ECE 602 Fall 2010

A Review of the Laplace and $\mathcal{Z}$ Transforms

by
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1 The Laplace Transform Defintion

The Laplace transform is an operator that transforms a function of time, $f(t)$, into a new function of complex variable, $F(s)$, where $s = \sigma + j\omega$, as illustrated in Figure 1. The operator $\mathcal{L}$ denotes that the time function $f(t)$ has been transformed to its Laplace transform, denoted $F(s)$. The Laplace transform is very useful in solving linear differential equations and hence

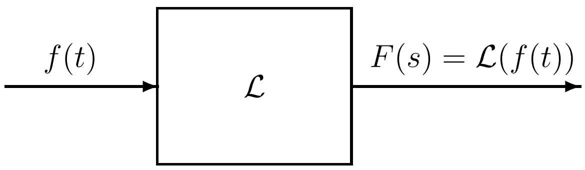

Figure 1: Schematic representation of the Laplace transform operator.

in analyzing control systems.

To obtain the Laplace transform of the given function of time, $f(t)$,

1. multiply $f(t)$ by a converging factor e^{-st}. This is a factor that decreases to zero as time increases to infinity;

2. Integrate $f(t)e^{-st}$ with respect to time between the limits 0^- and ∞ to obtain the Laplace transform of $f(t)$,

$$\boxed{F(s) = \mathcal{L}(f(t)) = \int_{0^-}^{\infty} f(t)e^{-st}dt}$$

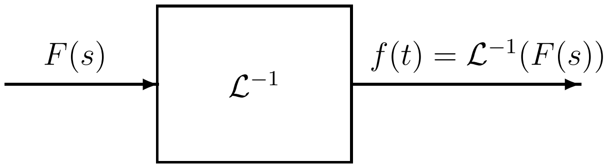

Figure 2: Schematic representation of the inverse Laplace transform operation.

The lower limit of integration is 0^-, rather than 0, to account for the effect of "instantaneous energy transfer".

The above definition of the Laplace transform is also referred to as the *one-sided* or *unilateral* Laplace transform. In the two-sided, or bilateral, Laplace transform, the lower limit is $-\infty$. For our purposes the one-sided Laplace transform is sufficient.

If we want to reverse the operation and take the inverse transform, back to the time domain, we write

$$\mathcal{L}^{-1}(F(s)) = f(t).$$

Taking the inverse Laplace transform is illustrated in Figure 2.

Because we are using the one-sided Laplace transform, we define all functions, whose Laplace transforms we compute, to be zero for $t < 0^-$. To proceed, we recall the definition of the unit step function denoted $1(t)$,

$$1(t) = \begin{cases} 1 & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0. \end{cases}$$

The unit step function is also called the Heaviside function.

Example 1 Find the Laplace transform of

$$f(t) = e^{-at}1(t),$$

where a is a real constant. A graph of $f(t)$ for $a = 3$ is shown in Figure 3. We have

$$\mathcal{L}(e^{-at}1(t)) = \int_{0^-}^{\infty} e^{-at}e^{-st}1(t)dt = \int_{0^-}^{\infty} e^{-(a+s)t}dt.$$

The above integral exists if

$$\Re(a+s) > 0.$$

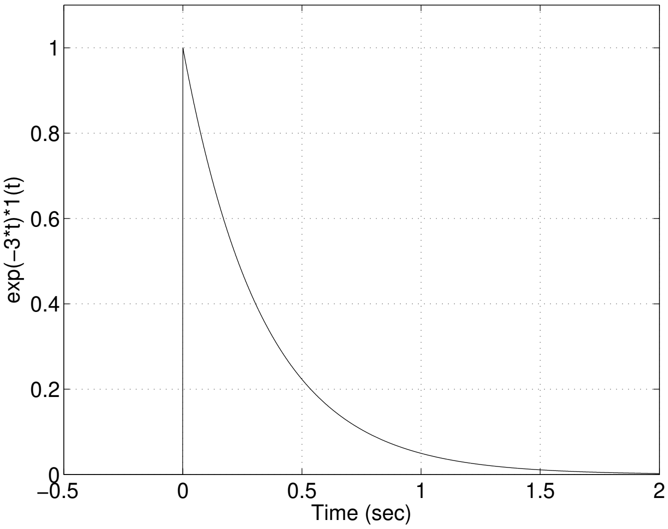

Figure 3: A plot of $e^{-3t}1(t)$.

