gain observers instead of high-order sliding-mode observers. High-gain observers are another type of observers for systems with uncertainties. They were used in [24], [25] to develop output feedback controllers stabilizing feedback linearizable uncertain systems. The applications of high-gain observers in adaptive
control or nonlinear control of uncertain systems can be found in [26], [27]. The incorporation of high-gain observers into sliding-mode control was initially proposed in [28] and then in [29]. In this paper, high-gain observers are used as approximate differentiators [30] to obtain the estimates of those auxiliary outputs. The proposed high-gain approximate differentiator based sliding-mode observer can achieve similar state estimation performance to that of the high-order sliding-mode exact differentiator based sliding-mode observer. The major advantage of the proposed high-gain observers over high-order sliding-mode observers used in [21] is the simplicity of the overall observer architecture. To the best of our knowledge, it is the first time that the sliding-mode observer presented in [12] is applied to the state observation for linear systems without the observer matching condition being satisfied. It is also the first time that the high-gain observer is used in such an application.

The remainder of this paper is organized as follows. The system description and the problem statement are given in Section II. The sliding-mode exact differentiator is reviewed in Section III, where it is then incorporated into the sliding-mode observer design. In Section IV, the high-gain approximate differentiator based sliding-mode observer is proposed and analyzed. Simulation results using the proposed high-gain approximate differentiator based sliding-mode observer are included in Section V. Conclusions are in Section VI.

II. SYSTEM DESCRIPTION AND PROBLEM STATEMENT

Let \( B_1 \in \mathbb{R}^{n \times m_1}, \ B_2 \in \mathbb{R}^{n \times m_2} \) and \( C \in \mathbb{R}^{p \times n} \) be known constant matrices, with \( B_2 \) and \( C \) being of full rank, that is, \( \text{rank } B_2 = m_2 \) and \( \text{rank } C = p \), and \( m_2 \leq p \). We consider the following class of linear time-invariant systems with unknown inputs:

\[
\begin{align*}
\dot{x} &= Ax + B_1 u_1 + B_2 u_2 \\
y &= C x,
\end{align*}
\]

where \( x \in \mathbb{R}^n, \ y \in \mathbb{R}^p, \ u_1 \in \mathbb{R}^{m_1} \) and \( u_2 \in \mathbb{R}^{m_2} \) are the state, output, known and unknown input vectors. Let \( \| \cdot \| \) denote the standard Euclidean norm. We assume that there is \( \rho > 0 \) such that \( \| u_2(t) \| \leq \rho \) for all \( t \). We also assume that the invariant zeros of the system model given by the triple \((A, B_2, C)\) are in the open left-hand complex plane, or equivalently, \( \text{rank } \begin{bmatrix} sI_n - A & B_2 \\ C & O_{p \times m_2} \end{bmatrix} = n + m_2. \) (2)
Fig. 1. Diagram of the sliding-mode observer.

for all \( s \) such that \( \Re(s) \geq 0 \).

For the system modeled by (1), if the observer matching condition [20] is satisfied, that is,

\[
\text{rank } B_2 = \text{rank}(CB_2) = m_2, \tag{3}
\]

we can construct the following sliding-mode observer first proposed in [12],

\[
\dot{x} = A\hat{x} + B_1u_1 + L(y - \hat{y}) - B_2E(y, \hat{y}, \eta) \tag{4}
\]

with \( \hat{y} = C\hat{x} \) and

\[
E(y, \hat{y}, \eta) = \begin{cases} 
\eta \|F(y - \hat{y})\| & \text{if } F(y - \hat{y}) \neq 0 \\
0 & \text{if } F(y - \hat{y}) = 0,
\end{cases} \tag{5}
\]

where \( \eta \) is a positive design parameter, \( L \in \mathbb{R}^{n \times p} \) and \( F \in \mathbb{R}^{m_2 \times p} \) are matrices such that

\[
(A - LC)^\top P + P(A - LC) = -2Q < 0
\]

and

\[
FC = B_2^\top P
\]

for some symmetric positive definite \( P \in \mathbb{R}^{n \times n} \) and \( Q \in \mathbb{R}^{n \times n} \). The architecture of the above sliding-mode observer is illustrated in Fig. [1] A design algorithm that is adapted from [17] is summarized in Appendix [A].

However, the observer matching condition (3) is sometimes too restrictive for the practical applications of the above observer. Many physical systems that can be modeled by (1) do not
satisfy the observer matching condition (3). In the following, we first construct a sliding-mode observer for the systems for which the observer matching condition does not hold. We do this by incorporating sliding-mode exact differentiators proposed in [21]. Then we use high-gain approximate differentiators that have similar performance but have lower implementation complexity than sliding-mode exact differentiators.

III. SLIDING-MODE EXACT DIFFERENTIATOR

In this section, we incorporate the sliding-mode exact differentiator proposed in [21] into the sliding-mode observer design presented in Section II to relax the restriction imposed by the observer matching condition (3).

A. Auxiliary Output Signal Generation

We first describe the sliding-mode exact differentiators that are used in [21] to generate auxiliary outputs. Let \( c_i \) be the \( i \)-th row of the output matrix \( C \). Recall that the relative degree of the \( i \)-th output \( y_i \) with respect to the unknown input \( u_2 \) is defined to be the smallest positive integer \( r_i \) such that

\[
\begin{bmatrix}
    c_1 A^k B_2 = 0, & k = 0, \ldots, r_i - 2 \\
    c_i A^{r_i} B_2 \neq 0.
\end{bmatrix}
\]

We can choose integers \( \gamma_i \) (1 \( \leq \gamma_i \leq r_i \)) such that

\[
C_a = \begin{bmatrix}
    c_1 \\
    \vdots \\
    c_1 A^{\gamma_1 - 1} \\
    \vdots \\
    c_p \\
    \vdots \\
    c_p A^{\gamma_p - 1}
\end{bmatrix}
\]

is of full rank with \( \text{rank}(C_a B_2) = \text{rank} B_2 \). It is proved in [21] that the system zeros of the system model given by the triple \((A, B_2, C_a)\) are in the open left-hand complex plane if the
triple \((A, B_2, C)\) satisfies (2). Thus, we can construct the sliding-mode observer of the form (4) for the following system model

\[
\begin{align*}
\dot{x} &= Ax + B_1u_1 + B_2u_2 \\
y_a &= C_ax,
\end{align*}
\]

if the output \(y_a = C_ax\) is available. However, the auxiliary outputs in \(y_a\) are not measurable and additional observers are required to estimate them.

To proceed, let

\[
y_{ij} = c_iA^{j-1}x, \quad i = 1, \ldots, p, \quad j = 1, \ldots, \gamma_i.
\]

Thus, we have \(y_a = [y_a^\top \cdots y_{ap}^\top]^\top\), where \(y_{ai} = [y_{i1} \cdots y_{i\gamma_i}]^\top\). The dynamics of \(y_{ij}, j = 1, \ldots, \gamma_i - 1\), are given by

\[
\begin{align*}
\dot{y}_{i1} &= y_{i2} + c_iB_1u_1 \\
&\quad \vdots \\
\dot{y}_{i(\gamma_i-2)} &= y_{i(\gamma_i-1)} + c_iA^{\gamma_i-3}B_1u_1 \\
\dot{y}_{i(\gamma_i-1)} &= c_iA^{\gamma_i-2}x + c_iA^{\gamma_i-2}B_1u_1,
\end{align*}
\]

where \(x\) and \(u_1\) can be viewed, respectively, as unknown input and known input vectors. We assume as in [21] that \(x\) and \(\dot{x}\) are bounded and

\[
|y_{ij}| \leq d_{ij}, \quad i = 1, \ldots, p \quad \text{and} \quad j = 1, \ldots, \gamma_i,
\]

which implies that \(u_1\) is bounded. Let \(\nu(\cdot)\), the injection term, be defined by the following super twisting algorithm [22],

\[
\begin{align*}
\nu(\cdot) &= \phi(\cdot) + \lambda \cdot |\frac{1}{2} \text{sign}(\cdot) \\
\dot{\phi}(\cdot) &= \alpha \text{sign}(\cdot),
\end{align*}
\]

where \(\lambda\) and \(\alpha\) are positive design parameters. For the system (7), which has triangular input observable form [31], a second-order sliding-mode observer can be constructed as follows:

\[
\begin{align*}
\dot{\hat{y}}_{i1} &= \nu(y_{i1} - \hat{y}_{i1}) + c_iB_1u_1 \\
\dot{\hat{y}}_{i2} &= E_{i1}\nu(\hat{y}_{i2} - \hat{y}_{i2}) + c_iAB_1u_1 \\
&\quad \vdots \\
\dot{\hat{y}}_{i(\gamma_i-1)} &= E_{i(\gamma_i-2)}\nu\left(\hat{y}_{i(\gamma_i-1)} - \hat{y}_{i(\gamma_i-1)}\right) + c_iA^{\gamma_i-2}B_1u_1,
\end{align*}
\]
where $\hat{y}_{i1} = y_{i1}$ and $\hat{y}_{ij} = \nu(\hat{y}_{i(j-1)} - \hat{y}_{i(j-1)})$, $j = 2, \ldots, \gamma_i - 1$, and $E_{ij}$, $j = 1, \ldots, \gamma_i - 2$, are defined as

$$E_{ij} = \begin{cases} 1 & \text{if } |\hat{y}_{ik} - \hat{y}_{ik}| = 0 \text{ for all } k < j \\ 0 & \text{otherwise.} \end{cases}$$

