ECE 602 Fall 2010

Vocabulary and Description of Dynamic Systems

by
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> Early amplifiers were made of crude materials that tended to disintegrate over use, causing the amp to "run away." Not only would an aging relay amplify the phone signal, it would mistakenly compound any tiny deviation from the range it expected until the mushrooming error filled and killed the system. What was needed was Heron's *regula*, a counter signal to rein in the chief signal, to dampen the effect of the perpetual recycling. Black came up with a *negative* feedback loop, which was designated negative in contrast to the snowballing positive loop of the amplifier. Conceptually, the electrical negative feedback loop is a toilet flusher or thermostat. This braking circuit keeps the amplifier honed in on a steady amplification in the same way a thermostat hones in on a steady temperature. But instead of metallic levers, a weak train of electrons talks to itself. [6, page 116]

1 What Is a System?

A *system* is a collection of interacting components. An electric motor, an airplane, as well as a biological unit such as the human arm are examples of systems. A system is characterized by two properties. They are:

1. the interrelations between the components that are contained within the system,
2. the system boundaries that separate the components within the system from the components outside.

The system boundaries can be real or imagined. They are elastic in the sense that we may choose, at any stage of the system analysis, to consider only a part of the original system as a system on its own. We call it a *subsystem* of the original system. On the other hand, we may decide to expand the boundaries of the original system to include new components. In Figure 1, we represent system's boundaries using a box.

The interactions between the system components may be governed, for example, by physical, biological, or economical laws. In dealing with systems, we are interested in the

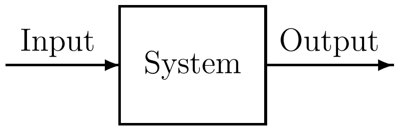

Figure 1: Representation of a system.

effects of external quantities upon the behavior of the system quantities. We refer to the external quantities acting on the system as the *inputs* to the system. The *output* of the system is its response to the input. The outputs are internal quantities available for measurement.

A system is a *continuous-time* system if it accepts continuous-time signals as its inputs and produces continuous-time signals as its outputs. We use the lowercase italic $u(t)$ to denote the input of the continuous-time single-input system, where the time t is assumed to range from $-\infty$ to ∞. For the multi-input continuous-time system, we use the boldface italic $\boldsymbol{u}(t)$ to denote its input, where $\boldsymbol{u}(t)$ is an $m \times 1$ column vector, that is,

$$\boldsymbol{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}.$$

Similarly, the output of the single-output continuous-time system will be denoted as $y(t)$, while the output of the multi-output continuous-time system will be denoted as

$$\boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix},$$

that is, the output $\boldsymbol{y}(t)$ is a column vector composed of p components.

A system is a *discrete-time* system if it accepts discrete-time signals as its inputs and produces discrete-time signals as its outputs. An example of a discrete-time signal is shown in Figure 2.

A discrete-time signal can be viewed as a sequence of points obtained from measurements at successive times. Thus a discrete-time sinal has a natural temporal ordering. We consider only the case where all discrete-time signals have the same sampling period denoted h.

We use the lowercase italic $u[k]$ to denote the input of the discrete-time single-input system, where k denotes discrete time instant and is assumed to range from $-\infty$ to ∞.

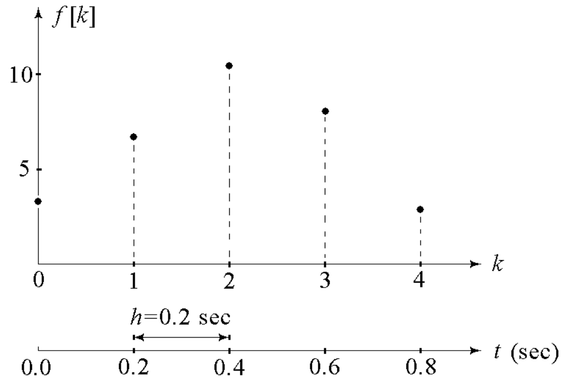

Figure 2: An example of a discrete-time signal with the sampling period $h = 0.2$ sec.