The region of the s-plane for which the Laplace transform exists is called the *Region of Convergence* and abbreviated ROC. Since $s = \sigma + j\omega$, the ROC, in our example, is the region in the s-plane where

$$0 < \Re(a+s) = \Re(a+\sigma+j\omega) = a+\sigma,$$

that is, the region where $\sigma > -a$. Proceeding with the integration, we obtain

$$\begin{aligned}\mathcal{L}(e^{-at}1(t)) &= \left(-\frac{1}{a+s}e^{-(a+s)t}\right)\Bigg|_{0^-}^{\infty} \\ &= \left(-\frac{1}{a+s}e^{-(a+s)\infty}\right) - \left(-\frac{1}{a+s}e^{-(a+s)0^-}\right).\end{aligned}$$

For s in the ROC, the first term tends to zero. Hence,

$$\boxed{\mathcal{L}(e^{-at}1(t)) = \frac{1}{s+a}}$$

Using the Laplace transform of the exponential function, we can easily find the Laplace transform of the unit step. Indeed, if $a = 0$, then $f(t) = e^{-at}1(t) = 1(t)$. Hence,

$$\boxed{\mathcal{L}(1(t)) = \frac{1}{s}} \tag{1}$$

2 Linearity Property of the Laplace Transform

The Laplace transform of the sum, or difference, of two functions of time is equal to the sum, or difference, of the transforms of each function, that is,

$$\mathcal{L}\left(f_1(t) \pm f_2(t)\right) = \mathcal{L}\left(f_1(t)\right) \pm \mathcal{L}\left(f_2(t)\right) \tag{2}$$

Indeed,

$$\begin{aligned}
\mathcal{L}\left(f_1(t) \pm f_2(t)\right) &= \int_{0^-}^{\infty} \left(f_1(t) \pm f_2(t)\right) e^{-st} dt \\
&= \int_{0^-}^{\infty} f_1(t) e^{-st} dt \pm \int_{0^-}^{\infty} f_2(t) e^{-st} dt \\
&= \mathcal{L}\left(f_1(t)\right) \pm \mathcal{L}\left(f_2(t)\right).
\end{aligned}$$

The Laplace transform of the product of a real or complex constant K and a time function $f(t)$ is equal to the product of the constant and the transform of the time function, that is,

$$\mathcal{L}\left(Kf(t)\right) = K\mathcal{L}\left(f(t)\right). \tag{3}$$

Indeed,

$$\mathcal{L}\left(Kf(t)\right) = \int_{0^-}^{\infty} Kf(t)e^{-st}dt = K\int_{0^-}^{\infty} f(t)e^{-st}dt = K\mathcal{L}\left(f(t)\right).$$

The above two properties can be represented in the form

$$\boxed{\mathcal{L}\left(Kf_1(t) \pm f_2(t)\right) = K\mathcal{L}\left(f_1(t)\right) \pm \mathcal{L}\left(f_2(t)\right)}$$

Example 2 To find the Laplace transform of

$$f(t) = K1(t),$$

where K is a constant, we can use (3) and the Laplace transform of the unit step, given by (1), to obtain

$$\mathcal{L}\left(K1(t)\right) = K\mathcal{L}\left(1(t)\right) = \frac{K}{s}.$$

Example 3 We will use the linear property of the Laplace transform to find the Laplace transform of

$$f(t) = \sin \omega t.$$

First, recall the Euler formula,

$$e^{j\omega t} = \cos \omega t + j \sin \omega t.$$

Hence,

$$e^{-j\omega t} = \cos \omega t - j \sin \omega t.$$

Subtracting the second of the above expressions from the first one and dividing the result by $2j$ gives