Let $\tilde{y}_{ij} = y_{ij} - \hat{y}_{ij}$. It follows from (7) and (9) that

$$\begin{align*}
\hat{y}_{i1} &= y_{i1} - \nu(y_{i1} - \hat{y}_{i1}) \\
\hat{y}_{i2} &= y_{i2} - E_{i1}\nu(\hat{y}_{i2} - \hat{y}_{i2}) \\
& \vdots \\
\hat{y}_{i(\gamma_i-2)} &= y_{i(\gamma_i-2)} - E_{i(\gamma_i-3)}\nu(\hat{y}_{i(\gamma_i-2)} - \hat{y}_{i(\gamma_i-2)}) \\
\hat{y}_{i(\gamma_i-1)} &= c_iA^{\gamma_i-1}x - E_{i(\gamma_i-2)}\nu(\hat{y}_{i(\gamma_i-1)} - \hat{y}_{i(\gamma_i-1)}).
\end{align*}$$

By choosing sufficiently large $\lambda$ and $\alpha$, (see [20]), the system (10) enters sliding mode on the manifold $\tilde{y}_{i1} = \cdots = \tilde{y}_{i(\gamma_i-1)} = 0$ after a finite time $T_i > 0$, which implies that

$$\hat{y}_{i1} = \cdots = \hat{y}_{i(\gamma_i-1)} = 0. \quad (11)$$

Thus, it follows from (6), (10) and (11) that

$$\begin{align*}
\nu(y_{i1} - \hat{y}_{i1}) &= c_iA^1x \\
\nu(y_{i2} - \hat{y}_{i2}) &= c_iA^2x \\
& \vdots \\
\nu(\hat{y}_{i(\gamma_i-1)} - \hat{y}_{i(\gamma_i-1)}) &= c_iA^{\gamma_i-1}x. 
\end{align*}$$

Let $y_s = [y_{s1}^T \cdots y_{sp}^T]^T$ with

$$y_{si} = \begin{bmatrix} y_{i1} \\ \nu(y_{i1} - \hat{y}_{i1}) \\ \nu(\hat{y}_{i2} - \hat{y}_{i2}) \\
& \vdots \\
& \nu(\hat{y}_{i(\gamma_i-1)} - \hat{y}_{i(\gamma_i-1)}) \end{bmatrix}. \quad (13)$$

It follows from (12) and (13) that $y_s = C_\alpha x$, that is, $y_s = y_a$ after a finite time $T = \max_{1 \leq i \leq p} T_i$.

Then, in [21], the signal $y_s$ derived from the sliding-mode exactor differentiators, described by (9), is directly used to construct a sliding-mode observer presented in [15]. However, the transient response of the constructed sliding-mode observer during the time interval $[t_0, t_0 + T]$, where $t_0$ is the initial time, is not analyzed. In the following subsection, we apply the above sliding-mode exact differentiators to construct the sliding-mode observer.
B. Sliding-Mode Observer Construction

We first introduce the saturation of the signal $y_s$ as $y^s_s = [y^s_{s1} \cdots y^s_{sp}]^\top$ with

$$y^s_{si} = \begin{bmatrix} S_{i1} \text{ sat} \left( \frac{y_{s1}}{S_{i1}} \right) \\ S_{i2} \text{ sat} \left( \frac{y_{s2} - y_{i1}}{S_{i2}} \right) \\ \vdots \\ S_{i\gamma_i} \text{ sat} \left( \frac{y_{(\gamma_i-1)s} - y_{(\gamma_i-1)i}}{S_{i\gamma_i}} \right) \end{bmatrix},$$

where sat$(\cdot)$ is the saturation function and $S_{ij} > d_{ij}$ with $d_{ij}$ defined in (8). Then, we construct the sliding-mode observer of the form

$$\dot{\hat{x}} = A\hat{x} + B_1u_1 + L_a (y^s_s - \hat{y}_a) - B_2E_a(y^s_s, \hat{y}_a, \eta)$$

with $\hat{y}_a = C_a\hat{x}$ and

$$E_a(y^s_s, \hat{y}_a, \eta) = \begin{cases} \eta \frac{F_a(y_a - y^*_s)}{\|F_a(y_a - y^*_s)\|} & \text{if } F_a(\hat{y}_a - y^*_a) \neq 0 \\ 0 & \text{if } F_a(\hat{y}_a - y^*_a) = 0, \end{cases}$$

where $L_a \in \mathbb{R}^{n \times \gamma}$ and $F_a \in \mathbb{R}^{m_2 \times \gamma}$ are matrices such that

$$(A - L_a C_a)^\top P_a + P_a (A - L_a C_a) = -2Q_a < 0$$

and

$$F_a C_a = B_2^\top P_a$$

for some symmetric positive definite $P_a \in \mathbb{R}^{n \times n}$ and $Q_a \in \mathbb{R}^{n \times n}$. The matrices $L_a$, $F_a$ and $P_a$ can be calculated using the design algorithm presented in Appendix A.

Remark 1: In practice, even though $y_{ij}$ is bounded, we may not know the bound of $y_{ij}$, that is, the exact value of $d_{ij}$. However, this does not restrict the choice of $S_{ij}$ significantly. In fact, we are free to select a very large $S_{ij}$ because the purpose of saturation is to guarantee the boundedness of the observer state vector $\dot{\hat{x}}(t)$ during the time interval $[0, t_0 + T]$.

Let $e = \hat{x} - x$ denote the state estimation error. Then the dynamics of the state estimation error are given by

$$\dot{e} = Ae + L_a (y^s_s - \hat{y}_a) - B_2u_2 - B_2E_a(y^s_s, \hat{y}_a, \eta).$$

August 22, 2008 DRAFT
In the following, we analyze the performance of the sliding-mode exact differentiator based sliding-mode observer (14).

**Theorem 1:** For the dynamical system (1) and the sliding-mode observer (14) with sliding-mode exact differentiators (9), if \( \eta \geq \rho \), then \( \lim_{t \to \infty} e(t) = 0 \). Specifically, for any given positive real \( R \), there exists a finite time \( T_f(R) \) such that \( \|e(t)\| \leq R \) for \( t \geq t_0 + T_f(R) \).

**Proof:** For \( t_0 \leq t \leq t_0 + T \), it is guaranteed that the observer state vector \( \hat{x}(t) \) in (14) is bounded because \( u_1, y_s^a \) and \( E_a(y_s^a, \hat{y}_a, \eta) \) are bounded and \( A - L_a C_a \) is Hurwitz. Thus, we know that \( e(t) \) is bounded for \( t_0 \leq t \leq t_0 + T \). For \( t \geq t_0 + T \), we have \( y_s(t) = y_a(t) \) and thus the dynamics of the state estimation error (16) become

\[
\dot{e} = Ae + L_a (y_s - \hat{y}_a) - B_2 u_2 - B_2 E_a(y_s, \hat{y}_a, \eta) \\
= (A - L_a C_a) e - B_2 u_2 - B_2 E_a(y_s, \hat{y}_a, \eta). \tag{17}
\]

Consider the Lyapunov function candidate, \( V = \frac{1}{2} e^\top P_a e \) for \( t \geq t_0 + T \), where \( P_a \) is defined in (III-B). Evaluating the time derivative of \( V \) on the solutions of (17), we obtain

\[
\dot{V} = e^\top P_a (A - L_a C_a) e - e^\top P_a B_2 u_2 - e^\top P_a B_2 E_a(y_s, \hat{y}_a, \eta) \\
= -e^\top Q_a e - (F_a C_a e)^\top u_2 - \eta (F_a C_a e)^\top E_a(y_s, \hat{y}_a, \eta).
\]

If \( F_a C_a e = 0 \), we have

\[
-(F_a C_a e)^\top u_2 - (F_a C_a e)^\top E_a(y_s, \hat{y}_a, \eta) = 0. \tag{18}
\]

On the other hand, if \( F_a C_a e \neq 0 \), we have

\[
-(F_a C_a e)^\top u_2 - (F_a C_a e)^\top E_a(y_s, \hat{y}_a, \eta) \\
= - (F_a C_a e)^\top u_2 - \eta (F_a C_a e)^\top \frac{F_a C_a e}{\|F_a C_a e\|} \\
\leq - (\eta - \rho) \|F_a C_a e\| \\
\leq 0. \tag{19}
\]

It follows from (18) and (19) that in both cases we have

\[
\dot{V} \leq -e^\top Q_a e \leq -\lambda_{\min}(Q_a) \|e\|^2, \tag{20}
\]

which implies that \( \dot{V} < 0 \) and, hence, \( \lim_{t \to \infty} e(t) = 0 \).
We can rewrite (20) as
\[
\dot{V} \leq -\lambda_{\min}(Q_a)\|e\|^2 \leq -2\mu_a V,
\]
where \( \mu_a = \lambda_{\min}(Q_a)/\lambda_{\max}(P_a) \). It follows from (21) and the comparison lemma (see, for example, [32, p. 7] or [33, p. 85]) that
\[
V(t) \leq \exp(-2\mu_a (t - t_0 - T)) V(t_0 + T),
\]
and thus
\[
\|e(t)\| \leq \exp(-\mu_a (t - t_0 - T)) \sqrt{\lambda_{\max}(P_a)/\lambda_{\min}(P_a)} \|e(t_0 + T)\|,
\]
where \( e(t_0 + T) \) is bounded. If \( \sqrt{\lambda_{\max}(P_a)/\lambda_{\min}(P_a)} \|e(t_0 + T)\| > R \), then for any given positive real \( R \), we can find the finite time \( T_f(R) \) such that \( \|e(t)\| \leq R \) for \( t \geq t_0 + T_f(R) \), where \( T_f(R) \) is the solution to the equation
\[
\exp(-\mu_a (T_f(R) - T)) \sqrt{\lambda_{\max}(P_a)/\lambda_{\min}(P_a)} \|e(t_0 + T)\| = R.
\]
Solving the above gives
\[
T_f(R) = T + \frac{1}{\mu_a} \ln \left( \frac{\sqrt{\lambda_{\min}(P_a)} \|e(t_0 + T)\|}{R} \right).
\]
On the other hand, if \( \sqrt{\lambda_{\max}(P_a)/\lambda_{\min}(P_a)} \|e(t_0 + T)\| \leq R \), then \( \|e(t)\| \leq R \) for \( t \geq t_0 + T \). In such a case, we can choose \( T_f(R) = T \). Therefore, there exists a finite time \( T_f(R) \) such that \( \|e(t)\| \leq R \) for \( t \geq t_0 + T_f(R) \), which concludes the proof of the theorem.