Thus,

$$u[k] := u(kh),$$

where the symbol ":=" denotes arithmetic assignment. Thus the statement "$u[k] := u(kh)$" should be interpreted to mean "$u[k]$ becomes $u(kh)$." For the multi-input discrete-time system, we use the boldface italic $\boldsymbol{u}[k]$ to denote its input, where $\boldsymbol{u}[k] := \boldsymbol{u}(kh)$ is an $m \times 1$ vector, that is,

$$\boldsymbol{u}[k] = \begin{bmatrix} u_1[k] \\ u_2[k] \\ \vdots \\ u_m[k] \end{bmatrix}.$$

Similarly, the output of the single-output discrete-time system will be denoted as $y[k]$, where $y[k] := y(kh)$. The output of the multi-output discrete-time system will be denoted as

$$\boldsymbol{y}[k] = \begin{bmatrix} y_1[k] \\ y_2[k] \\ \vdots \\ y_p[k] \end{bmatrix},$$

that is, the output $\boldsymbol{y}[k]$ is a column vector composed of p components.

1.1 Memoryless, Causal, and Lumped Systems

A system is a *memoryless system* if its output at time t_0 depends only on the input applied at t_0 and is independent of the input applied before t_0, that is, current output of a memoryless system depends only on its current input; it is independent of past inputs. An example of memoryless system is a circuit consisting of only resistors.

In a *causal* or *non-anticipatory* system its current output may depend on current and past inputs but not on future inputs. In a *non-causal* system, its current output depends on future input. Such a system can predict or anticipate what inputs will be applied in the future. Physical systems have no such capability, that is, physical systems are causal.

To proceed, we consider, following Kailath [4, p. 63], a simple system consisting of a capacitor and a voltage source as shown in Figure 3. The input u is a current through the

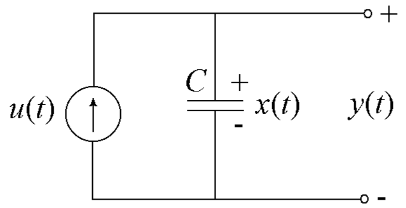

Figure 3: A circuit where the the past input from $-\infty$ up to t affects the current output at time t.

capacitor and the output y is a voltage x across the capacitor. The relation between voltage and current for a capacitor has the form,

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} u(\tau)d\tau.$$

It follows from the above that the system output is affected by its past input from $-\infty$ to time t. Tracking the past inputs may be impractical if not impossible. To alleviate the problem, the concept of state is introduced that we discuss next.

The condition or the state of the system are described by the *state variables* denoted x_i. The state variables provide the information that together with the knowledge of the system inputs enable us to determine the future state of the system. In other words, "A *dynamical system* consists of a set of possible states, together with a rule that determines the present state in terms of past states" [1, page 1].

Definition 1 [2, p. 6] *The state $\boldsymbol{x}(t_0)$ of a system at time t_0 is the information at t_0 that, together with the input $\mathbf{u}(t)$, for $t \geq t_0$, determines uniquely the output $\boldsymbol{y}(t)$ for all $t \geq t_0$.*

Returning to the example above, we select the voltage across the capacitor as the system state, which is also its output. Then, we write

$$\begin{aligned} y(t) &= x(t) \\ &= \frac{1}{C}\int_{-\infty}^{t} u(\tau)d\tau \\ &= \frac{1}{C}\int_{-\infty}^{t_0} u(\tau)d\tau + \frac{1}{C}\int_{t_0}^{t} u(\tau)d\tau \\ &= x(t_0) + \frac{1}{C}\int_{t_0}^{t} u(\tau)d\tau. \end{aligned}$$

Thus, if we know the initial state $x(t_0)$ and the current input $u(t)$, then the past input is irrelevant in the process of calculating the output $y(t)$ for $t \geq t_0$ because the initial state "summarizes" the past input.

A system is a *lumped* system if the number of state variables is finite, that is, the state vector $\boldsymbol{x}$ is composed of finite number of components. A simple example of lumped system is the circuit shown in Figure 3. More examples of lumped systems are given in Section **??**.

A system is *distributed* if its state consists of infinitely many components. We now give an example of a distributed system.

Example 1 [2, p. 7] Consider the unit-time delay system model of the form,

$$y(t) = u(t-1).$$

In this example, the output is the input delayed by one time unit. To determine $y(t)$ for $t > t_0$, we need the information about $u(t)$ on the time interval $[t_0 - 1,\ t_0]$. Thus the initial state of this time-delay system is the set of points, $\{u(t) : t \in [t_0 - 1,\ t_0]\}$. However, there are infinitely many points $u(t)$ on this interval. Thus, the above time-delay system is a distributed system because its state is infinitely dimensional.

