ECE 602 Fall 2010

State Equation Solution

by

Stanislaw H. Żak

Linear systems can be modeled using convolutions and in the case of lumped systems, we can use state-space equations. Analytical computation of convolutions can be quite involved. An efficient method of obtaining a solution to a model of a lumped system is to convert the transfer function into a state-space model and then solve the state equation. We begin with a time-invariant case and then proceed with the time-varying case.

1 Solution of Uncontrolled System

Given a dynamical system's time-invariant model of the form

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) \tag{1}$$

subject to an initial condition

$$\boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{2}$$

where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$. The solution to (1) subject to (2) is

$$\boxed{\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0}$$

The above is a generalization of the situation when $\boldsymbol{A}$ is a scalar. The exponential matrix $e^{\boldsymbol{A}t}$ is defined as

$$e^{\boldsymbol{A}t} = \boldsymbol{I}_n + t\boldsymbol{A} + \frac{t^2}{2!}\boldsymbol{A}^2 + \frac{t^3}{3!}\boldsymbol{A}^3 + \cdots$$

Observe that

$$\frac{d}{dt}e^{\boldsymbol{A}t} = \boldsymbol{A}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{A}.$$

Suppose now that the initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ is replaced with a more general condition

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0,$$

where t_0 is a given initial time instant. Then, the solution to $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$ subject to $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ is

$$\boxed{\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}_0}$$

The matrix $e^{\boldsymbol{A}(t-t_0)}$ is often written as

$$e^{\boldsymbol{A}(t-t_0)} = \boldsymbol{\Phi}\left(t, t_0\right)$$

and is called the *state transition matrix*, because it relates the state at any instant of time t_0 to the state at any other time t. There are many methods for calculating the state transition matrix. For an overview of some of them, we refer the reader to the classic paper on the subject by Moler and Van Loan [1]. In the case when the matrix $\boldsymbol{A}$ is of low dimensions, we can use the formula that results from applying the Laplace transform to the time-invariant system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \quad \boldsymbol{x}(0) = \boldsymbol{x}_0$. The Laplace transform of the above matrix differential equation is

$$s\boldsymbol{X}(s) - \boldsymbol{x}_0 = \boldsymbol{A}\boldsymbol{X}(s).$$

Hence

$$\boldsymbol{X}(s) = \left(s\boldsymbol{I}_n - \boldsymbol{A}\right)^{-1}\boldsymbol{x}_0,$$

and the inverse transform yields

$$\boxed{e^{\boldsymbol{A}t} = \mathcal{L}^{-1}\left\{\left(s\boldsymbol{I}_n - \boldsymbol{A}\right)^{-1}\right\}} \tag{3}$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform operator.

2 Solution of Controlled System

Consider a model of a controlled dynamic system,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{4}$$

subject to a given initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$, where $\boldsymbol{B}$ is an $n \times m$ matrix, and $m < n$ as is usually in practice. Premultiplying both sides of (4) by the integrating factor, $e^{-\boldsymbol{A}t}$, yields

$$e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t) = e^{-\boldsymbol{A}t}\boldsymbol{A}\boldsymbol{x}(t) + e^{-\boldsymbol{A}t}\boldsymbol{B}\boldsymbol{u}(t).$$

We then rearrange the above equation as

$$e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t) - e^{-\boldsymbol{A}t}\boldsymbol{A}\boldsymbol{x}(t) = e^{-\boldsymbol{A}t}\boldsymbol{B}\boldsymbol{u}(t). \tag{5}$$

Noting that

$$\frac{d}{dt}\left(e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\right) = -\boldsymbol{A}e^{-\boldsymbol{A}t}\boldsymbol{x}(t) + e^{-\boldsymbol{A}t}\dot{\boldsymbol{x}}(t),$$

we can represent (5) in the form

$$\frac{d}{dt}\left(e^{-\boldsymbol{A}t}\boldsymbol{x}(t)\right) = e^{-\boldsymbol{A}t}\boldsymbol{B}\boldsymbol{u}(t).$$

Integrating both sides of the resulting equation gives

$$e^{-\boldsymbol{A}t}\boldsymbol{x}(t) - \boldsymbol{x}(0) = \int_0^t e^{-\boldsymbol{A}\tau}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau.$$

Hence

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau.$$

If the initial condition is $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, then integration from t_0 to t yields

$$\boxed{\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0) + \int_{t_0}^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau} \tag{6}$$

As in the case of the uncontrolled system, we can also use the Laplace transform technique to obtain the solution to (4) subject to the initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$. Indeed, taking the Laplace transform of (4) and performing some manipulations yields

$$\boldsymbol{X}(s) = (sI_n - \boldsymbol{A})^{-1}\boldsymbol{x}(0) + (sI_n - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s).$$

Hence

$$\boldsymbol{x}(t) = \mathcal{L}^{-1}\left\{(sI_n - \boldsymbol{A})^{-1}\boldsymbol{x}(0)\right\} + \mathcal{L}^{-1}\left\{(sI_n - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s)\right\}.$$

If $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$, then

$$\boldsymbol{x}(t) = \mathcal{L}^{-1}\left\{e^{-t_0 s}(sI_n - \boldsymbol{A})^{-1}\boldsymbol{x}(t_0)\right\} + \mathcal{L}^{-1}\left\{e^{-t_0 s}(sI_n - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s)\right\}.$$

3 From Continuous to Discrete Models

Our objective here is to present a method for constructing linear discrete-time models from given linear continuous-time models.