**Corollary 1:** For sufficiently large \( \eta \), the sliding surface: \( \{e : \sigma = F_a C_a e = 0\} \), is invariant in the state estimation error space and is reached in finite time.

**Proof:** For a given positive real \( R \), it follows from Theorem 1 that there exists a finite time \( T_f(R) \) such that \( e(t) \) is bounded, that is, \( \sigma(t) \) is bounded for \( t_0 \leq t \leq t_0 + T(R) \) and \( \|e(t)\| \leq R \) for \( t \geq t_0 + T_f(R) \). For \( t \geq t_0 + T_f(R) \), we obtain, using (15) and (17),

\[
\sigma^T \dot{\sigma} = \sigma^T (F_a C_a (A - L_a C_a)e - F_a C_a B_2 u_2 - F_a C_a B_2 E_a(y_s, \hat{y}_a, \eta))
\leq \|F_a C_a (A - L_a C_a)\| \|e\| \|\sigma\| - \sigma^T (B_2^T P_a B_2) u_2 - \eta \sigma^T (B_2^T P_a B_2) \|\sigma\|
\leq R \|F_a C_a (A - L_a C_a)\| \|\sigma\| + \lambda_{\max}(B_2^T P_a B_2) \|u_2\| \|\sigma\| - \eta \lambda_{\min}(B_2^T P_a B_2) \|\sigma\|
\leq - \left( \eta - \frac{\rho \lambda_{\max}(B_2^T P_a B_2) + R \|F_a C_a (A - L_a C_a)\|}{\lambda_{\min}(B_2^T P_a B_2)} \right) \lambda_{\min}(B_2^T P_a B_2) \|\sigma\|. \tag{22}
\]
The matrix $B_2^T P_a B_2$ is symmetric positive definite because $B_2$ is of full rank and $P_a$ is positive definite. Therefore, $\lambda_{\min}(B_2^T P_a B_2) > 0$. If we choose $\eta$ such that

$$\eta \geq \frac{\rho \lambda_{\max}(B_2^T P_a B_2) + R \| F_a C_a (A - L_a C_a) \|}{\lambda_{\min}(B_2^T P_a B_2)} + \varepsilon,$$

where $\varepsilon$ is a small positive constant, then we obtain

$$\sigma^T \dot{\sigma} \leq -\varepsilon \| \sigma \|, \quad (23)$$

which implies that $\{ e : F_a C_a e = 0 \}$ is invariant.

Using the same arguments as in [15, p. 53], we rewrite (23) as

$$\frac{1}{2} \frac{d}{dt} \| \sigma \|^2 \leq -\varepsilon \| \sigma \|. \quad (24)$$

Integrating (24) from $t_0 + T_f(R)$ to $t$, we obtain

$$\| \sigma(t) \| - \| \sigma(t_0 + T_f(R)) \| \leq -\varepsilon (t - t_0 - T_f(R)).$$

Let $T_s$ denote the time the sliding surface is reached. We have

$$\| \sigma(T_s) \| - \| \sigma(t_0 + T_f(R)) \| \leq -\varepsilon (T_s - t_0 - T_f(R)),$$

which implies that

$$T_s \leq t_0 + T_f(R) + \frac{\| \sigma(t_0 + T_f(R)) \|}{\varepsilon}.$$ 

Thus, the proof of the corollary is complete. 

It follows from Corollary [1] that the state estimation error, $e$, enters sliding mode along $\{ e : \sigma = 0 \}$ after a finite time, and therefore

$$\dot{\sigma} = F_a C_a (A - L_a C_a) e - F_a C_a B_2 u_2 - F_a C_a B_2 E_a(y_s, \hat{y}_a, \eta) = 0. \quad (25)$$

We have $\lim_{t \to \infty} e(t) = 0$ and then it follows from (25) that as $t \to \infty$,

$$F_a C_a B_2 u_2 = -F_a C_a B_2 E_a(y_s, \hat{y}_a, \eta). \quad (26)$$

Taking into account (15), we can rewrite (26) as

$$B_2^T P_a B_2 u_2 = -B_2^T P_a B_2 E_a(y_s, \hat{y}_a, \eta).$$

Because $B_2^T P_a B_2$ is invertible, we obtain

$$u_2 = -E_a(y_s, \hat{y}_a, \eta).$$

Thus we can estimate the unknown input $u_2$ with increasing accuracy as $t \to \infty$. 

August 22, 2008
IV. HIGH-GAIN APPROXIMATE DIFFERENTIATOR

Although we can obtain the exact augmented output $y_a$ after a finite time by using sliding-mode exact differentiators, the implementation of second-order sliding-mode observers becomes complicated for large $\gamma_i$. On the other hand, it is also difficult to choose appropriate $\lambda$ and $\alpha$ for the super twisting algorithm. In this section, we propose to use high-gain observers to generate estimates of those auxiliary outputs in the $y_a$.

A. High-Gain Observer Construction

The dynamics of $y_{ai}$, $i = 1, \ldots, p$, are given by

$$
\begin{align*}
\dot{y}_{i1} &= y_{i2} + c_iB_1u_1 \\
& \quad \vdots \\
\dot{y}_{i(\gamma_i-1)} &= y_{i\gamma_i} + c_iA^{\gamma_i-2}B_1u_1 \\
\dot{y}_{i\gamma_i} &= f_i(x, u_2) + c_iA^{\gamma_i-1}B_1u_1,
\end{align*}
$$

which can be written as

$$
\begin{align*}
\dot{y}_{ai} &= \tilde{A}_i y_{ai} + \tilde{b}_{i1} f_i(x, u_2) + \tilde{b}_{i2}u_1 \\
y_{i1} &= \tilde{c}_i y_{ai},
\end{align*}
$$

(27)

where the pair $(\tilde{A}_i, \tilde{b}_{i1})$ is in canonical controllable form which represents the chain of $\gamma_i$ integrators,

$$
f_i(x, u_2) = c_i A^{\gamma_i} x + c_i A^{\gamma_i-1}B_1u_2,
$$

(28)

$\tilde{b}_{i2} = [c_iB_1 \cdots c_iA^{\gamma_i-1}B_1]^T$ and $\tilde{c}_i = [1\ 0 \ \cdots \ 0]$. We also assume for the above system (27) that $x$ and $u_1$ are bounded and $y_{ij}$ satisfies (8). If $\gamma_i > 1$, we construct the following high-gain observers, see Fig. [2]

$$
\begin{align*}
\dot{\hat{y}}_{i1} &= \hat{y}_{i2} + \frac{\alpha_{i1}}{\epsilon} (y_{i1} - \hat{y}_{i1}) + c_iB_1u_1 \\
& \quad \vdots \\
\dot{\hat{y}}_{i(\gamma_i-1)} &= \hat{y}_{i\gamma_i} + \frac{\alpha_{i(\gamma_i-1)}}{\epsilon^{\gamma_i}} (y_{i1} - \hat{y}_{i1}) + c_iA^{\gamma_i-2}B_1u_1 \\
\dot{\hat{y}}_{i\gamma_i} &= \frac{\alpha_{i\gamma_i}}{\epsilon^{\gamma_i}} (y_{i1} - \hat{y}_{i1}) + c_iA^{\gamma_i-1}B_1u_1,
\end{align*}
$$

(29)

where $\epsilon \in (0, 1)$ is a design parameter and $\alpha_{ij}$, $j = 1, \ldots, \gamma_i$, are selected so that the roots of the equation, $s^{\gamma_i} + \alpha_{i1}s^{\gamma_i-1} + \cdots + \alpha_{i(\gamma_i-1)}s + \alpha_{i\gamma_i} = 0$, have negative real parts. Let $y_{hi} =$
Fig. 2. Diagram of the high-gain observer.

\[ \begin{bmatrix} \hat{y}_{i1} & \cdots & \hat{y}_{i\gamma_i} \end{bmatrix}^T \text{ and } l_i = \begin{bmatrix} \alpha_{i1}/\epsilon & \cdots & \alpha_{i\gamma_i}/\epsilon^{\gamma_i} \end{bmatrix}^T. \]

We can rewrite (29) as

\[ \dot{y}_{hi} = \bar{A}_i y_{hi} + l_i \bar{c}_i (y_{ai} - y_{hi}) + \bar{b}_i u_1. \] (30)

If \( \gamma_i = 1 \), we do not need to construct the above high-gain observer (30) because of the availability of \( y_{i1} \). In such a case, we have \( y_{hi} = y_{ai} = y_{i1} \). To proceed, let \( \zeta_i = 0 \) if \( \gamma_i = 1 \) and let \( \zeta_i = \begin{bmatrix} \zeta_{i1} & \cdots & \zeta_{i\gamma_i} \end{bmatrix}^T \) if \( \gamma_i > 1 \), where

\[ \zeta_{ij} = \frac{y_{ij} - \hat{y}_{ij}}{\epsilon^{\gamma_i-j}}, \quad j = 1, \ldots, \gamma_i. \] (31)

It follows from (27) and (30) that

\[ \epsilon \dot{\zeta}_i = \bar{A}_i \zeta_i + \epsilon \bar{b}_i f_i(x, u_2), \] (32)

where \( \bar{A}_i = \epsilon^{-1} (\bar{A}_i - l_i \bar{c}_i) D \) is a Hurwitz matrix independent of \( \epsilon \).

