ECE 602 Fall 2010

# State-Space Realization Examples

by

Stanislaw H. Żak

**Example 1** We find a state-space realization of the transfer function matrix,

$$\boldsymbol{G}(s)=\frac{1}{s^3+3s^2+2s}\begin{bmatrix} s+1 & 2s^2+s-1 & s^2-1\\ -s^2-s & -s^2+s & s\end{bmatrix}. \tag{1}$$

The above transfer function is strictly proper; therefore

$$\boldsymbol{D}=\boldsymbol{G}(\infty)=\begin{bmatrix} 0&0&0\\ 0&0&0\end{bmatrix}.$$

We proceed to represent $\boldsymbol{G}(s)$ as

$$\begin{aligned}\boldsymbol{G}(s) &= \frac{1}{s^3+3s^2+2s}\left(\begin{bmatrix} 0&2&1\\ -1&-1&0\end{bmatrix}s^2+\begin{bmatrix} 1&1&0\\ -1&1&1\end{bmatrix}s+\begin{bmatrix} 1&-1&-1\\ 0&0&0\end{bmatrix}\right)\\ &= \frac{1}{s^3+3s^2+2s}\left(\boldsymbol{N}_2s^2+\boldsymbol{N}_1s+\boldsymbol{N}_0\right).\end{aligned}$$

Using realization formulas found in the proof of Theorem 1, we obtain

$$\begin{aligned}\boldsymbol{A} &= \begin{bmatrix} \boldsymbol{O} & \boldsymbol{I}_3 & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{I}_3\\ \boldsymbol{O} & -2\boldsymbol{I}_3 & -3\boldsymbol{I}_3\end{bmatrix}\\ &= \left[\begin{array}{ccccccccccc} 0&0&0&\vdots&1&0&0&\vdots&0&0&0\\ 0&0&0&\vdots&0&1&0&\vdots&0&0&0\\ 0&0&0&\vdots&0&0&1&\vdots&0&0&0\\ \cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\ 0&0&0&\vdots&0&0&0&\vdots&1&0&0\\ 0&0&0&\vdots&0&0&0&\vdots&0&1&0\\ 0&0&0&\vdots&0&0&0&\vdots&0&0&1\\ \cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\ 0&0&0&\vdots&-2&0&0&\vdots&-3&0&0\\ 0&0&0&\vdots&0&-2&0&\vdots&0&-3&0\\ 0&0&0&\vdots&0&0&-2&\vdots&0&0&-3\end{array}\right].\end{aligned} \tag{2}$$

The input matrix $\boldsymbol{B}$ is

$$\boldsymbol{B}=\left[\begin{array}{c}\boldsymbol{O}\\ \boldsymbol{O}\\ \boldsymbol{I}_3\end{array}\right]=\left[\begin{array}{ccc}0&0&0\\0&0&0\\0&0&0\\ \cdots&\cdots&\cdots\\0&0&0\\0&0&0\\0&0&0\\ \cdots&\cdots&\cdots\\1&0&0\\0&1&0\\0&0&1\end{array}\right], \tag{3}$$

and the matrix $\boldsymbol{C}$ has the form

$$\boldsymbol{C}=\left[\begin{array}{ccc}\boldsymbol{N}_0 & \boldsymbol{N}_1 & \boldsymbol{N}_2\end{array}\right]=\left[\begin{array}{ccccccccccc}1&-1&-1&\vdots&1&1&0&\vdots&0&2&1\\0&0&0&\vdots&-1&1&1&\vdots&-1&-1&0\end{array}\right]. \tag{4}$$

The dimension of this realization is 9.

---

We can obtain another realization of $\boldsymbol{G}(s)$ using Theorem 1. We begin by constructing a realization for $\boldsymbol{G}(s)^\top$ using realization formulas from the proof of Theorem 1 to obtain

$$\boldsymbol{G}(s)^\top=\tilde{\boldsymbol{C}}(s\boldsymbol{I}-\tilde{\boldsymbol{A}})^{-1}\tilde{\boldsymbol{B}}+\tilde{\boldsymbol{D}}. \tag{5}$$

We next transpose both sides of (5) to obtain

$$\begin{aligned}\left(\boldsymbol{G}(s)^\top\right)^\top &= \boldsymbol{G}(s)\\ &= \left(\tilde{\boldsymbol{C}}(s\boldsymbol{I}-\tilde{\boldsymbol{A}})^{-1}\tilde{\boldsymbol{B}}+\tilde{\boldsymbol{D}}\right)^\top\\ &= \tilde{\boldsymbol{B}}^\top\left(s\boldsymbol{I}-\tilde{\boldsymbol{A}}^\top\right)^{-1}\tilde{\boldsymbol{C}}^\top+\tilde{\boldsymbol{D}}^\top.\end{aligned}$$