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}.$$

Therefore,

$$\mathcal{L}(\sin \omega t) = \mathcal{L}\left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right) = \mathcal{L}\left(\frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)\right).$$

Applying to the above (3) yields

$$\mathcal{L}(\sin \omega t) = \frac{1}{2j}\mathcal{L}\left(e^{j\omega t} - e^{-j\omega t}\right).$$

Applying now (2), we obtain

$$\mathcal{L}(\sin \omega t) = \frac{1}{2j}\left(\mathcal{L}\left(e^{j\omega t}\right) - \mathcal{L}\left(e^{-j\omega t}\right)\right).$$

We then use twice the formula for the Laplace transform of the exponential to get

$$\mathcal{L}(\sin \omega t) = \frac{1}{2j}\left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right).$$

Performing algebraic manipulations gives

$$\boxed{\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}}$$

Similarly, we obtain

$$\mathcal{L}(\cos \omega t) = \mathcal{L}\left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right).$$

Hence,

$$\boxed{\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}}$$

3 Laplace Transforms of Derivatives

We will now show that

$$\boxed{\mathcal{L}\left(\frac{df(t)}{dt}\right) = sF(s) - f\left(0^-\right)}$$

where $F(s) = \mathcal{L}(f(t))$ and $f\left(0^-\right)$ is the initial value of f, that is, the value of f at 0^-. Thus, differentiating in the time domain corresponds to multiplying $F(s)$ by s and then subtracting the initial value of $f(t)$. To derive the above formula, we apply the definition of the Laplace transform,

$$\mathcal{L}\left(\frac{df(t)}{dt}\right) = \int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st} dt, \tag{4}$$

and then evaluate the integral by integrating by parts. Recall that the formula for integrating by parts can easily be derived from the formula for the derivative of the product of two functions. We use the notation,

$$u' = \frac{du(t)}{dt}.$$

Then, we have

$$(uv)' = u'v + uv'.$$

Integrating the above and rearranging gives

$$\int u'v = uv - \int uv'. \tag{5}$$

We let

$$u' = \frac{df}{dt} \quad \text{and} \quad v = e^{-st}.$$

Then applying (5) to (4), we obtain

$$\mathcal{L}\left(\frac{df(t)}{dt}\right) = e^{-st} f(t) \,\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t)\left(-se^{-st}\right) dt.$$

Assuming that $f(t)$ is Laplace transformable, the value of $e^{-st}f(t)$ at $t = \infty$ is zero. Hence, the right-hand side of the above reduces to

$$\mathcal{L}\left(\frac{df(t)}{dt}\right) = -f\left(0^-\right) + s\int_{0^-}^{\infty} f(t)e^{-st} dt = sF(s) - f\left(0^-\right).$$

Thus, we showed that differentiation in the time domain corresponds to an algebraic operation in the s domain.

Using the Laplace transform of the first derivative of a time function, we can easily determine the Laplace transform of higher-order derivatives. Indeed, to find the Laplace transform of $\frac{d^2 f(t)}{dt^2}$, we let

$$g(t) = \frac{df(t)}{dt}.$$

Then,

$$\mathcal{L}(g(t)) = G(s) = \mathcal{L}\left(\frac{df(t)}{dt}\right) = sF(s) - f\left(0^-\right). \tag{6}$$

Note that

$$\frac{d^2 f(t)}{dt^2} = \frac{dg(t)}{dt}.$$

Hence,

$$\mathcal{L}\left(\frac{df^2(t)}{dt^2}\right) = \mathcal{L}\left(\frac{dg(t)}{dt}\right) = sG(s) - g\left(0^-\right).$$

Substituting into the above (6) gives

$$\mathcal{L}\left(\frac{df^2(t)}{dt^2}\right) = s\left(sF(s) - f\left(0^-\right)\right) - g\left(0^-\right).$$

Observing that $g\left(0^-\right) = \frac{df\left(0^-\right)}{dt}$, we get

$$\boxed{\mathcal{L}\left(\frac{df^2(t)}{dt^2}\right) = s^2F(s) - sf\left(0^-\right) - \frac{df\left(0^-\right)}{dt}}$$