**Proposition 1:** For the high-gain observer (30), there exists a finite time \( T_i(\epsilon) \) such that \( ||\zeta_i(t)|| \leq \beta_i \epsilon \) for some positive constant \( \beta_i \) and \( t \geq t_0 + T_i(\epsilon) \). Moreover, \( T_i(\epsilon) \) approaches zero when \( \epsilon \) approaches to zero, that is, \( \lim_{\epsilon \to 0^+} T_i(\epsilon) = 0 \).

**Proof:** See Appendix B. \( \square \)

It follows from (31) that \( y_{ai} - y_{hi} = D_i \zeta_i \), where \( D_i = \text{diag}[\epsilon^{\gamma_i-1} \epsilon^{\gamma_i-2} \cdots 1] \). Let \( y_h = [y_{h1}^T \cdots y_{hp}^T]^T, \quad D = \text{diag}[D_1 \cdots D_p] \) and \( \zeta = [\zeta_1^T \cdots \zeta_p^T]^T \). We have

\[ y_a - y_h = D \zeta. \] (33)
Note that the induced Euclidean norm of $D$ is 1, that is, $\|D\| = 1$. Let $\beta_i = 0$ and $T_i(\epsilon) = 0$ if $\gamma_i = 1$. Thus, it follows from Proposition 1 that $\|\zeta\| \leq \beta \epsilon$, where $\beta = (\sum_{i=1}^{p} \beta_i^2)^{\frac{1}{2}}$, after a finite time $T(\epsilon) = \max_{1 \leq i \leq p} T_i(\epsilon)$, and $\lim_{\epsilon \to 0} T(\epsilon) = 0$.

**B. State Estimation Performance Analysis**

In order to eliminate the peaking phenomena that accompanies the operation of the above high-gain observer [24], we introduce the saturation of the signal $y_h$ such that $y_h^s = [y_{h1}^s \ldots y_{hp}^s]^T$, where $y_{hi}^s = y_{ai} = y_{i1}$ if $\gamma_i = 1$ and

$$y_{hi}^s = \begin{bmatrix} S_{i1} \text{sat} \left( \frac{y_{i1}}{S_{i1}} \right) \\ \vdots \\ S_{i\gamma_i} \text{sat} \left( \frac{y_{i\gamma_i}}{S_{i\gamma_i}} \right) \end{bmatrix}$$

with $S_{ij} > d_{ij}$ if $\gamma_i > 1$. Then we construct the following sliding-mode observer,

$$\dot{x} = A \dot{x} + B_1 u_1 + L_a (y_h^s - \hat{y}_a) - B_2 E_a (y_h^s, \hat{y}_a, \eta),$$

where $\dot{y}_a = C_a \dot{x}$ and

$$E_a (y_h^s, \hat{y}_a, \eta) = \begin{cases} \frac{F_a (y_a - y_h^s)}{\|F_a (y_a - y_h^s)\|} & \text{if } F_a (\hat{y}_a - y_h^s) \neq 0 \\ 0 & \text{if } F_a (\hat{y}_a - y_h^s) = 0. \end{cases}$$

The matrices $L_a$ and $F_a$ are identical to those in (14). A block diagram of the above sliding-mode observer is given in Fig. 3. The It follows from (1) and (34) that
Consider the Lyapunov function candidate, $V$ for positive constants specifically, with high-gain approximate differentiators (23), we analyze the performance of the proposed high-gain approximate differentiator based sliding-mode observer given by (23).

**Theorem 2:** For the dynamical system (1) and the associated sliding-mode observer (23) with high-gain approximate differentiators (30), there exists a constant $\epsilon^* \in (0, 1)$ such that if $\epsilon \in (0, \epsilon^*)$ and $\eta \geq \rho$, then the state estimation error $e(t)$ is uniformly ultimately bounded. Specifically, $\|e(t)\| \leq \kappa(\epsilon)$ after a finite time $T_f(\epsilon)$, where

$$\kappa(\epsilon) = \frac{\kappa_1\epsilon + \sqrt{\kappa_1^2\epsilon^2 + 4\mu_a\kappa_1\epsilon}}{2\mu_a} \sqrt{\frac{2}{\lambda_{\min}(P_a)}}$$

for positive constants $\mu_a$, $\kappa_1$ and $\kappa_2$.

**Proof:** It follows from Proposition 1 that $\|\zeta(t)\| \leq \beta \epsilon$ for $t \geq t_0 + T(\epsilon)$. Then, it follows from (33) that $\|y(\epsilon) - y(t)\| \leq \beta \epsilon$ for $t \geq t_0 + T(\epsilon)$. There exists a constant $\bar{\epsilon}$ such that if $\|y(\epsilon) - y(t)\| \leq \beta \bar{\epsilon}$, then $y(t)$ is not saturated, that is, $y^*(t) = y(t)$. Thus, we can choose $\epsilon^* = \min\{\bar{\epsilon}, 1\}$ such that if $\epsilon \in (0, \epsilon^*)$, then $\|\zeta(t)\| \leq \beta \epsilon$ and $y^*(t) = y(t)$ after a finite time $T(\epsilon)$.

For $t_0 \leq t \leq t_0 + T(\epsilon)$, it is guaranteed that the observer state vector $\hat{x}(t)$ in (34) is bounded because $u_1$, $y^*_h$ and $E_a(y^*_h, \hat{y}_a, \eta)$ are bounded and $A - L_aC_a$ is Hurwitz. Thus, $e(t)$ is bounded for $t_0 \leq t \leq t_0 + T(\epsilon)$. For $t \geq t_0 + T(\epsilon)$, because $y^*_h(t) = y_h(t)$ and $y_h = y_a - D\zeta$, the dynamics of the state estimation error (35) become

$$\dot{e} = A e + L_a (y_h - \hat{y}_a) - B_2 u_2 - B_2 E_a(y_h, \hat{y}_a, \eta)$$

$$= A e + L_a (y_a - D\zeta - \hat{y}_a) - B_2 u_2 - B_2 E_a(y_h, \hat{y}_a, \eta)$$

$$= (A - L_a C_a) e - L_a D\zeta - B_2 u_2 - B_2 E_a(y_h, \hat{y}_a, \eta).$$

Consider the Lyapunov function candidate, $V = \frac{1}{2} e^T P_a e$ for $t \geq t_0 + T(\epsilon)$. Evaluating the time derivative of $V$ on the solutions of (36), we obtain

$$\dot{V} = e^T P (A - L_a C_a) e - e^T P_a L_a D\zeta - e^T P_a B_2 u_2 - e^T P_a B_2 E_a(y_h, \hat{y}_a, \eta)$$

$$= -e^T Q_a e - e^T P_a L_a D\zeta - (F_a C_a e)^T u_2 - (F_a C_a e)^T E_a(y_h, \hat{y}_a, \eta)$$

$$= -e^T Q_a e - e^T P_a L_a D\zeta - (F_a C_a e + F D\zeta - F_a D\zeta)^T u_2$$

$$- (F_a C_a e + F_a D\zeta - F_a D\zeta)^T E_a(y_h, \hat{y}_a, \eta)$$
\[ -e^\top Q_a e - e^\top P_a L_a D \zeta + (F_a D \zeta)^\top u_2 + (F_a D \zeta)^\top E_a(y_h, \hat{y}_a, \eta) \]
\[ - (F_a C_a e + F D \zeta)^\top u_2 - (F_a C_a e + F_a D \zeta)^\top E_a(y_h, \hat{y}_a, \eta). \]

If \( F_a(C_a e + D \zeta) = 0 \), then
\[ - (F_a C_a e + F_a D \zeta)^\top u_2 - (F_a C_a e + F_a C_a e)^\top E_a(y_a, \hat{y}_a, \eta) = 0. \]

On the other hand, if \( F_a(C_a e + D \zeta) \neq 0 \), then
\[ - (F_a C_a e + F_a D \zeta)^\top u_2 - (F_a C_a e + F_a C_a e)^\top E_a(y_a, \hat{y}_a, \eta) \]
\[ = - (F_a C_a e + F_a D \zeta)^\top u_2 - \eta(F_a C_a e + F_a D \zeta)^\top \frac{F_a C_a e + F_a D \zeta}{\|F_a C_a e + F_a D \zeta\|} \]
\[ \leq - (\eta - \rho) \|F_a C_a e + F_a D \zeta\| \]
\[ \leq 0. \]

It follows from (37) and (38) that in both cases we have
\[ \dot{V} \leq - e^\top Q_a e - e^\top P_a L_a D \zeta + (F_a D \zeta)^\top u_2 + (F_a D \zeta)^\top E_a(y_h, \hat{y}_a, \eta). \]

Performing some manipulations gives
\[ \dot{V} \leq - \lambda_{\min}(Q_a) \|e\|^2 + \|P_a L_a\| \|D\| \|\zeta\| \|e\| + (\eta + \rho) \|F_a\| \|D\| \|\zeta\| \]
\[ \leq - \lambda_{\min}(Q_a) \|e\|^2 + \beta \|P_a L_a\| \|e\| + (\eta + \rho) \beta \|F_a\| \]
\[ = - 2\mu_a V + \kappa_1 \epsilon \sqrt{V} + \kappa_2 \epsilon, \]
where
\[ \kappa_1 = \sqrt{\frac{\beta \|P_a L_a\|}{\lambda_{\max}(P_a)}} \quad \text{and} \quad \kappa_2 = (\eta + \rho) \beta \|F_a\|. \]

It follows from (40) that
\[ \dot{V} \leq - \mu_a V - \mu_a V + \kappa_1 \epsilon \sqrt{V} + \kappa_2 \epsilon \]
\[ = - \mu_a V - \left( \sqrt{V} - R_- \right) \left( \sqrt{V} - R_+ \right), \]
where
\[ R_- = \frac{\kappa_1 \epsilon - \sqrt{\kappa_1^2 \epsilon^2 + 4 \mu_a \kappa_2 \epsilon}}{2 \mu_a} < 0 \quad \text{and} \quad R_+ = \frac{\kappa_1 \epsilon + \sqrt{\kappa_1^2 \epsilon^2 + 4 \mu_a \kappa_1 \epsilon}}{2 \mu_a} > 0. \]