The resulting realization has the form

$$\left.\begin{aligned}\dot{\tilde{\boldsymbol{x}}} &= \left[\begin{array}{cccccc}\boldsymbol{O}&\boldsymbol{O}&\boldsymbol{O}&\cdots&\boldsymbol{O}&-\alpha_0\boldsymbol{I}_p\\ \boldsymbol{I}_p&\boldsymbol{O}&\boldsymbol{I}_m&\cdots&\boldsymbol{O}&-\alpha_1\boldsymbol{I}_p\\ \vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\ \boldsymbol{O}&\boldsymbol{O}&\boldsymbol{O}&\cdots&\boldsymbol{O}&-\alpha_{r-2}\boldsymbol{I}_p\\ \boldsymbol{O}&\boldsymbol{O}&\boldsymbol{O}&\cdots&\boldsymbol{I}_p&-\alpha_{r-1}\boldsymbol{I}_m\end{array}\right]\tilde{\boldsymbol{x}}+\left[\begin{array}{c}\boldsymbol{N}_0\\ \boldsymbol{N}_1\\ \vdots\\ \boldsymbol{N}_{r-2}\\ \boldsymbol{N}_{r-1}\end{array}\right]\boldsymbol{u}\\ \boldsymbol{y} &= \left[\begin{array}{cccccc}\boldsymbol{O}&\boldsymbol{O}&\boldsymbol{O}&\cdots&\boldsymbol{O}&\boldsymbol{I}_p\end{array}\right]\tilde{\boldsymbol{x}}+\boldsymbol{G}(\infty)\boldsymbol{u}.\end{aligned}\right\} \tag{6}$$

Note that although the realization (6) is based on the realization found in the proof of Theorem 1, their dimensions will be different if $p \neq m$.

**Example 2** We use the above method to construct a state-space realization of the transfer function matrix given by (1). We first apply realization formulas of Theorem 1 to $\boldsymbol{G}(s)^\top$, where

$$\begin{aligned}
\boldsymbol{G}(s)^\top &= \frac{1}{s^3+3s^2+2s}\begin{bmatrix} s+1 & -s^2-s \\ 2s^2+s-1 & -s^2+s \\ s^2-1 & s \end{bmatrix} \\
&= \frac{1}{s^3+3s^2+2s}\left(\begin{bmatrix} 0 & -1 \\ 2 & -1 \\ 1 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ -1 & 0 \end{bmatrix}\right) \\
&= \frac{1}{s^3+3s^2+2s}\left(\boldsymbol{N}_2^\top s^2 + \boldsymbol{N}_1^\top s + \boldsymbol{N}_0^\top\right).
\end{aligned}$$

Applying to the above (6), we obtain another state-space realization of the transfer function (1), where

$$\begin{aligned}
\hat{\boldsymbol{A}} &= \begin{bmatrix} \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{I}_2 & \boldsymbol{O} & -2\boldsymbol{I}_2 \\ \boldsymbol{O} & \boldsymbol{I}_2 & -3\boldsymbol{I}_3 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & \vdots & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & \vdots & 0 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & \vdots & 0 & 0 & \vdots & -2 & 0 \\ 0 & 1 & \vdots & 0 & 0 & \vdots & 0 & -2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \vdots & 1 & 0 & \vdots & -3 & 0 \\ 0 & 0 & \vdots & 0 & 1 & \vdots & 0 & -3 \end{bmatrix}.
\end{aligned} \tag{7}$$

The input matrix is

$$\hat{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{N}_0 \\ \boldsymbol{N}_1 \\ \boldsymbol{N}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ \cdots & \cdots & \cdots \\ 1 & 1 & 0 \\ -1 & 1 & 1 \\ \cdots & \cdots & \cdots \\ 0 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}, \tag{8}$$

and the output matrix has the form

$$\hat{\boldsymbol{C}} = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{I}_2 \end{bmatrix} = \left[\begin{array}{cc:cc:cc} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]. \tag{9}$$

The obtained realization is of lower dimension than the previous one. This does not mean that the alternative method will always yield lower dimension realizations than the first method.