Successively applying the above arguments, we can obtain the Laplace transform of the n-th time derivative,

$$\boxed{\mathcal{L}\left(\frac{df^n(t)}{dt^n}\right) = s^nF(s) - s^{n-1}f\left(0^-\right) - s^{n-2}\frac{df\left(0^-\right)}{dt} - \cdots - s\frac{d^{n-2}f\left(0^-\right)}{dt^{n-2}} - \frac{d^{n-1}f\left(0^-\right)}{dt^{n-1}}}$$

4 Solving Differential Equations Using Laplace Transform

We now illustrate how to use the Laplace transform and inverse Laplace transform to solve linear ordinary differential equations. After taking the Laplace transform of both sides of a differential equation and performing required manipulations in the s domain, we need to reconstruct the solution in the time domain. This is achieved using the inverse Laplace transform.

Example 4 We will solve the following differential equation using the Laplace transform,

$$2\frac{d^2x(t)}{dt^2} + 7\frac{dx(t)}{dt} + 5x(t) = 101(t),$$

subject to the initial conditions

$$\frac{dx\left(0^-\right)}{dt} = 1, \quad x\left(0^-\right) = -2,$$

where $1(t)$ is the unit step function.

We take the Laplace transform of both sides of the differential equation to get

$$2\mathcal{L}\left(\frac{d^2x(t)}{dt^2}\right)+7\mathcal{L}\left(\frac{dx(t)}{dt}\right)+5\mathcal{L}\left(x(t)\right)=10\mathcal{L}\left(1(t)\right).$$

Using Laplace transforms of time derivatives of a function of time and the Laplace transform of the unit step function, we obtain

$$2\left(s^2X(s)-sx\left(0^-\right)-\frac{dx\left(0^-\right)}{dt}\right)+7\left(sX(s)-x\left(0^-\right)\right)+5X(s)=\frac{10}{s}.$$

Substituting into the above the initial conditions and rearranging gives

$$\left(2s^2+7s+5\right)X(s)=\frac{10}{s}-4s-12=\frac{10-4s^2-12s}{s}.$$

Hence,

$$X(s)=\frac{-4s^2-12s+10}{s\left(2s^2+7s+5\right)}=\frac{-2s^2-6s+5}{s\left(s^2+3.5s+2.5\right)}=\frac{-2s^2-6s+5}{s\left(s+2.5\right)\left(s+1\right)}. \tag{7}$$

We now need to reconstruct the solution of the differential equation in the time domain. In other words, given $X(s)$ we want to obtain $x(t)$ in which $x(t)$ is zero for $t<0$ such that $X(s)=\mathcal{L}(x(t))$. Thus,

$$x(t)=\mathcal{L}^{-1}\left(X(s)\right).$$

To proceed, note that $X(s)$ as given by (7) is a rational function of s, that is, $X(s)$ is a ratio of two polynomials in s, in which the degree of the numerator with respect to s is smaller than the degree of the denominator. Such a rational function is called strictly proper and can be expanded into a sum of partial fractions by writing a term or a series of terms for each zero of the denominator. We obtain

$$X(s)=\frac{K_1}{s}+\frac{K_2}{s+2.5}+\frac{K_3}{s+1}. \tag{8}$$

We will now compute the constants K_i's. We can do so by representing the right-hand side of the above as a rational function and then comparing the resulting numerator with the numerator of (7). We represent (8) as

$$\begin{aligned} X(s) &= \frac{K_1\left(s+2.5\right)\left(s+1\right)+K_2s\left(s+1\right)+K_3s\left(s+2.5\right)}{s\left(s+2.5\right)\left(s+1\right)} \\ &= \frac{\left(K_1+K_2+K_3\right)s^2+\left(3.5K_1+K_2+2.5K_3\right)s+2.5K_1}{s\left(s+2.5\right)\left(s+1\right)} \end{aligned} \tag{9}$$