We conclude from (41) that \( \dot{V} < 0 \) when \( \|e\| > R_+ \). In summary, the state estimation error \( e \) is uniformly ultimately bounded with respect to any closed ball of radius greater than \( R_+ \).
Hence, as long as $\sqrt{V} > R_+$, that is, for $V > R^2_+$, we have
\[
\left(\sqrt{V} - R_-\right)\left(\sqrt{V} - R_+\right) < 0.
\]
Therefore, if $V(t_0 + T(\epsilon)) = V(e(t_0 + T(\epsilon))) > R^2_+$ and $V(t) > R^2_+$ for $t \geq t_0 + T_f(\epsilon)$, then
\[
\dot{V} \leq -\mu_a V,
\]
which implies that
\[
V(t) \leq \exp\left(-\mu_a (t - t_0 - T(\epsilon))\right) V(t_0 + T(\epsilon)).
\]
Thus, we can find a finite time $T_f(\epsilon)$ such that $V(t) \leq R^2_+$ for $t \geq t_0 + T(\epsilon)$, where $T_f(\epsilon)$ is the solution to the equation
\[
V(t_0 + T(\epsilon)) \exp\left(-\mu_a (T_f(\epsilon) - T(\epsilon))\right) = R^2_+,
\]
and has the form,
\[
T_f(\epsilon) = T(\epsilon) + \frac{1}{\mu_a} \ln \left(\frac{V(t_0 + T(\epsilon))}{R^2_+}\right).
\]
On the other hand, if $V(t_0 + T(\epsilon)) \leq R^2_+$, then $V(t) \leq R^2_+$ for $t \geq t_0 + T(\epsilon)$. In such a case, we can choose $T_f(\epsilon) = T(\epsilon)$. Therefore, there exists a finite time $T_f(\epsilon)$ such that $V(t) \leq R^2_+$ for $t \geq t_0 + T_f(\epsilon)$, which implies $\|e(t)\| \leq \kappa(\epsilon)$, where
\[
\kappa(\epsilon) = \frac{\kappa_1 \epsilon + \sqrt{\kappa_1^2 \epsilon^2 + 4 \mu_a \kappa_1 \epsilon}}{2 \mu_a} \sqrt{\frac{2}{\lambda_{\text{min}}(P_a)}}.
\]
The proof of the theorem is complete.

Remark 2: It follows from Theorem 2 that the state estimation error enters the closed ball \( \{ e : \|e\| \leq \kappa(\epsilon) \} \) after a finite time $T_f(\epsilon)$. It is easy to verify that
\[
\lim_{\epsilon \to 0^+} T_f(\epsilon) = \begin{cases} \infty & \text{if } V(t_0) \neq 0 \\ 0 & \text{if } V(t_0) = 0, \end{cases}
\]
because $\lim_{\epsilon \to 0^+} T(\epsilon) = 0$ and $\lim_{\epsilon \to 0^+} R^+ = 0$.

Moreover, the radius of the above closed ball can be adjusted by the design parameter $\epsilon$ and because $\lim_{\epsilon \to 0^+} \kappa(\epsilon) = 0$, the state estimation error $e$ converges to the origin as $\epsilon$ goes to zero.

Corollary 2: The hyperplane, \( \{ (e, \zeta) : \sigma = F_a(C_a e + D\zeta) = 0 \} \), is invariant in the $(e, \zeta)$-space and is reached in finite time for sufficiently large $\eta$. 

August 22, 2008 DRAFT
Proof: For \( t \geq t_0 + T_f(\epsilon) \), it follows from (36) that
\[
\sigma^T \dot{\sigma} = \sigma^T \left( F_a C_a \dot{e} + F_a D \dot{\zeta} \right)
\]
\[
= \sigma^T \left( F_a C_a (A - L_a C_a) e - F_a C_a L_a D \zeta - F_a C_a B_2 u_2 \right)
\]
\[
- F_a C_a B_2 E_a (y_h, \dot{y}_a, \eta) + \frac{1}{\epsilon} F_a D \dot{A}_a \zeta + F_a D \dot{B}_1 f (x, u_2)
\]
\[
\leq \| F_a C_a (A - L_a C_a) \| \| e \| \| \sigma \| + \| F_a C_a L_a \| \| D \| \| \zeta \| \| \sigma \| + \sigma^T (B_2 P_a B_2) u_2
\]
\[
- \eta \sigma^T (B_2 P_a B_2) \frac{\sigma}{\| \sigma \|} + \frac{1}{\epsilon} \| F_a \dot{A}_a \| \| D \| \| \zeta \| \| \sigma \| + \| F_a \dot{B}_1 \| \| f (x, u_2) \| \| \sigma \|
\]
\[
\leq \kappa (\epsilon) \| F_a C_a (A - L_a C_a) \| \| \sigma \| + \beta \epsilon \| F_a C_a L_a \| \| \sigma \| + \lambda_{\max} (B_2^T P_a B_2) \| u_2 \| \| \sigma \|
\]
\[
- \eta \lambda_{\min} (B_2^T P_a B_2) \| \sigma \| + \beta \| F_a \dot{A}_a \| \| \sigma \| + \beta_1 \| F_a \dot{B}_1 \| \| \sigma \|
\]
\[
= - \left( \eta - \frac{\kappa_3 + \kappa_4 + \kappa_5 + \kappa_6 + \kappa_7}{\lambda_{\min} (B_2^T P_a B_2)} \right) \lambda_{\min} (B_2^T P_a B_2) \| \sigma \|,
\] (42)
where \( \kappa_3 = \kappa (\epsilon) \| F_a C_a (A - L_a C_a) \|, \kappa_4 = \beta \epsilon \| F_a C_a L_a \|, \kappa_5 = \rho \lambda_{\max} (B_2^T P_a B_2), \kappa_6 = \beta \| F_a \dot{A}_a \|, \text{ and } \kappa_7 = \| F_a \dot{B}_1 \| \| f (x, u_2) \| \). It follows from (42) that if we choose \( \eta \) such that
\[
\eta \geq \frac{\kappa_3 + \kappa_4 + \kappa_5 + \kappa_6 + \kappa_7}{\lambda_{\min} (B_2^T P_a B_2)} + \epsilon,
\]
where \( \epsilon \) is a small positive constant, then
\[
\sigma^T \dot{\sigma} \leq - \epsilon \| \sigma \|,
\] (43)
which implies the above hyperplane is invariant. On the other hand, it can be shown that the sliding surface will be reached in finite time using the same arguments as in the proof of Corollary 1. This concludes the proof of the corollary. \( \blacksquare \)

C. Unknown Input Reconstruction

Our objective in this subsection is to show that we can use the proposed architecture to estimate the unknown input \( u_2 \). We will show that
\[
u_2 \approx - E_a (y_h, \dot{y}_a, \eta),
\] (44)
after a finite time for sufficiently small \( \epsilon \). We proceed as follows. First, by Corollary 2 we note that the manifold \( \{(e, \zeta) : \sigma = 0\} \) is invariant and is reached after a finite time, and therefore
\[
\dot{\sigma} = F_a C_a (A - L_a C_a) e - F_a C_a L_a D \zeta - F_a C_a B_2 u_2
\]
\[
- F_a C_a B_2 E_a (y_h, \dot{y}_a, \eta) + F_a D \dot{\zeta} = 0.
\] (45)
uniformly continuous function is not necessarily uniformly continuous. The square wave on
Note that a uniformly continuous function is weakly uniformly continuous. However, a weakly
always be made left-continuous by changing the function values at the points of discontinuity.

We then show that for sufficiently small \( \epsilon \), \( \|e(t)\| \), \( \|\zeta(t)\| \) and \( \||\zeta|| \) become negligible after a finite time, which result in (44).

By Proposition 1, we have \( \|\zeta(t)\| \leq \beta \epsilon \) for \( t \geq t_0 + T(\epsilon) \). By Theorem 2, we have \( \|e(t)\| \leq \kappa(\epsilon) \) for \( t \geq t_0 + T_f(\epsilon) \), where \( \lim_{\epsilon \to 0^+} \kappa(\epsilon) = 0 \). Thus, it remains to show that, for sufficiently small \( \epsilon \), \( \|\zeta(t)\| \) becomes negligible after a finite time. In what follows, we perform preliminary manipulations before formally proceeding with the proof. Recall that \( \zeta = [\zeta^T_1 \cdots \zeta^T_p]^T \). Thus, we only need to prove that, for sufficiently small \( \epsilon \), \( \|\zeta_i(t)\|, i = 1, \ldots, p \), becomes negligible after a finite time. We first rewrite (32) as

\[
\dot{\zeta}_i(t) = \frac{1}{\epsilon} \bar{A}_{ci} \zeta_i(t) + \bar{b}_{i1} f_i(x(t), u_2(t))
= \frac{1}{\epsilon} \bar{A}_{ci} \zeta_i(t) + v_i(t),
\]

where \( v_i(t) = \bar{b}_{i1} f_i(x(t), u_2(t)) \). Because \( x(t) \) and \( u_2(t) \) are bounded, it follows from (28) that \( f_i(x(t), u_2(t)) \) is bounded. Thus, \( v_i(t) \) is a bounded measurable function. It is well known the the solution to (47) has the form

\[
\zeta_i(t) = \exp \left( \frac{1}{\epsilon} \bar{A}_{ci} (t - t_0) \right) \zeta_i(t_0) + \int_{t_0}^t \exp \left( \frac{1}{\epsilon} \bar{A}_{ci} (t - s) \right) v_i(s) ds.
\]

Performing a change of variables in the integral of (48) by \( z = (t - s) / \epsilon \), we obtain

\[
\zeta_i(t) = \exp \left( \frac{1}{\epsilon} \bar{A}_{ci} (t - t_0) \right) \zeta_i(t_0) + \epsilon \int_0^{(t-t_0)/\epsilon} \exp (\bar{A}_{ci} z) v_i(t - \epsilon z) dz.
\]

To proceed, two notions regarding the function \( v_i(t) \) are defined.