In practice it is often not possible or too expensive to measure or determine the values of all of the state variables. Instead, only their subset or combination can be measured. The system quantities whose behavior can be measured or observed constitute the system's outputs.

1.2 Formulation of the Control Problem

In engineering applications, when dealing with dynamic systems, we are interested in specifying the system inputs that force the system states or outputs to behave with time in some pre-specified manner. That is, we are interested in *controlling* the system states or outputs. This is accomplished by means of a *controller* whose task is to produce the required system's inputs that in turn result in the desired system's outputs. An interconnection of the system and a controller is called a *control system*. In Figure 4 and 5, we show two different types of control systems that we discuss in the next section. Constructing a controller is a part of the *control problem*. The essential elements of the control problem, as described by Owens [9, page 181], are:

1. a dynamic system to be controlled,
2. a specified objective for the system,
3. a set of admissible controllers, and
4. a means of measuring the performance of any given control strategy to evaluate its effectiveness.

We now examine these elements one by one. The first step in the controller design procedure is the construction of a *truth model* of the dynamics of the process to be controlled. The truth model is a *simulation model* that includes all the relevant characteristics of the process. The truth model is too complicated for use in the controller design. Thus, we need to develop a simplified model that can be used to design a controller. Such a simplified model is labeled by Friedland [3] as the *design model*. The design model should capture the essential feature of the process. During the modeling process, properties of the system, physical constraints, and technical requirements should be taken into account. We discuss in more detail the process of constructing mathematical models of physical systems in sections **??** and **??**.

The objective of a control system is to complete some specified task. This can be expressed as a combination of constraints on the output or state variables and limits on the time available to complete the control objective. For example, the objective of a controller might be to force the output, of a single output system, to settle within a certain percentage of the given value after pre-specified time.

Control input signals are obtained from physical devices capable of providing only a limited amount of energy. We term the class of controllers that can be considered for the given control design problem as the set of admissible controllers.

We can evaluate the performance of any given control law by visual inspection of the transient characteristics of the controlled system after the design is complete. But this method of assessment of the effectiveness of the given control strategy is highly inaccurate. We therefore try to quantify the means of evaluating performance in the form of a performance index or cost functional at the beginning of the design process. We construct a numerical index of performance whose value reflects the quality of any admissible controller in accomplishing the system objective. This performance index is assumed to have the property that its numerical value decreases as the quality of the controller increases. This is the reason why we sometimes refer to the performance index as the penalty functional. The "best" controller then is the one that produces the smallest value of the performance index. We call an admissible controller that simultaneously ensures the completion of the system objective and the minimization of the performance index an optimal controller for the system.

1.3 Open-Loop Versus Closed-Loop

We distinguish between two types of control systems. They are:

- open-loop control systems, and
- closed-loop control systems.

An open-loop control system usually contains:

1. A process to be controlled, labeled *plant*,
2. The controlling variable of the plant, called the *plant input*, or just *input* for short,
3. The controlled variable of the plant, called the *plant output*, or just *output* for short,
4. A *reference input*, which dictates the desired value of the output,
5. A *controller* that acts upon the reference input in order to form the system input which would force the behavior of the output in accordance with the reference signal.

Note that the plant and controller themselves can be considered as systems on their own. A schematic representation of an open-loop system is depicted in Figure 4. In an open-loop control system the output has no influence on the input or reference signal. The controller operates without taking into account the output. Thus, the plant input is formed with no influence of the output. A household appliance such as an iron is a simple example of an open-loop control system. In this example, we consider the iron itself as the plant or process

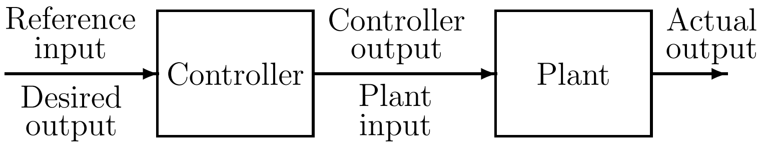

Figure 4: Open-loop control system.