---

We now present yet another method of constructing a state-space realization of a given transfer function matrix. This method is being used by the MATLAB's function `tf2ss`. We begin by denoting the $i$-th column of a proper transfer function matrix $\boldsymbol{G}(s)$ as $\boldsymbol{g}_i(s)$. We observe that the input-output relation of an MIMO system can be expressed as

$$\begin{aligned} \boldsymbol{Y}(s) &= \boldsymbol{G}(s)\boldsymbol{U}(s) \\ &= \boldsymbol{g}_1(s)U_1(s) + \boldsymbol{g}_2(s)U_2(s) + \cdots + \boldsymbol{g}_m(s)U_m(s), \end{aligned}$$

where $U_i(s)$ is the $i$-th component of the input vector $\boldsymbol{U}(s)$. We construct a state-space realization of each column $\boldsymbol{g}_i(s)$ and then combine these realizations to obtain the state-space realization of $\boldsymbol{G}(s)$ as follows. If the quadruple $(\boldsymbol{A}_i, \boldsymbol{b}_i, \boldsymbol{C}_i, \boldsymbol{d}_i)$ is the realization of $\boldsymbol{g}_i(s)$, then a realization $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ of $\boldsymbol{G}(s)$ has the form,

$$\left.\begin{aligned} \boldsymbol{A} &= \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{O} & \cdots & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{A}_2 & \cdots & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{A}_m \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{b}_2 & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{b}_m \end{bmatrix}, \\ \boldsymbol{C} &= \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{C}_2 & \cdots & \boldsymbol{C}_m \end{bmatrix}, \quad \text{and} \quad \boldsymbol{D} = \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \cdots & \boldsymbol{d}_m \end{bmatrix}. \end{aligned}\right\} \tag{10}$$

**Example 3** We use the column-by-column method to construct a realization of the transfer function matrix,

$$\boldsymbol{G}(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+1} \\ \frac{-1}{s^2+3s+2} & \frac{1}{s+2} \end{bmatrix}. \tag{11}$$

This transfer function matrix is strictly proper, therefore

$$\boldsymbol{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

There are two columns, that is,

$$\boldsymbol{G}(s)=\boldsymbol{g}_1(s)+\boldsymbol{g}_2(s)=\begin{bmatrix}\frac{1}{s+1}\\ \frac{-1}{s^2+3s+2}\end{bmatrix}+\begin{bmatrix}\frac{2}{s+1}\\ \frac{1}{s+2}\end{bmatrix}.$$

We apply the realization formulas from the proof of Theorem 1 to

$$\boldsymbol{g}_1(s)=\begin{bmatrix}\frac{1}{s+1}\\ \frac{-1}{s^2+3s+2}\end{bmatrix}=\frac{1}{s^2+3s+2}\begin{bmatrix}s+2\\ -1\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}s+\begin{bmatrix}2\\ -1\end{bmatrix}=\boldsymbol{N}_1 s+\boldsymbol{N}_0.$$

We obtain

$$\boldsymbol{A}_1=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix},\quad \boldsymbol{b}_1=\begin{bmatrix}0\\ 1\end{bmatrix},\quad \boldsymbol{C}_1=\begin{bmatrix}2 & 1\\ -1 & 0\end{bmatrix},\quad \boldsymbol{d}_1=\begin{bmatrix}0\\ 0\end{bmatrix}.$$

We next concern ourselves with the second column of $\boldsymbol{G}(s)$,

$$\boldsymbol{g}_2(s)=\begin{bmatrix}\frac{2}{s+1}\\ \frac{1}{s+2}\end{bmatrix}=\frac{1}{s^2+3s+2}\begin{bmatrix}2(s+2)\\ s+1\end{bmatrix}=\begin{bmatrix}2\\ 1\end{bmatrix}s+\begin{bmatrix}4\\ 1\end{bmatrix}=\boldsymbol{N}_1 s+\boldsymbol{N}_0.$$

We apply realization formulas from the proof of Theorem 1 to obtain

$$\boldsymbol{A}_2=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix},\quad \boldsymbol{b}_2=\begin{bmatrix}0\\ 1\end{bmatrix},\quad \boldsymbol{C}_2=\begin{bmatrix}4 & 2\\ 1 & 1\end{bmatrix},\quad \boldsymbol{d}_2=\begin{bmatrix}0\\ 0\end{bmatrix}.$$