**Definition 1:** A function \( v_i(t) \) is left-continuous if \( \lim_{\epsilon \to 0^+} v_i(t - \epsilon) = v_i(t) \) for all \( t \).

**Definition 2:** A function \( v_i(t) \) defined on \( S \subset \mathbb{R} \) is weakly uniformly continuous if for every \( \nu > 0 \), there exists a \( \delta > 0 \) such that for each interval \( \Omega \subset S \) with length less than \( \delta \), \( \|v_i(s) - v_i(t)\| < \nu \) for \( s, t \in \Omega \).

**Remark 3:** A function with simple jump discontinuities, for example, a square wave, can always be made left-continuous by changing the function values at the points of discontinuity. Note that a uniformly continuous function is weakly uniformly continuous. However, a weakly uniformly continuous function is not necessarily uniformly continuous. The square wave on
the complement of its switching points is an example of a function that is weakly uniformly continuous but not uniformly continuous. If $S$ is connected, then two notions of uniformly continuity are equivalent.

In the following, we use $S_1 + S_2$, where $S_1, S_2 \subset \mathbb{R}$, to denote the set $\{s_1 + s_2 : s_1 \in S_1, s_2 \in S_2\}$. If $S_1$ or $S_2$ is empty, then $S_1 + S_2$ is defined to be empty.

**Theorem 3:** Consider the dynamics given by (47), where $A_{ci}$ is Hurwitz and $v(t)$ is bounded. Let $J$ denote the set of points at which $v(t)$ is discontinuous and let $\tau > t_0 > 0$. If $v(t)$ is left-continuous, then $\lim_{\epsilon \to 0^+} \hat{\zeta}_i(t) = 0$ for each $t > t_0 \geq 0$. Moreover, if $v(t)$ is also weakly uniformly continuous on $[\tau, \infty) \setminus J$, then the convergence of $\hat{\zeta}_i(t)$ to 0 as $\epsilon \to 0^+$ is uniform on $[\tau, \infty) \setminus (J + (0, \xi))$ for each $\xi > 0$. In particular, if $v_i(t)$ is uniformly continuous, then the convergence is uniform on $[\tau, \infty)$.

**Proof:** It follows from (49) that

$$\frac{\zeta_i(t)}{\epsilon} = \exp \left( \frac{1}{\epsilon} A_{ci} (t - t_0) \right) \frac{\zeta_i(t_0)}{\epsilon} + \int_0^{(t-t_0)/\epsilon} \exp \left( \bar{A}_{ci} z \right) v_i(t - \epsilon z) \, dz. \quad (50)$$

Note that the matrix $\bar{A}_{ci}$ is Hurwitz. Let $-\lambda, \lambda > 0$, denote the maximum of the real parts of its eigenvalues. The first term on the right hand side of (50) satisfies

$$\left\| \exp \left( \frac{1}{\epsilon} A_{ci} (t - t_0) \right) \frac{\zeta_i(t_0)}{\epsilon} \right\| \leq \frac{M_1}{\epsilon} \| \zeta_i(t_0) \| \exp \left( -\frac{1}{\epsilon} \lambda (t - t_0) \right), \quad (51)$$

because

$$\left\| \exp \left( \frac{1}{\epsilon} A_{ci} (t - t_0) \right) \right\| \leq M_1 \exp \left( -\frac{1}{\epsilon} \lambda (t - t_0) \right),$$

for some $M_1 > 0$. Because the initial conditions for the high-gain observers are always bounded, there exists $M_2 > 0$ such that

$$\| \zeta_i(t_0) \| \leq \frac{M_2}{e^{\gamma \tau - 1}}.$$

Substituting the above into (51), we obtain

$$\left\| \exp \left( \frac{1}{\epsilon} A_{ci} (t - t_0) \right) \frac{\zeta_i(t_0)}{\epsilon} \right\| \leq \frac{M_1 M_2}{e^{\gamma \tau}} \exp \left( -\frac{1}{\epsilon} \lambda (t - t_0) \right). \quad (52)$$

By calculus, for each $t > t_0 \geq 0$, we have

$$\lim_{\epsilon \to 0^+} \frac{M_1 M_2}{e^{\gamma \tau}} \exp \left( -\frac{1}{\epsilon} \lambda (t - t_0) \right) = 0,$$

and thus,

$$\lim_{\epsilon \to 0^+} \exp \left( \frac{1}{\epsilon} A_{ci} (t - t_0) \right) \frac{\zeta_i(t_0)}{\epsilon} = 0. \quad (53)$$

August 22, 2008
We next consider the second term on the right hand side of (50). Let

\[ g_i(z) = \exp(\overline{A}_cz) \mathbf{v}_i(t - \varepsilon z)I_{[0, (t-t_0)/\varepsilon]}(z), \]

where \( I_{[0, (t-t_0)/\varepsilon]}(z) \) denotes the indicator function for the interval \([0, (t-t_0)/\varepsilon)\). Observer that for each \( z \geq 0 \), if \( \mathbf{v}_i(t) \) is left-continuous, then

\[ \lim_{\varepsilon \to 0^+} g_i(z) = \exp(\overline{A}_cz) \mathbf{v}_i(t). \]

Because \( \mathbf{v}_i(t) \) is bounded, there exists \( M_3 > 0 \) such that \( \|\mathbf{v}_i(t)\| \leq M_3 \) for all \( t > t_0 \). Thus, for each \( z \geq 0 \) and \( \varepsilon \in (0, 1) \),

\[ \|g_i(z)\| \leq M_1M_3\exp(-\lambda z). \]

Because \( \lambda > 0 \), the function \( \exp(-\lambda z) \) is integrable on \([0, \infty)\). Then we can thus apply Lebesgue’s dominated convergence theorem \([34, \text{page 45}]\) (to each component) such that for each \( t > t_0 \geq 0 \),

\[ \lim_{\varepsilon \to 0^+} \int_0^{(t-t_0)/\varepsilon} \exp(\overline{A}_cz) \mathbf{v}_i(t - \varepsilon z)dz = \lim_{\varepsilon \to 0^+} \int_0^{\infty} g_i(z)dz \]

\[ = \int_0^{\infty} \exp(\overline{A}_cz) \mathbf{v}_i(t)dz \]

\[ = \left( \int_0^{\infty} \exp(\overline{A}_cz)dz \right) \mathbf{v}_i(t) \]

\[ = \overline{A}_c^{-1} \left( \lim_{z \to \infty} \exp(\overline{A}_cz) - I_{\gamma_1} \right) \mathbf{v}_i(t) \]

\[ = -\overline{A}_c^{-1} \mathbf{v}_i(t). \quad (54) \]

We have \( \lim_{z \to \infty} \exp(\overline{A}_cz) = 0 \) because \( \overline{A}_c \) is Hurwitz. Therefore, it follows from (50), (53) and (54) that

\[ \lim_{\varepsilon \to 0^+} \frac{\zeta_i(t)}{\varepsilon} = -\overline{A}_c^{-1} \mathbf{v}_i(t). \quad (55) \]

Combining (47) and (55), we conclude that

\[ \lim_{\varepsilon \to 0^+} \frac{\dot{\zeta}_i(t)}{\varepsilon} = \overline{A}_c \lim_{\varepsilon \to 0^+} \frac{\zeta_i(t)}{\varepsilon} + \mathbf{v}_i(t) = 0 \]

for each \( t > t_0 \geq 0 \).

Now we consider the case when \( \mathbf{v}_i(t) \) is also weakly uniformly continues on \([\tau, \infty)\) \( \setminus J \).

Let \( \nu > 0 \) and \( \tau > t_0 > 0 \). We can use (50) to estimate the difference between \( \zeta_i(t)/\varepsilon \) and \(-\overline{A}_c^{-1} \mathbf{v}_i(t)\). For \( t \geq \tau \), it follows from (52) that there exists a constant \( \mu_1 \in (0, 1) \) such that

\[ \left\| \exp\left(\frac{1}{\varepsilon} \overline{A}_c(t - t_0) \right) \frac{\zeta_i(t_0)}{\varepsilon} \right\| \leq \frac{\nu}{2} \]
for $\epsilon \in (0, \mu_1)$. We next analyze the difference between the second term on the right hand side of (50) and $-A_{ci}^{-1}v_i(t)$. Let $t \geq \tau$. Then we have
\[
\left\| \int_0^{(t-\delta_0)/\epsilon} \exp(A_{ci}z) v_i(t - \epsilon z)dz + A_{ci}^{-1}v_i(t) \right\|
\]
\[
= \left\| \int_0^{(t-\delta_0)/\epsilon} \exp(A_{ci}z) [v_i(t - \epsilon z) - v_i(t)]dz \right. \\
+ \left. \int_0^{(t-\delta_0)/\epsilon} \exp(A_{ci}z) v_i(t)dz + A_{ci}^{-1}v_i(t) \right\|
\]
\[
\leq M_1 \int_0^{(t-\delta_0)/\epsilon} \exp(-\lambda z) \|v_i(t - \epsilon z) - v_i(t)\|dz \\
+ \left\| \int_0^{(t-\delta_0)/\epsilon} \exp(A_{ci}z) v_i(t)dz + A_{ci}^{-1}v_i(t) \right\|.
\]
We analyze the terms in the above sum separately.

Let $S = [\tau, \infty) \setminus (J + (0, \xi))$. The set $S$ is closed and it may be empty. For each $t \in S$, the distance from $t$ to the nearest point of discontinuity less than $t$ is at least $\xi$ since all points with distance less than $\xi$ to the right of a point of discontinuity are removed. Note that it is possible for $S$ to contain a point of discontinuity.