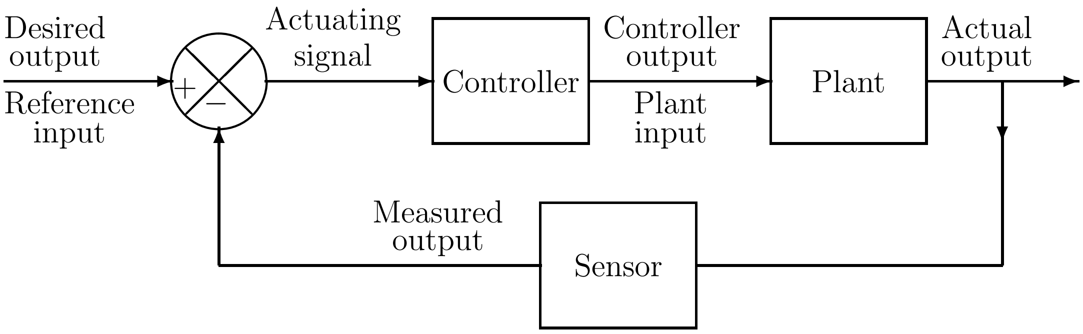

Figure 5: Closed-loop control system.

to be controlled. Here we set the temperature control mechanism for the desired temperature. This setting, for example very hot, is the reference input. We can view the potentiometer, inside the iron, as a controller, or a control actuator. It regulates the temperature of the iron by allowing the necessary amount of resistance to produce the desired temperature. The plant's output is the temperature of the heated iron.

Not every control system is an open-loop system. Another type of a control system is a closed-loop system. We can convert an open-loop system into a closed-loop system by adding, to an open-loop system, the following components:

6. The *feedback loop* where the output signal is measured with a sensor and then the measured signal is fed back to the *summing junction*,

7. The summing junction, where the measured output signal is subtracted from the reference, command, input signal in order to generate an *error signal*, also labeled as an *actuating signal*.

A schematic of a closed-loop control system, or feedback system, is shown in Figure 5. In a closed-loop system the error signal causes an appropriate action of the controller, which

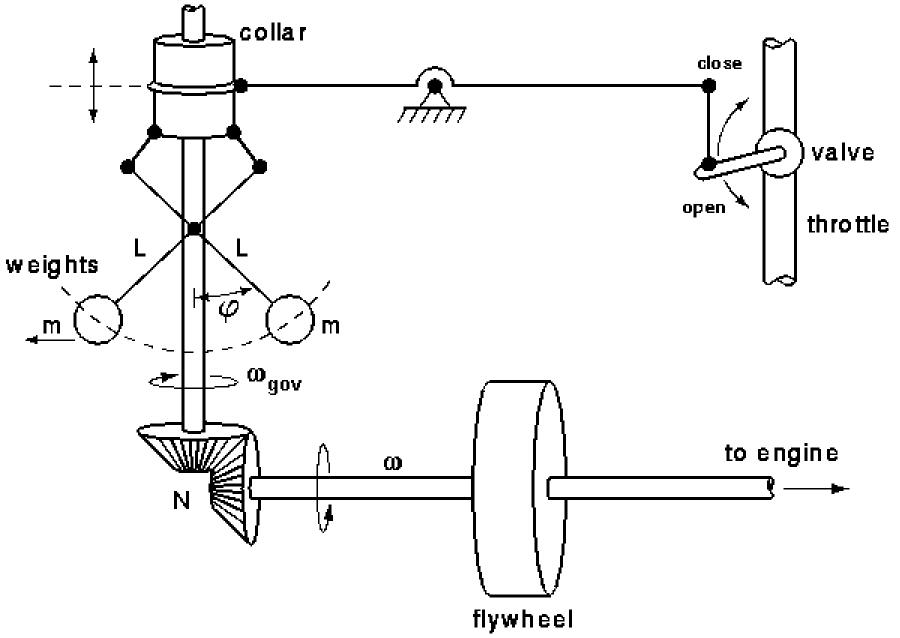

Figure 6: A simplified schematic of the flyball governor.

in turn "instructs" the plant to behave in a certain way in order to approach the desired output, as specified by the reference input signal. Thus, in the closed-loop system, output information is fed back to the controller, and the controller then appropriately modifies the plant output behavior. A controller, also called a *compensator*, can be placed either in the *forward loop*, or in the *feedback loop*. Thus, a central component of the closed-loop system is feedback. Norbert Wiener defined feedback in 1954 as follows: "Feedback is a method of controlling a system by reinserting into it the results of its past performance." [6, p. 439].

If the goal of the controller is to maintain the output at a constant value, then we have a *regulator* control system. A "centrifugal speed regulator", or the "flyball governor," commonly known as Watt's governor, is an example of a regulator in a closed-loop system. A schematic of Watt's governor is shown in Figure 6.