Comparing coefficients of like powers of the numerators of (7) and (9) gives three algebraic equations in three unknowns,

$$\left.\begin{array}{rcl} K_1 + K_2 + K_3 &=& -2 \\ 3.5K_1 + K_2 + 2.5K_3 &=& -6 \\ 5 &=& 2.5K_1 \end{array}\right\}$$

Solving the above system of equations gives

$$K_1 = 2, \quad K_2 = 2, \quad K_3 = -6.$$

Substituting the above into (8) yields

$$X(s) = \frac{2}{s} + \frac{2}{s+2.5} - \frac{6}{s+1}.$$

Applying the inverse Laplace transform to the above, we obtain

$$\begin{array}{rcl} x(t) &=& \mathcal{L}\left(\frac{2}{s} + \frac{2}{s+2.5} - \frac{6}{s+1}\right) \\ &=& \mathcal{L}\left(\frac{2}{s}\right) + \mathcal{L}\left(\frac{2}{s+2.5}\right) - \mathcal{L}\left(\frac{6}{s+1}\right) \\ &=& \left(2 + 2e^{-2.5t} - 6e^{-t}\right) 1(t). \end{array}$$

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5 Laplace Transforms of Integrals

We first consider taking the Laplace transform of

$$\int_{0^-}^{t} f(x)dx. \tag{10}$$

Let $F(s)$ denote the Laplace transform of $f(t)$, that is, $F(s) = \mathcal{L}(f(t))$. We find the Laplace transform of (10) using integration by parts to obtain

$$\mathcal{L}\left(\int_{0^-}^{t} f(x)dx\right) = \int_{0^-}^{\infty}\left(\int_{0^-}^{t} f(x)dx\right) e^{-st}dt. \tag{11}$$

Recall the formula for integration by parts,

$$\int u'v = uv - \int uv'.$$

Let

$$u = -\frac{1}{s}e^{-st} \quad \text{and} \quad v = \int_{0^-}^{t} f(x)dx.$$

Note that

$$\frac{dv}{dt} = \frac{d}{dt}\int_{0^-}^{t} f(x)dx = f(t).$$

Integrating (11) by parts gives

$$\mathcal{L}\left(\int_{0^-}^{t} f(x)dx\right) = -\frac{1}{s}e^{-st}\int_{0^-}^{t} f(x)dx\,\Big|_{0^-}^{\infty} + \frac{1}{s}\int_{0^-}^{\infty} e^{-st}f(t)dt.$$

The first term on the right-hand side is zero at both the lower and upper limits. The value at the lower limit, $t = 0^-$, is clearly zero. The value at the upper limit, $t = \infty$, is zero because we assumed that $f(t)$ is Laplace transformable. Hence,

$$\boxed{\mathcal{L}\left(\int_{0^-}^{t} f(x)dx\right) = \frac{F(s)}{s}} \tag{12}$$

We next find the Laplace transform of

$$\int_{-\infty}^{t} f(x)dx. \tag{13}$$

Applying the definition and using the fact that the Laplace transform is linear yields

$$\mathcal{L}\left(\int_{-\infty}^{t} f(x)dx\right) = \mathcal{L}\left(\int_{-\infty}^{0^-} f(x)dx\right) + \mathcal{L}\left(\int_{0^-}^{t} f(x)dx\right) = F_1(s) + F_2(s).$$

The limits of the first integral are constant, therefore the integral will be a constant. Hence,

$$F_1(s) = \frac{\int_{-\infty}^{0^-} f(x)dx}{s},$$

while by (12),

$$F_2(s) = \frac{F(s)}{s}.$$

Therefore,

$$\boxed{\mathcal{L}\left(\int_{-\infty}^{t} f(x)dx\right) = \frac{F(s)}{s} + \frac{\int_{-\infty}^{0^-} f(x)dx}{s}} \tag{14}$$

Example 5 We find the Laplace transform of the voltage, $v_C(t)$, across a capacitor C. We have