Let $t \in S$. By assumption, $v_i(t)$ is weakly uniformly continuous on $[\tau, \infty) \setminus J$. Because $(t - \xi, t) \subset [\tau, \infty) \setminus J$, we can choose $\delta$, independent of $t$, such that $0 < \delta < \xi$ and $\|v_i(s) - v_i(w)\| \leq \lambda \nu/(8M_1)$ for $s, w \in (t - \delta, t)$. Because $v_i(t)$ is left-continuous, we conclude, by letting $w \to t^-$, that $\|v_i(s) - v_i(t)\| \leq \lambda \nu/(8M_1)$ for $s \in (t - \delta, t]$. Then we have
\[
M_1 \int_0^{(t-\delta_0)/\epsilon} \exp(-\lambda z) \|v_i(t - \epsilon z) - v_i(t)\|dz \\
= M_1 \left( \int_0^{\delta/\epsilon} + \int_{\delta/\epsilon}^{(t-\delta_0)/\epsilon} \exp(-\lambda z) \|v_i(t - \epsilon z) - v_i(t)\|dz \right) \\
< \frac{\lambda \nu}{8} \int_0^{\delta/\epsilon} \exp(-\lambda z) dz + 2M_1M_3 \int_{\delta/\epsilon}^{(t-\delta_0)/\epsilon} \exp(-\lambda z) dz \\
< \frac{\lambda \nu}{8} \int_0^{\infty} \exp(-\lambda z) dz + 2M_1M_3 \int_{\delta/\epsilon}^{\infty} \exp(-\lambda z) dz \\
= \frac{\nu}{8} + \frac{2M_1M_3}{\lambda} \exp\left(-\frac{\lambda \delta}{\epsilon}\right).
\]
Choose $\mu_2 \in (0, 1)$ such that

$$\frac{2M_1M_3}{\lambda} \exp \left( -\frac{\lambda \delta}{\epsilon} \right) \leq \frac{\nu}{8}$$

for $\epsilon \in (0, \mu_2)$. It follows that for $t \in S$ and $\epsilon \in (0, \mu_2)$,

$$M_1 \int_0^{(t-t_0)/\epsilon} \exp \left( -\lambda z \right) \|v_i(t-\epsilon z) - v_i(t)\|dz \leq \frac{\nu}{4}.$$

On the other hand, we have

$$\left\| \int_0^{(t-t_0)/\epsilon} \exp \left( \bar{A}_{ci}z \right) v_i(t)dz + \bar{A}_{ci}^{-1}v_i(t) \right\|
\leq M_1M_3 \left\| \bar{A}_{ci}^{-1} \right\| \exp \left( -\frac{\lambda(t-t_0)}{\epsilon} \right)
\leq M_1M_3 \left\| \bar{A}_{ci}^{-1} \right\| \exp \left( -\frac{\lambda(\tau-t_0)}{\epsilon} \right),$$

because $\lambda > 0$, $\epsilon > 0$ and $t \geq \tau$. Choose $\mu_3 \in (0, 1)$ such that

$$M_1M_3 \left\| \bar{A}_{ci}^{-1} \right\| \exp \left( -\frac{\lambda(\tau-t_0)}{\epsilon} \right) < \frac{\nu}{4}$$

for $\epsilon \in (0, \mu_3)$. Let $\mu = \min\{\mu_1, \mu_2, \mu_3\}$. Then, combining the above inequalities, we conclude that for $t \in S$ and $\epsilon \in (0, \mu)$,

$$\left\| \frac{\zeta_i(t)}{\epsilon} + \bar{A}_{ci}^{-1}v_i(t) \right\| \leq \nu,$$

which implies that $\zeta_i(t)/\epsilon$ converges uniformly to $-\bar{A}_{ci}^{-1}v_i(t)$ on $S$. The uniform convergence of $\dot{\zeta}_i(t)$ to 0 on $[\tau, \infty) \setminus (J + (0, \xi))$ follows immediately.

If $v_i(t)$ is uniformly continuous, then $J$ is empty which implies that $J + (0, \xi)$ is empty. Thus, it follows that the convergence of $\dot{\zeta}_i(t)$ to 0 is uniform on $[\tau, \infty)$ and the proof of the theorem is complete.

The estimation of the unknown inputs illustrated in Section V will show that one cannot expect uniform convergence in an immediate neighborhood of a jump discontinuity and a small interval to the right of such a discontinuity must be excised to guarantee uniform convergence. On the other hand, the above theorem does not hold if the function $v_i(t)$ is only weakly uniformly continuous on $[\tau, T] \setminus J$ or uniformly continuous on $[\tau, T]$ for $T \geq \tau$. For example, if the
unknown input $u_2(t)$ contains signals of the form $\sin(t^2)$, then $v_i(t)$ is only uniformly continuous on $[\tau, T]$ for $T > \tau$. Thus, we have the following local version of Theorem 3.

**Corollary 3:** Consider the dynamics given by (47), where $\bar{A}_i$ is Hurwitz and $v_i(t)$ is bounded. Let $J$ denote the set of discontinuities of $v_i(t)$ and let $0 < t_0 < \tau < T$. Suppose $v_i(t)$ is left-continuous and weakly uniformly continuous on $[\tau, T] \setminus J$ (or uniformly continuous on $[\tau, T]$). Then for each $\xi > 0$, the convergence of $\dot{\zeta}_i(t)$ to 0 as $\epsilon \to 0^+$ is uniform on $[\tau, T] \setminus (J + (0, \xi))$ (or $[\tau, T]$).

V. NUMERICAL EXAMPLES

In this section, we illustrate the effectiveness of our proposed high-gain approximate differentiator based sliding-mode observer with two numerical examples. Our simulations demonstrate that its performance is quite similar to that of the high-order sliding-mode exact differentiator based sliding-mode observer. Due to lack of space, we only show simulations with the high-gain approximate differentiator based sliding-mode observer.

**Example 1:** We first consider a linear time invariant system (1) determined by

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
-1 & -5 & -10 & -10 & -5
\end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & -1 \\
1 & 0 \\
0 & 0
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}.
\]

The initial condition is $x(0) = [0.5 \ 0.5 \ 0.5 \ -0.5 \ -0.5]^T$, and the known input $u_1$ is set to be zero vector. The unknown input $u_2$ consists of a square wave with amplitude 1 and frequency 1Hz, and a sawtooth signal with amplitude 2 and frequency 1Hz.

It is easy to check that for this system $\text{rank}(CB_2) \neq \text{rank} B_2$ because $c_1B_2 = 0$. Thus, we choose $\gamma_1 = r_1 = 3$ such that

\[
C_a = \begin{bmatrix}
c_1 \\
c_1A \\
c_1A^2 \\
c_2
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
is of full rank with \( \text{rank}(C_aB_2) = \text{rank} B_2 \). We employ a high-gain observer to estimate the auxiliary outputs \( y_{12} = c_1 A \dot{x} \) and \( y_{13} = c_1 A^2 \dot{x} \). The design parameters of the high-gain observer are selected to be \( \alpha_{11} = 3, \alpha_{12} = 3, \alpha_{13} = 1 \) and \( \epsilon = 0.001 \). The estimated and true values of the auxiliary outputs are shown in Fig. 4.

Now we use the estimates of the auxiliary outputs to construct the sliding-mode observer described by (34). By the algorithm described in Appendix A, we use \( \kappa = 2.0659 \) and \( \eta = 50 \) to obtain

\[
L_a = \begin{bmatrix}
6 & 1 & 0 & 0 \\
0 & 7 & 0 & 0 \\
0 & 0 & 2.0659 & 0 \\
0 & 0 & 0 & 2.0659 \\
0 & 0 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
F_a = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix}.
\]

We set the initial states of the sliding-mode observer to be zero, that is, \( \hat{x}(0) = 0 \), and select \( S_{11} = S_{12} = S_{13} = 1.5 \). In Fig. [5], we show the state estimation performance. The reconstruction of the unknown inputs is illustrated in Fig. [6]. Note that the spikes at the right hand side of the estimated unknown inputs are caused by non-uniform convergence in the immediate neighborhood to the right of jump discontinuities. The estimated unknown inputs will converge pointwise to the their true values at each fixed \( t \) as the spike moves to the left when \( \epsilon \) decreases.

\textit{Example 2}: In this example, we replace the output matrix \( C \) in Example [1] by a new output

\[\text{(34)}\]
Fig. 5. Real and estimated system states in Example [1]
Following the algorithm described in Appendix [A] with $\kappa = 158.3395$ and $\eta = 100$, we obtain

$$L_a = \begin{bmatrix} 1.9276 & 0 & -1.3947 & 0 \\ 2.5626 & 0 & 2.2961 & 0 \\ -5.1252 & 10 & -4.5923 & 0 \\ 2.2264 & -9 & 1.9168 & -1 \\ 0.7994 & 0 & 1.4967 & 0 \end{bmatrix}$$

and

$$F_a = \begin{bmatrix} 0 & 0 & 0 & -10 \\ 0 & -1 & 0 & 9 \end{bmatrix}.$$
The initial states of the sliding-mode observer is the same as in Example 1, $S_{11} = S_{12} = S_{21} = 3$ and $S_{22} = 8$. The state estimation performance is similar to that of Example 1 as shown in Fig. 5. The reconstruction of the unknown inputs is illustrated in Fig. 8.