Combining the above realizations gives an overall realization of $\boldsymbol{G}(s)$ of the form,

$$\boldsymbol{A} = \begin{bmatrix}\boldsymbol{A}_1 & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{A}_2\end{bmatrix}=\left[\begin{array}{cc:cc}0 & 1 & 0 & 0\\ -2 & -3 & 0 & 0\\ \hdashline 0 & 0 & 0 & 1\\ 0 & 0 & -2 & -3\end{array}\right],\quad \boldsymbol{B}=\begin{bmatrix}\boldsymbol{b}_1 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{b}_2\end{bmatrix}=\left[\begin{array}{c:c}0 & 0\\ 1 & 0\\ \hdashline 0 & 0\\ 0 & 1\end{array}\right],$$

$$\boldsymbol{C} = \begin{bmatrix}\boldsymbol{C}_1 & \boldsymbol{C}_2\end{bmatrix}=\left[\begin{array}{cc:cc}2 & 1 & 4 & 2\\ -1 & 0 & 1 & 1\end{array}\right],\quad \text{and}\quad \boldsymbol{D}=\begin{bmatrix}\boldsymbol{d}_1 & \boldsymbol{d}_2\end{bmatrix}=\left[\begin{array}{c:c}0 & 0\\ 0 & 0\end{array}\right].$$

---

**Example 4** We now apply the method resulting in the state-space realization given by (6) to the transfer function (11) from the previous example. We begin by transposing $\boldsymbol{G}(s)$ to obtain

$$\boldsymbol{G}(s)^\top=\begin{bmatrix}\frac{1}{s+1} & \frac{-1}{s^2+3s+2}\\ \frac{2}{s+1} & \frac{1}{s+2}\end{bmatrix}=\tilde{\boldsymbol{g}}_1(s)+\tilde{\boldsymbol{g}}_2(s)=\begin{bmatrix}\frac{1}{s+1}\\ \frac{2}{s+1}\end{bmatrix}+\begin{bmatrix}\frac{-1}{s^2+3s+2}\\ \frac{1}{s+2}\end{bmatrix}.$$

The column $\tilde{\boldsymbol{g}}_1(s)$ can be represented as

$$\tilde{\boldsymbol{g}}_1(s) = \frac{1}{s+1}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Hence, we have one-dimensional realization of $\tilde{\boldsymbol{g}}_1(s)$,

$$\tilde{\boldsymbol{A}}_1 = \begin{bmatrix} -1 \end{bmatrix}, \quad \tilde{\boldsymbol{b}}_1 = \begin{bmatrix} 1 \end{bmatrix}, \quad \tilde{\boldsymbol{C}}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \text{nad} \quad \tilde{\boldsymbol{d}}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{12}$$

We then find a realization of $\tilde{\boldsymbol{g}}_2(s)$, which we first represent as

$$\tilde{\boldsymbol{g}}_2(s) = \begin{bmatrix} \frac{-1}{s^2+3s+2} \\ \frac{1}{s+2} \end{bmatrix} = \frac{1}{s^2+3s+2}\begin{bmatrix} -1 \\ s+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \tilde{\boldsymbol{N}}_1 s + \tilde{\boldsymbol{N}}_0.$$

We obtain a second-order realization of $\tilde{\boldsymbol{g}}_2(s)$, where

$$\tilde{\boldsymbol{A}}_2 = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \tilde{\boldsymbol{b}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \tilde{\boldsymbol{C}}_2 = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad \tilde{\boldsymbol{d}}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{13}$$

Combining (12) and (13) and applying the transposition operation we obtain a third-order realization of (11),

$$\begin{aligned} \hat{\boldsymbol{A}} &= \begin{bmatrix} \tilde{\boldsymbol{A}}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \tilde{\boldsymbol{A}}_2 \end{bmatrix}^\top = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix}, \quad \hat{\boldsymbol{B}} = \begin{bmatrix} \tilde{\boldsymbol{C}}_1 & \tilde{\boldsymbol{C}}_2 \end{bmatrix}^\top = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}, \\ \hat{\boldsymbol{C}} &= \begin{bmatrix} \tilde{\boldsymbol{b}}_1 & 0 \\ \boldsymbol{0} & \tilde{\boldsymbol{b}}_2 \end{bmatrix}^\top = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \hat{\boldsymbol{D}} = \begin{bmatrix} \tilde{\boldsymbol{D}}_1 & \tilde{\boldsymbol{D}}_2 \end{bmatrix}^\top = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

---

In the above examples, we obtained realizations of different dimensions for the same transfer function. A natural questions arises of how to find a minimal realization for a given transfer function matrix. To answer this question we need notions of controllability and observability. We will prove that a realization is minimal if and only if it is both reachable and observable.