$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i(x)dx.$$

Using (14) gives

$$\mathcal{L}\left(v_C(t)\right) = \frac{1}{C}\frac{I(s)}{s} + \frac{\frac{1}{C}\int_{-\infty}^{0^-} i(x)dx}{s}.$$

Since $\int_{-\infty}^{0^-} i(x)dx$ is the charge, q, on the capacitor at $t = 0^-$ and $v = q/C$, we obtain

$$\boxed{\mathcal{L}\left(v_C(t)\right) = \frac{1}{C}\frac{I(s)}{s} + \frac{v_C\left(0^-\right)}{s}}$$

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6 More Properties of the Laplace Transform

We will now show that translation in the time domain corresponds to multiplication by an exponential in the s domain, that is,

$$\boxed{\mathcal{L}\left(f(t-a)u(t-a)\right) = e^{-sa}F(s), \qquad a > 0}$$

where $F(s) = \mathcal{L}(f(t))$. We have

$$\mathcal{L}\left(f(t-a)u(t-a)\right) = \int_{0^-}^{\infty} f(t-a)u(t-a)e^{-st}dt = \int_{a^-}^{\infty} f(t-a)e^{-st}dt,$$

because $u(t-a) = 0$ for $t < a$. Next, we change the variable of integration. We let

$$x = t - a.$$

Then, $x = 0^-$ when $t = a^-$, $x = \infty$ when $t = \infty$, and $dx = dt$. Substituting the above into (15) yields

$$\mathcal{L}\left(f(t-a)u(t-a)\right) = \int_{0^-}^{\infty} f(x)e^{-s(x+a)}dx = e^{-sa}\int_{0^-}^{\infty} f(x)e^{-sx}dx = e^{-as}F(s),$$

which is the desired result.

We will now prove the time/frequency scaling property,

$$\boxed{\mathcal{L}(f(at)) = \frac{1}{a}F\left(\frac{s}{a}\right), \quad a > 0}$$

Indeed, applying the Laplace transform yields

$$\mathcal{L}(f(at)) = \int_{0^-}^{\infty} f(at)e^{-st}dt. \tag{15}$$

Let $x = at$. Then, $dt = \frac{1}{a}dx$. Substituting the above into (15) gives

$$\mathcal{L}(f(at)) = \frac{1}{a}\int_{0^-}^{\infty} f(x)e^{-\left(\frac{s}{a}\right)x}dx = \frac{1}{a}F\left(\frac{s}{a}\right),$$

which was to be shown.

Using the sifting property we can show that

$$\boxed{\mathcal{L}(\delta(t)) = 1}$$

Indeed, applying the definition of the Laplace transform and using the sifting property, we obtain

$$\mathcal{L}(\delta(t)) = \int_{0^-}^{\infty} \delta(t)e^{-st}dt = e^0 \int_{0^-}^{0^+} \delta(t)dt = 1.$$

7 The z-Transform Definition

The z-transform is an operator denoted $\mathcal{Z}(\cdot)$ that, when applied to a sequence of numbers produces a function of a variable z.

Definition 1 *For a given sequence of numbers*

$$y(0), y(1), \ldots, y(k), \ldots$$

its z-transform is the series defined as

$$Y(z) = \mathcal{Z}(y(k)) = \sum_{k=0}^{\infty} \frac{y(k)}{z^k}.$$

Example 6 We find the $\mathcal{Z}$-transform of the sampled unit step, $f(kT) = 1$. The $\mathcal{Z}$-transform of this sequence is

$$\begin{aligned} F(z) &= \sum_{k=0}^{\infty} f(kT)z^{-k} \\ &= 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots \end{aligned}$$

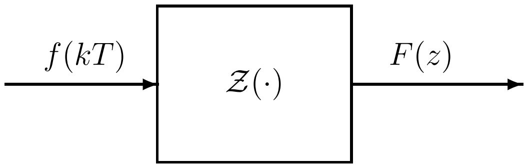

Figure 4: The $\mathcal{Z}$-transform operator is acting on a sequence of numbers resulting in a function of the variable z.