VI. CONCLUSIONS

The main contributions of this paper are novel sliding-mode observer architectures for state estimation and unknown input reconstruction for systems with unknown inputs. Rigorous proofs of the convergence of the state estimation error and the unknown input reconstruction are provided. The proposed architectures consist of dynamical systems for generating auxiliary
outputs that are used by a sliding-mode observer to estimate the states and reconstruct the unknown inputs. Two methods for generating auxiliary outputs are analyzed: high-order sliding-mode exact differentiator and high-gain approximate differentiator. The first method yields asymptotic state estimation, whereas the second method achieves uniform ultimate boundedness of the state estimation error. The high-gain approximate differentiators result in a simpler overall observer architecture than the high-order sliding-mode exact differentiator. In addition, it is shown that the high-gain differentiator based observer performs comparably to the high-order differentiator based observer.

APPENDIX A

SLIDING-MODE OBSERVER DESIGN ALGORITHM

1) Check if the conditions (2) and (3) are satisfied. If not, the sliding-mode observer in (4) cannot be constructed.

2) Find the nonsingular matrices $T \in \mathbb{R}^{n \times n}$ and $S \in \mathbb{R}^{p \times p}$ such that the triple $(A, B_2, C)$ is transformed into the form

$$
\hat{A} = TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},
$$

$$
\hat{B}_2 = TB_2 = \begin{bmatrix} B_{21} \\ O_{(n-m_2) \times m_2} \end{bmatrix},
$$

$$
\hat{C} = SCT^{-1} = \begin{bmatrix} I_{m_2} & O_{m_2 \times (p-m_2)} \\ O_{(p-m_2) \times m_2} & C_{22} \end{bmatrix},
$$

where $A_{11} \in \mathbb{R}^{m_2 \times m_2}$, $A_{22} \in \mathbb{R}^{(n-m_2) \times (n-m_2)}$, $B_{21} \in \mathbb{R}^{m_2 \times m_2}$ and $C_{22} \in \mathbb{R}^{(p-m_2) \times (p-m_2)}$;

3) Choose $L_{22} \in \mathbb{R}^{(n-m_2) \times (p-m_2)}$ such that $A_{22} - L_{22}C_{22}$ is Hurwitz. Solve $P_{22}$ for

$$(A_{22} - L_{22}C_{22})^\top P_{22} + P_{22} (A_{22} - L_{22}C_{22}) = -2Q_{22},$$

where $Q_{22}$ is a symmetric positive definite matrix.

4) Choose $\kappa$ such that

$$\kappa > \frac{1}{2} \lambda_{\text{max}} \left( A_{11}^\top + A_{11} + (A_{12} + A_{21}^\top P_{22}) \hat{Q}_{22}^{-1} (A_{12}^\top + P_{22} A_{21}) \right);$$
5) Construct $\hat{P}$, $\hat{F}$ and $\hat{L}$ as

$$
\hat{P} = \begin{bmatrix}
I_{m_2} & O_{m_2 \times (p-m_2)} \\
O_{(p-m_2) \times m_2} & P_{22}
\end{bmatrix},
$$

$$
\hat{F} = \begin{bmatrix}
B_{21}^\top & O_{m_2 \times (p-m_2)}
\end{bmatrix},
$$

$$
\hat{L} = \begin{bmatrix}
\kappa I_{m_2} & O_{m_2 \times (p-m_2)} \\
O_{(n-m_2) \times m_2} & L_{22}
\end{bmatrix},
$$

and compute $P = T^\top \hat{P} T$, $F = \hat{F} S$, $L = T^{-1} \hat{L} S$;

6) Construct the sliding-mode observer (4).

**APPENDIX B**

**PROOF OF PROPOSITION 1**

The idea of this proof comes from [35]. Let $P_{ci}$ be the solution to the continuous Lyapunov equation $\bar{A}_{ci} P_{ci} + P_{ci} \bar{A}_{ci} = -2Q_{ci}$ for some $Q_{ci} = Q_{ci}^\top > 0$. Consider the Lyapunov function candidate for $t \geq t_0$,

$$
V_i = \frac{1}{2} \zeta_i^\top P_{ci} \zeta_i. \tag{56}
$$

Evaluating the time derivative of $V_i$ on the solutions of (32), yields

$$
\dot{V}_i = \zeta_i^\top P_{ci} \left( \frac{1}{\epsilon} \bar{A}_{ci} \zeta_i + \bar{b}_i f_i(x, u_2) \right)
\leq -\frac{1}{\epsilon} \zeta_i^\top Q_{ci} \zeta_i + \| \zeta_i^\top P_{ci} \bar{b}_i \| \| f_i(x, u_2) \|. \tag{57}
$$

Because $x$ and $u_2$ are bounded, it follows from (28) that $|f_i(x, u_2)| \leq \beta_{i1}$ for some $\beta_{i1} > 0$. Taking into account the above and performing some manipulations on (57) gives

$$
\dot{V}_i = -\frac{1}{\epsilon} \zeta_i^\top Q_{ci} \zeta_i + \beta_{i1} \| \zeta_i^\top P_{ci} \bar{b}_i \|
\leq -\frac{1}{\epsilon} \lambda_{\text{min}}(Q_{ci}) \| \zeta_i \|^2 + \beta_{i1} \| P_{ci} \bar{b}_i \| \| \zeta_i \|
\leq -\frac{1}{\epsilon} \lambda_{\text{min}}(Q_{ci}) \lambda_{\text{max}}(P_{ci}) \| \zeta_i \|^2 + \frac{\sqrt{2} \beta_{i1} \| P_{ci} \| \| \bar{b}_i \| \sqrt{\frac{1}{2} \lambda_{\text{min}}(P_{ci}) \| \zeta_i \|^2}}{\lambda_{\text{max}}(P_{ci})}
\leq -\frac{1}{\epsilon} \lambda_{\text{max}}(P_{ci}) V_i + \frac{\sqrt{2} \beta_{i1} \| P_{ci} \| \| \bar{b}_i \| \sqrt{\lambda_{\text{min}}(P_{ci})}}{\sqrt{\lambda_{\text{min}}(P_{ci})}} \sqrt{V_i}
\leq -\frac{1}{\epsilon} \lambda_{\text{min}}(Q_{ci}) V_i + \frac{\sqrt{2} \beta_{i1} \| P_{ci} \| \| \bar{b}_i \| \sqrt{V_i}}{\sqrt{\lambda_{\text{min}}(P_{ci})}}
\leq -\frac{\mu_{ci}}{\epsilon} V_i - \sqrt{V_i} \left( \frac{\mu_{ci}}{\epsilon} \sqrt{V_i} - \beta_{i2} \right),
$$

August 22, 2008 DRAFT
where
\[ \mu_{ci} = \frac{\lambda_{\min}(Q_{ci})}{\lambda_{\max}(P_{ci})}, \quad \beta_{i2} = \frac{\sqrt{2} \beta_{i1} \|P_{ci}\bar{b}_{i1}\|}{\sqrt{\lambda_{\min}(P_{ci})}}. \]

It is easy to verify that as long as
\[ V_i \geq \left( \frac{\beta_{i2} \epsilon}{\mu_{ci}} \right)^2 = \beta_{i3} \epsilon^2, \]
where \( \beta_{i3} = \frac{\beta_{i2}^2}{\mu_{ci}^2} \), then
\[ \frac{\mu_{ci}}{\epsilon} \sqrt{V_i - \beta_{i2}} \geq 0. \]

Because the initial conditions for the high-gain observers are bounded and \( \epsilon \) is a design parameter such that \( 0 < \epsilon < 1 \), it follows from (56) that
\[ V_i(t_0) \leq \frac{\beta_{i4}}{\epsilon^{2\gamma_i - 2}}, \]
for some \( \beta_{i4} > 0 \).

If \( V_i(t_0) > \beta_{i3} \epsilon^2 \) and \( V_i(t) \geq \beta_{i3} \epsilon^2 \) for \( t \geq t_0 \), then
\[ \dot{V}_i \leq -\frac{\mu_{ci}}{\epsilon} V_i, \]
and, hence, it follows from the comparison lemma that
\[ V_i(t) \leq \exp \left( -\frac{\mu_{ci}}{\epsilon} (t - t_0) \right) V_i(t_0) \leq \frac{\beta_{i4}}{\epsilon^{2\gamma_i - 2}} \left( -\frac{\mu_{ci}}{\epsilon} (t - t_0) \right). \]
Thus, we can find a finite time \( T_i(\epsilon) \) such that \( V_i(t) \leq \beta_{i3} \epsilon^2 \) for \( t \geq t_0 + T_i(\epsilon) \), where \( T_i(\epsilon) \) is the solution to the equation,
\[ \frac{\beta_{i4}}{\epsilon^{2\gamma_i - 2}} \exp \left( -\frac{\mu_{ci}}{\epsilon} T_i(\epsilon) \right) = \beta_{i3} \epsilon^2, \]
of the form,
\[ T_i(\epsilon) = \frac{\epsilon}{\mu_{ci}} \ln \left( \frac{\beta_{i4}}{\beta_{i3} \epsilon^{2\gamma_i}} \right). \] (58)

It follows from (58) that \( \lim_{\epsilon \to 0^+} T_i(\epsilon) = 0 \). On the other hand, if \( V_i(t_0) \leq \beta_{i3} \epsilon^2 \), then \( V_i(t) \leq \beta_{i3} \epsilon^2 \) for \( t \geq t_0 \). In such a case, we can choose \( T_i(\epsilon) = 0 \). Therefore, there exists a finite time \( T_i(\epsilon) \) such that \( V_i(t) \leq \beta_{i3} \epsilon^2 \) for \( t \geq t_0 + T_i(\epsilon) \) and \( \lim_{\epsilon \to 0^+} T_i(\epsilon) = 0 \). It follows that \( \|\zeta(t)\| \leq \beta_i \epsilon \) for \( t \geq t_0 + T_i(\epsilon) \), where \( \beta_i = \sqrt{2\beta_{i3}/\lambda_{\min}(P_{ci})} \). Thus, we concludes the proof of the proposition.
REFERENCES


[20] T. Floquet and J. P. Barbot, “A canonical form for the design of unknown input sliding mode observers,” in *Advances in