The task of the centrifugal speed regulator is to control setting of a throttle valve automatically to maintain the desired engine speed. The nominal speed is set by adjusting the valve in the throttle. As the engine speeds up the weights are thrown outwards, the throttle partially closes and the engine slows down. As the engine slows down below its nominal speed, the throttle is opened up and the engine gains its speed. The operation of an engine equipped with a centrifugal speed regulator can be represented using the block diagram of Figure 5. The reference input to the closed-loop system is the desired engine speed. The desired engine speed is obtained by appropriately setting the valve. The system output is the actual engine speed. The flyball governor is our controller, while the engine represents the plant. The control signal generated by the controller is the setting of the throttle valve.

We mention at this juncture that Watt's application of the centrifugal speed governor in 1788, to regulate the speed of the steam engine, marks the starting point for the development of automatic control, and in particular feedback control, as a science [7]. Before 1788, steam engines were controlled manually using the throttle valve. The flyball governor spread widely and quickly. It has been estimated that by 1868 there were about 75,000 flyball governors working in England alone [7]. The flyball governor is credited for the hatching of the industrial revolution. We devise a mathematical model of this revolutionary invention in Subsection **??**, and in Section **??** we perform a quantitative analysis of its dynamical behavior.

If the goal of the controller is to force the output to follow a desired trajectory, then we have a *servomechanism* or *tracking* control system. As we mentioned before, to design a controller, we need first to analyze the plant quantitatively. The analysis requires a mathematical or linguistic description of the interrelations between the system quantities themselves as well as the interrelations between system quantities and system inputs.

2 Linearity

It follows from Definition 1 that a system accepts the input, $\boldsymbol{u}(t)$, and the initial state, $\boldsymbol{x}(t_0)$, to produce its output, $\boldsymbol{y}(t)$. We thus can view a system as an operator that acts on two inputs, $\boldsymbol{u}(t)$ and $\boldsymbol{x}(t_0)$ to produce an output, $\boldsymbol{y}(t)$. We use the symbol L to represent a system. Then its operation can be described as

$$L(\boldsymbol{u}(t), \boldsymbol{x}(t_0)) = \boldsymbol{y}(t).$$

A system L is said to possess an *additivity property* if for any t_0 and any $\boldsymbol{x}_i(t_0)$ and $\boldsymbol{u}_i(t)$, $t \geq t_0$, if

$$L(\boldsymbol{u}_i(t), \boldsymbol{x}_i(t_0)) = \boldsymbol{y}_i(t), \quad i = 1, 2,$$

then

$$L(\boldsymbol{u}_1(t) + \boldsymbol{u}_2(t), \boldsymbol{x}_1(t_0) + \boldsymbol{x}_2(t_0)) = \boldsymbol{y}_1(t) + \boldsymbol{y}_2(t).$$

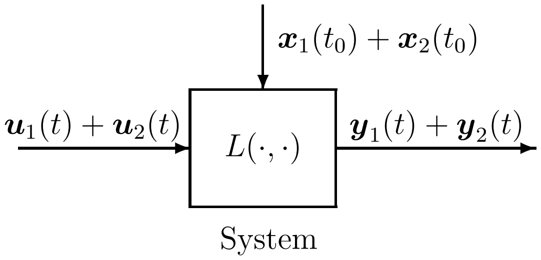

Figure 7: Illustration of the additivity property.

We illustrate the above property in Figure 7.

A system L is said to possess a *homogeneity property* if for any t_0 and any $\boldsymbol{x}(t_0)$ and $\boldsymbol{u}(t)$, $t \geq t_0$, if

$$L(\boldsymbol{u}(t), \boldsymbol{x}(t_0)) = \boldsymbol{y}(t),$$

then

$$L(c\boldsymbol{u}(t), c\boldsymbol{x}(t_0)) = c\boldsymbol{y}(t),$$

where c is real or complex constant. We illustrate the above property in Figure 8.

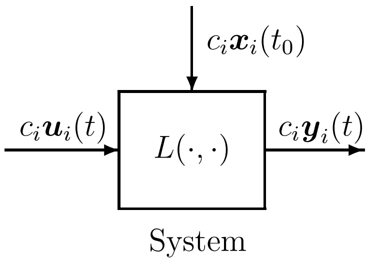

Figure 8: Illustration of the homogeneity property.

