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Constructing Robust Controllers Using Lyapunov's Theorem

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One of the applications of the Lyapunov theorem is to construct robust state-feedback controllers. A basic issue in the control of dynamic systems is the effect of uncertainties, or neglected nonlinearities, on the stability of the closed-loop system.

A controller is viewed robust if it maintains stability for all uncertainties in an expected range. We consider a class of dynamic systems modeled by

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{f}(t, \boldsymbol{x}(t)) + \boldsymbol{B}\boldsymbol{h}(t, \boldsymbol{x}(t), \boldsymbol{u}(t)), \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{u}(t) \in \mathbb{R}^m$, and the functions $\boldsymbol{h}$ and $\boldsymbol{f}$ model uncertainties, or nonlinearities, in the system. We refer to $\boldsymbol{h}$ as the *matched uncertainty* because it affects the system dynamics via the input matrix $\boldsymbol{B}$ in the same fashion as the input $\boldsymbol{u}$ does. In other words, the uncertainty $\boldsymbol{h}$ matches the system input $\boldsymbol{u}$. The vector $\boldsymbol{f}$ models the *unmatched uncertainty*. We assume that the uncertain elements $\boldsymbol{h}$ and $\boldsymbol{f}$ satisfy the following norm bounds:

1. $\|\boldsymbol{h}(t, \boldsymbol{x}, \boldsymbol{u})\| \leq \gamma_h \|\boldsymbol{u}\| + \alpha_h \|\boldsymbol{x}\|$,
2. $\|\boldsymbol{f}(t, \boldsymbol{x})\| \leq \alpha_f \|\boldsymbol{x}\|$,

where γ_h, α_h, and α_f are known nonnegative constants. We further assume that the matrix $\boldsymbol{A}$ is asymptotically stable. If this is not the case, we apply a preliminary state-feedback controller $\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{x}$ such that $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}$ is asymptotically stable. Such a feedback exists provided that the pair $(\boldsymbol{A}, \boldsymbol{B})$ is stabilizable. Our goal is to construct a linear state-feedback controller that would make the closed-loop system asymptotically stable for arbitrary $\boldsymbol{f}$ and $\boldsymbol{h}$ that satisfy the above norm bounds.

Theorem 1 *[1, p. 168] Suppose that $\boldsymbol{A}$ is asymptotically stable, and $\boldsymbol{P} = \boldsymbol{P}^\top > 0$ is the solution to the Lyapunov matrix equation $\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -2\boldsymbol{Q}$ for some $\boldsymbol{Q} = \boldsymbol{Q}^\top > 0$. Suppose also that*

$$\alpha_f < \frac{\lambda_{\min}(\boldsymbol{Q})}{\lambda_{\max}(\boldsymbol{P})} \quad \textit{and} \quad \gamma_h < 1.$$

Then, the state-feedback controller $\boldsymbol{u} = -\gamma \boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{x}$, where

$$\gamma > \frac{\alpha_h^2}{4\left(\lambda_{\min}(\boldsymbol{Q}) - \alpha_f \lambda_{\max}(\boldsymbol{P})\right)(1-\gamma_h)}$$

stabilizes the uncertain system model (1) for arbitrary $\boldsymbol{f}$ and $\boldsymbol{h}$ that satisfy the norm bounds.

Proof The time derivative of the positive definite function $V = \boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{x}$ evaluated on any trajectory of the closed-loop system is

$$\begin{aligned} \dot{V} &= 2\boldsymbol{x}^\top \boldsymbol{P}\dot{\boldsymbol{x}} \\ &= -2\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} - 2\gamma \boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{B}\boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{x} + 2\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{f} + 2\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{B}\boldsymbol{h}. \end{aligned}$$

Our goal is to determine $\tilde{\gamma} > 0$ such that if $\gamma > \tilde{\gamma}$, then $\dot{V} < 0$, which in turn implies the asymptotic stability of the closed-loop system. To proceed further, recall that if $\boldsymbol{Q}$ is a symmetric matrix, then

$$\lambda_{\min}(\boldsymbol{Q})\|\boldsymbol{x}\|^2 \leq \boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} \leq \lambda_{\max}(\boldsymbol{Q})\|\boldsymbol{x}\|^2,$$

and therefore

$$-\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} \leq -\lambda_{\min}(\boldsymbol{Q})\|\boldsymbol{x}\|^2.$$

Furthermore,

$$\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{B}\boldsymbol{B}^\top \boldsymbol{P}\boldsymbol{x} = \boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{B}\left(\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{B}\right)^\top = \|\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{B}\|^2.$$

For a symmetric positive definite matrix $\boldsymbol{P}$, its induced 2-norm is

$$\|\boldsymbol{P}\| = \lambda_{\max}(\boldsymbol{P}).$$

Taking the above relations into account, using the norm bounds on the uncertain elements, and performing some manipulations yields

$$\begin{aligned} \dot{V} \leq\ & -2\lambda_{\min}(\boldsymbol{Q})\|\boldsymbol{x}\|^2 - 2\gamma\|\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{B}\|^2 + 2\alpha_f \lambda_{\max}(\boldsymbol{P})\|\boldsymbol{x}\|^2 \\ & +2\|\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{B}\|\left(\alpha_h\|\boldsymbol{x}\| + \gamma_h \gamma \|\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{B}\|\right). \end{aligned} \tag{2}$$