We use a sample-and-hold device that transforms a continuous signal, $f(t)$, into the staircase signal,

$$f(kh), \quad kh \leq t < (k+1)h,$$

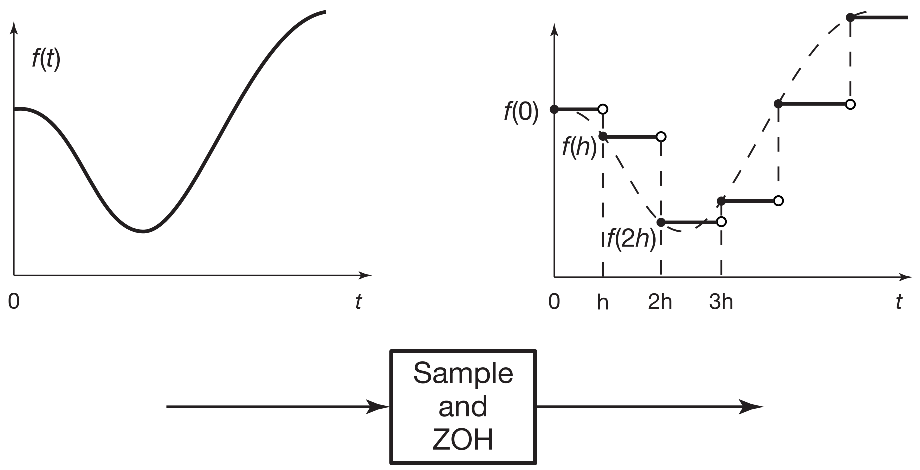

Figure 1: Sample and zero-order hold (ZOH) element operating on a continuous function.

where h is the sampling period. The sample and zero-order hold (ZOH) operation is illustrated in Figure 1

Suppose that we are given a continuous-time model,

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}_0 = \boldsymbol{x}(t_0) & (7) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t). & (8) \end{aligned}$$

The solution to the state equation is

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0) + \int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{u}(\tau)d\tau. \tag{9}$$

We assume that the input to the system is generated by a sample-and-hold device and has the form,

$$\boldsymbol{u}(t) = \boldsymbol{u}(k), \quad kh \le t < (k+1)h.$$

Let $t_0 = kh$ and $t = (k+1)h$ and let us use shorthand notation,

$$\boldsymbol{x}(kh) = \boldsymbol{x}(k)$$

Then taking into account that $\boldsymbol{u}(k)$ is constant on the interval $[kh, (k+1)h)$ we represent (9) as

$$\begin{aligned} \boldsymbol{x}(k+1) &= e^{\boldsymbol{A}h}\boldsymbol{x}(k) + \int_{kh}^{(k+1)h} e^{\boldsymbol{A}(kh+h-\tau)}\boldsymbol{B}\boldsymbol{u}(k)d\tau \\ &= e^{\boldsymbol{A}h}\boldsymbol{x}(k) + \int_{kh}^{(k+1)h} e^{\boldsymbol{A}(kh+h-\tau)}\boldsymbol{B}d\tau\, \boldsymbol{u}(k). & (10) \end{aligned}$$

Consider now the second term on the right-hand side of the above equation. Let

$$\eta = kh + h - \tau.$$

Then we can represent (10) as

$$\begin{aligned}
\boldsymbol{x}(k+1) &= e^{\boldsymbol{A}h}\boldsymbol{x}(k) + \int_{kh}^{(k+1)h} e^{\boldsymbol{A}(kh+h-\tau)}\boldsymbol{B}d\tau\,\boldsymbol{u}(k) \\
&= e^{\boldsymbol{A}h}\boldsymbol{x}(k) + \int_{0}^{h} e^{\boldsymbol{A}\eta}\boldsymbol{B}d\eta\,\boldsymbol{u}(k) \\
&= \boldsymbol{\Phi}\boldsymbol{x}(k) + \boldsymbol{\Gamma}\boldsymbol{u}(k), \qquad (11)
\end{aligned}$$

where

$$\boldsymbol{\Phi} = e^{\boldsymbol{A}h} \quad \text{and} \quad \boldsymbol{\Gamma} = \int_0^h e^{\boldsymbol{A}\eta}\boldsymbol{B}d\eta.$$

The discrete output equation has the form

$$\boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k).$$

Example 1 We construct an exact linear discrete state-space model for the continuous system model,

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{x}(k) + \begin{bmatrix} 2 \\ 1 \end{bmatrix}u(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix}\boldsymbol{x}(t).
\end{aligned}$$

The sampling period is $h = 1$ sec.

We first evaluate $\boldsymbol{\Phi} = e^{\boldsymbol{A}h}$. Note that $\boldsymbol{A}^2 = \boldsymbol{O}$. Hence

$$\begin{aligned}
\boldsymbol{\Phi} &= \boldsymbol{I}_2 + \boldsymbol{A}h \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

Next, we evaluate

$$\boldsymbol{\Gamma} = \int_0^h e^{\boldsymbol{A}\tau}\boldsymbol{B}d\tau.$$

We obtain

$$
\begin{aligned}
\boldsymbol{\Gamma} &= \int_0^h e^{\boldsymbol{A}\tau} \boldsymbol{B} d\tau \\
&= \int_0^h \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} d\tau \\
&= \int_0^1 \begin{bmatrix} 2+\tau \\ 1 \end{bmatrix} d\tau \\
&= \left. \begin{bmatrix} 2\tau + \frac{1}{2}\tau^2 \\ \tau \end{bmatrix} \right|_0^1 \\
&= \begin{bmatrix} 2.5 \\ 1 \end{bmatrix}.
\end{aligned}
$$

The discrte model then has the form

$$
\begin{aligned}
\boldsymbol{x}(k+1) &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 2.5 \\ 1 \end{bmatrix} \\
y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}(k).
\end{aligned}
$$

4 Summary and Notes

5 Exercises

References

[1] C. Moler and C. Van Loan. Nineteen dubious ways to compute the exponential of a matrix. *SIAM Review*, 20(4):801–836, October 1978.