A system L is said to possess a *superposition property* if for any t_0 and any $\boldsymbol{x}_i(t_0)$ and $\boldsymbol{u}_i(t)$, $t \geq t_0$, if

$$L(\boldsymbol{u}_i(t), \boldsymbol{x}_i(t_0)) = \boldsymbol{y}_i(t), \quad i = 1, 2,$$

then

$$L(c_1\boldsymbol{u}_1(t) + c_2\boldsymbol{u}_2(t), c_1\boldsymbol{x}_1(t_0) + c_2\boldsymbol{x}_2(t_0)) = c_1\boldsymbol{y}_1(t) + c_2\boldsymbol{y}_2(t).$$

We illustrate the above property in Figure 9.

Note that the superposition property is a combination of the additivity and homogeneity properties.

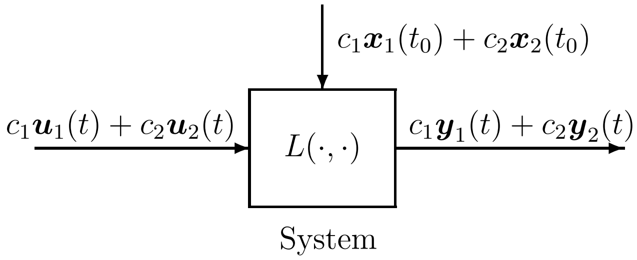

Figure 9: Illustration of the superposition property.

Definition 2 *A system is said to be linear if it possesses the superposition property. A system is said to be nonlinear if it does not have the superposition property.*

For an informative discussion on some subtleties in the definition of linearity, we recommend Kailath [4, Section 1.1].

If $\boldsymbol{u}(t) = \mathbf{0}$ for $t \geq 0$, then the system's output will be produced exclusively by the initial condition $\boldsymbol{x}(t_0)$. We call such an output the *zero-input response* and denote it as $\boldsymbol{y}_{zi}$.

If, on the other hand, the initial state $\boldsymbol{x}(t_0) = \mathbf{0}$, then the output will be produced exclusively by the input. This output is called the *zero-state response* and is denoted $\boldsymbol{y}_{zs}$.

If the system is linear, then the additivity property implies that the total response, $\boldsymbol{y}(t)$ is the sum of the zero-input and zero-state responses, that is,

$$\boldsymbol{y}(t) = \boldsymbol{y}_{zi}(t) + \boldsymbol{y}_{zs}(t), \quad t \geq t_0.$$

For linear systems the superposition property holds for zero-state and zero-input responses separately as well. In our discussion here, for the sake of simplicity, we do not display the second argument in the operator L when we set zero initial state or zero input.

Now, if we set the initial state to zero, $\boldsymbol{x}(t_0) = \mathbf{0}$ and if

$$L(\boldsymbol{u}_j(t)) = \boldsymbol{y}_{zs,j}(t), \quad j = 1, 2,$$

then

$$L(c_1\boldsymbol{u}_1(t) + c_2\boldsymbol{u}_2(t)) = c_1\boldsymbol{y}_{zs,1}(t) + c_2\boldsymbol{y}_{zs,2}(t).$$

Similarly, if we set $\boldsymbol{u}(t) = \mathbf{0}$ for $t \geq t_0$ and if

$$L(\boldsymbol{x}_j(t_0)) = \boldsymbol{y}_{zi,j}(t), \quad j = 1, 2,$$

then

$$L(c_1\boldsymbol{x}_1(t_0) + c_2\boldsymbol{x}_2(t_0)) = c_1\boldsymbol{y}_{zi,1}(t) + c_2\boldsymbol{y}_{zi,2}(t).$$

3 Input-Output Description of Linear Systems

In this Section, we develop a mathematical description of the zero-state response of linear systems. We assume that the initial state is zero and so the system's output is exclusively due to the system input.

Definition 3 *We say that the system is relaxed at* t_0 *if its initial condition at* t_0 *is zero, that is,* $\boldsymbol{x}(t_0) = \mathbf{0}$.

To proceed, we need certain facts about the impulse function, $\delta(t)$, that we now review.

3.1 The Impulse Function and Its Properties

The impulse function, denoted $\delta(t)$, also called the Dirac function, is a signal of infinite amplitude, zero duration, and unity area. We can construct an impulse function as the limit of pulse functions

$$p_{\varepsilon_i}(t) = \frac{1}{\varepsilon_i}\left(1(t) - 1(t - \varepsilon_i)\right)$$

as $\varepsilon_i \to 0$, where $1(t)$ is the unit step function defined as

$$1(t) = \begin{cases} 1 & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0. \end{cases}$$

The pulse p_{ε_i} is illustrated in Figure 10. Note that the pulse functions have the following features as $\varepsilon_i \to 0$:

1. the amplitude approaches infinity,
2. the duration of the pulses approaches zero,
3. the area under each pulse is constant; in our example the area equals unity.