Let

$$\tilde{\boldsymbol{Q}} = \begin{bmatrix} \lambda_{\min}(\boldsymbol{Q}) - \alpha_f \lambda_{\max}(\boldsymbol{P}) & -\frac{\alpha_h}{2} \\ -\frac{\alpha_h}{2} & \gamma(1-\gamma_h) \end{bmatrix} \in \mathbb{R}^{2\times 2}. \tag{3}$$

Then, we can represent (2) as

$$\dot{V} \le -2 \begin{bmatrix} \|\boldsymbol{x}\| & \|\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{B}\| \end{bmatrix} \tilde{\boldsymbol{Q}} \begin{bmatrix} \|\boldsymbol{x}\| \\ \|\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{B}\| \end{bmatrix}. \tag{4}$$

For $\dot{V}$ to be negative it is enough that the leading principal minors of the matrix $\tilde{\boldsymbol{Q}}$ given by (3) be positive. The first-order leading principal minor is positive by assumption. For the second-order leading principal minor to be positive it is sufficient that $\gamma_h < 1$ and

$$\gamma > \frac{\alpha_h^2}{4\left(\lambda_{\min}(\boldsymbol{Q}) - \alpha_f \lambda_{\max}(\boldsymbol{P})\right)(1-\gamma_h)}.$$

The proof is now complete.

□

Note that if $\gamma_h = 0$, then the closed-loop system is asymptotically stable if

$$\alpha_f < \frac{\lambda_{\min}(\boldsymbol{Q})}{\lambda_{\max}(\boldsymbol{P})} \quad \text{and} \quad \gamma > \frac{\alpha_h^2}{4\left(\lambda_{\min}(\boldsymbol{Q}) - \alpha_f \lambda_{\max}(\boldsymbol{P})\right)}.$$

Example 1 Consider a dynamic system model

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}(u + h(\boldsymbol{x}, u)) \\ &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} (u + h(\boldsymbol{x}, u)). \end{aligned}$$

Let $\boldsymbol{P}$ be the solution of the matrix Lyapunov equation $\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -2\boldsymbol{I}_2$. We will design a linear state-feedback controller $u = -\gamma \boldsymbol{b}^\top \boldsymbol{P}\boldsymbol{x}$ so that the origin of the closed-loop system is uniformly asymptotically stable for any $h(\boldsymbol{x}, u)$ such that

$$|h(\boldsymbol{x}, u)| \le 4\|\boldsymbol{x}\| + \frac{1}{2}|u|. \tag{5}$$

In particular, we will find a bound $\tilde{\gamma} > 0$ so that for any $\gamma > \tilde{\gamma}$ the closed-loop system is uniformly asymptotically stable in the face of uncertainties satisfying the norm bound (5).

Solving the Lyapunov equation $\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -2\boldsymbol{I}_2$ yields

$$\boldsymbol{P} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence,

$$u = -\gamma \boldsymbol{b}^\top \boldsymbol{P}\boldsymbol{x} = -\gamma \left(\frac{1}{2}x_1 + x_2 \right).$$

We obtain a bound on γ by evaluating $\frac{d}{dt}\left(\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{x}\right)$ on the trajectories of the closed-loop system. First, we evaluate

$$\begin{aligned}
\frac{d}{dt}\left(\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{x}\right) &= 2\boldsymbol{x}^\top \boldsymbol{P}\dot{\boldsymbol{x}} \\
&= 2\boldsymbol{x}^\top \boldsymbol{P}\left(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}(u + h(\boldsymbol{x}, u))\right) \\
&= -2\boldsymbol{x}^\top \boldsymbol{x} - 2\gamma \boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{b}\boldsymbol{b}^\top \boldsymbol{P}\boldsymbol{x} + 2\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{b}h.
\end{aligned}$$

Taking the norms and performing manipulations yields

$$\begin{aligned}
\frac{d}{dt}\left(\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{x}\right) &\leq -2\boldsymbol{x}^\top \boldsymbol{x} - 2\gamma \left|\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{b}\right|^2 + 2\left|\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{b}\right| \left(4\|\boldsymbol{x}\| + \frac{1}{2}\gamma \left|\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{b}\right|\right) \\
&= -2\|\boldsymbol{x}\|^2 - \gamma \left|\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{b}\right|^2 + 8\left|\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{b}\right| \|\boldsymbol{x}\| \\
&= -2 \begin{bmatrix} \|\boldsymbol{x}\| & \left|\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{b}\right| \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & \gamma/2 \end{bmatrix} \begin{bmatrix} \|\boldsymbol{x}\| \\ \left|\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{b}\right| \end{bmatrix}.
\end{aligned}$$

For $\frac{d}{dt}\left(\boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{x}\right)$ to be negative definite it is enough that the matrix

$$\begin{bmatrix} 1 & -2 \\ -2 & \gamma/2 \end{bmatrix}$$

be positive definite. This is the case whenever

$$\gamma > \tilde{\gamma} = 8.$$

References

[1] S. H. Żak. *Systems and Control*. Oxford University Press, New York, 2003.