If z is a complex variable such that $|z| > 1$, then the above sequence converges and we have

$$\begin{aligned} F(z) &= \frac{1}{1 - 1/z} \\ &= \frac{z}{z-1}. \end{aligned}$$

Example 7 We find the $\mathcal{Z}$-transform of the sampled unit ramp, $f(kT) = kT$. The $\mathcal{Z}$-transform of this sequence is

$$\begin{aligned} F(z) &= \sum_{k=0}^{\infty} kTz^{-k} \\ &= T\left(z^{-1} + 2z^{-2} + 3z^{-3} + \cdots\right) \end{aligned} \tag{16}$$

Multiplying both sides of the above by z gives

$$zF(z) = T\left(1 + 2z^{-1} + 3z^{-2} + \cdots\right) \tag{17}$$

Subtracting (16) from (17) yields

$$\begin{aligned} zF(z) - F(z) &= (z-1)F(z) \\ &= T\left(1 + z^{-1} + z^{-2} + \cdots\right) \\ &= \frac{1}{1 - z^{-1}} \\ &= \frac{z}{z-1}. \end{aligned} \tag{18}$$

Hence, the $\mathcal{Z}$-transform of $f(kT) = kT$ is

$$\boxed{\mathcal{Z}(kT) = T\frac{z}{(z-1)^2}} \tag{19}$$

Example 8 We now find the $\mathcal{Z}$-transform of the geometric sequence

$$f(k) = a^{kT}. \tag{20}$$

We obtain

$$\boxed{\mathcal{Z}(a^{kT}) = \sum_{k=0}^{\infty} \frac{a^{kT}}{z^k} = \frac{z}{z-a^T}} \tag{21}$$

for $|z| > \left|a^T\right|$.

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8 Some Properties of the $\mathcal{Z}$-Transform

We analyze now three basic properties that are especially useful when computing transforms of more complicated sequences.

Linearity property: The $\mathcal{Z}$-transform is a linear operator, that is,

(i) if $F_1(z) = \mathcal{Z}(f_1(kT))$ and $F_2(z) = \mathcal{Z}(f_2(kT))$ then

$$\mathcal{Z}(f_1(kT) + f_2(kT)) = F_1(z) + F_2(z); \text{ and}$$

(ii) if $F(z) = \mathcal{Z}(f(kT))$ then

$$\mathcal{Z}(cf(kT)) = cF(z),$$

where c is a real or complex number.

We can represent the above two properties as: if $F_1(z) = \mathcal{Z}(f_1(kT))$ and $F_2(z) = \mathcal{Z}(f_2(kT))$, and c is a real or complex number, then

$$\mathcal{Z}(f_1(kT) + cf_2(kT)) = F_1(z) + cF_2(z).$$

$\mathcal{Z}$-transform of a delayed sequence: If the sequence $f(kT)$ has the $\mathcal{Z}$-transform $F(z)$, then the unit delayed sequence

$$g(kT) = \begin{cases} f((k-1)T) & k \geq 1 \\ 0 & k = 0 \end{cases}$$

has the $\mathcal{Z}$-transform

$$G(z) = z^{-1}F(z). \tag{22}$$

We use the definition of the $\mathcal{Z}$-transform to prove the above property;

$$\begin{aligned} G(z) &= \sum_{k=0}^{\infty} z^{-k} g(kT) \\ &= \sum_{k=1}^{\infty} z^{-k} f((k-1)T) \\ &= z^{-1} \sum_{k=1}^{\infty} z^{-(k-1)} f((k-1)T) \\ &= z^{-1} \sum_{j=0}^{\infty} z^{-j} f(jT) \\ &= z^{-1} F(z). \end{aligned}$$

$\mathcal{Z}$-transform of an advanced sequence: If the sequence $f(kT)$ has the $\mathcal{Z}$-transform $F(z)$, then the unit advanced sequence $h(kT) = f((k+1)T)$, $k \geq 0$, has the transform

$$\mathcal{Z}(f((k+1)T)) = zF(z) - zf(0). \tag{23}$$

To prove the above, we apply the definition of the $\mathcal{Z}$-transform to obtain

$$\begin{aligned} H(z) &= \sum_{k=0}^{\infty} z^{-k} h(kT) \\ &= \sum_{k=0}^{\infty} z^{-k} f((k+1)T) \\ &= z \sum_{k=0}^{\infty} z^{-k-1} f((k+1)T) \\ &= z \sum_{j=1}^{\infty} z^{-j} f(jT) \\ &= z \sum_{j=0}^{\infty} z^{-j} f(jT) - zf(0) \\ &= zF(z) - zf(0). \end{aligned}$$

Example 9 We find the $\mathcal{Z}$-transform of the delayed geometric sequence,

$$f(kT) = \begin{cases} 0 & \text{for} \quad k = 0 \\ a^{(k-1)T} & \text{for} \quad k > 0. \end{cases} \tag{24}$$

Applying (22) to the geometric sequence (20) yields

$$\boxed{\mathcal{Z}(f(kT) = \frac{1}{z - a^T}} \tag{25}$$

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9 Solving Difference Equations Using the $\mathcal{Z}$-Transform

The $\mathcal{Z}$-transform is a very effective tool for solving linear, constant-coefficients difference equations. The basis for the method is the $\mathcal{Z}$-transform of an advanced sequence given by (23). Repeated application of this property gives

$$\mathcal{Z}(f((k+2)T)) = z^2 F(z) - z^2 f(0) - z f(T). \tag{26}$$

Applying again (23) to (26) gives

$$\mathcal{Z}(f((k+3)T)) = z^3 F(z) - z^3 f(0) - z^2 f(T) - z f(2T) \tag{27}$$

and so on.

Example 10 We solve the difference equation

$$x[k+1] + 3x[k] = 2^k, \quad k = 0, 1, \ldots,$$

where $x(0) = 0$. Applying the $\mathcal{Z}$-transform to both sides of the above difference equation gives

$$zX(z) - zx(0) + 3X(z) = \frac{z}{z-2}.$$

Hence,

$$X(z) = \frac{z}{(z+3)(z-2)},$$

which is a strictly proper rational function of z. We represent $X(z)$ in partial fraction form as

$$X(z) = \frac{A}{z+3} + \frac{B}{z-2},$$

where

$$A = (z+3)X(z)|_{z=-3} = \frac{3}{5} \quad \text{and} \quad B = (z-2)X(z)|_{z=2} = \frac{2}{5}.$$

Therefore,

$$X(z) = \frac{3/5}{z+3} + \frac{2/5}{z-2}.$$

We use (25) when applying the inverse $\mathcal{Z}$-transform to the above,

$$x[k] = \begin{cases} 0 & \text{for} \quad k = 0 \\ \frac{3}{5}(-3)^{k-1} + \frac{2}{5}2^{k-1} & \text{for} \quad k \geq 1 \end{cases}$$

Example 11 We solve the difference equation

$$x[k+1] + 3x[k] = 2^k, \quad k = 0, 1, \ldots,$$

where $x(0) = 5$. Applying the $\mathcal{Z}$-transform to both sides of the above difference equation gives

$$zX(z) - zx(0) + 3X(z) = \frac{z}{z-2}.$$

Hence,

$$X(z) = \frac{5z^2 - 9z}{(z+3)(z-2)} = \frac{z(5z-9)}{(z+3)(z-2)}.$$

The above rational function is not strictly proper so we use a different approach than in Example 10. In this example z is a factor of the numerator so we can represent the above as

$$\frac{X(z)}{z} = \frac{5z-9}{(z+3)(z-2)}$$

and express the right hand side in partial fraction format as

$$\frac{X(z)}{z} = \frac{24/5}{z+3} + \frac{1/5}{z-2}.$$

Thus

$$X(z) = \frac{24}{5}\frac{z}{z+3} + \frac{1}{5}\frac{z}{z-2}.$$

Taking into account (21) when applying the inverse $\mathcal{Z}$-transform yields

$$x[k] = \frac{24}{5}(-3)^k + \frac{1}{5}2^k, \quad k \geq 0$$