The unit impulse function is defined as

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \quad \text{and} \quad \delta(t) = 0 \ \text{ for } \ t \neq 0.$$

The above definition states that the area under the impulse function is constant. The area represents the strength of the impulse function. The impulse function of strength K is denoted $K\delta(t)$. Its graphical representation is depicted in Figure 11. The strength of the impulse is shown next to the arrow's head. The shifted impulse function of strength K is also shown in Figure 11. The unit impulse function can be thought of as a derivative of the

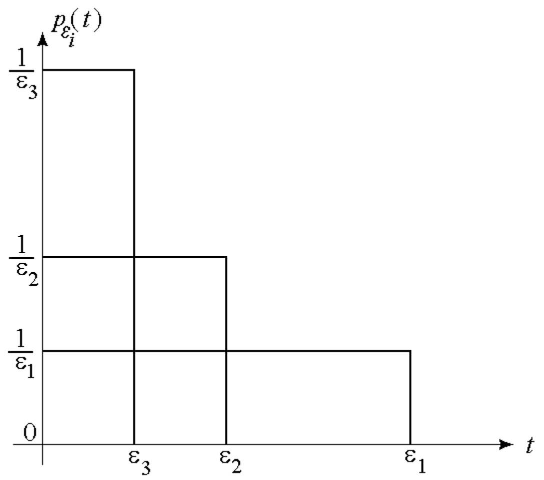

Figure 10: Generating the unit impulse function as the limit of the pulse functions.

unit step function, that is,

$$\delta(t) = \frac{d1(t)}{dt}.$$

The function shown in Figure 12 approaches the unit step function as $\varepsilon \to 0$. The function shown in Figure 13, which is the derivative of the function from Figure 12, approaches the unit impulse function as $\varepsilon \to 0$.

The impulse function has the *sifting property*,

$$\boxed{\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)}$$

where $f(t)$ is a continuous function of time. It follows from the above that the impulse function sifts out everything except the value of f at $t = a$—hence the name of the property. To verify the validity of the sifting property, we note that $\delta(t-a)$ is zero everywhere except at $t = a$. Hence, we can write,

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = \int_{a-\varepsilon}^{a+\varepsilon} f(t)\delta(t-a)dt.$$

By assumption, f is continuous at a. Therefore, it must take the value of $f(a)$ as $t \to a$. Thus,

$$\int_{a-\varepsilon}^{a+\varepsilon} f(t)\delta(t-a)dt = f(a)\int_{a-\varepsilon}^{a+\varepsilon} \delta(t-a)dt = f(a),$$

which was to be demonstrated.

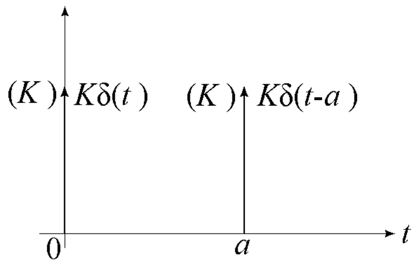

Figure 11: Graphical representation of the impulse $K\delta(t)$ and $K\delta(t-a)$.

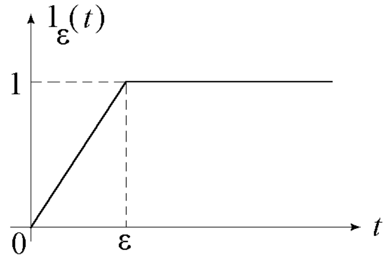

Figure 12: The function 1_ϵ approaches the unit step as $\varepsilon \to 0$.

We now discuss another important property of the impulse function—the sampling property. Since $\delta(t-a)=0$ for $t \neq a$,

$$f(t)\delta(t-a) = 0 \quad \text{for} \quad t \neq a$$

as is

$$f(a)\delta(t-a) = 0 \quad \text{for} \quad t \neq a.$$

However, when $t=a$, we have

$$f(t)\delta(t-a) = f(a)\delta(t-a) \quad \text{for} \quad t = a$$

provided that $f(a)$ exists. Therefore,

$$\boxed{f(t)\delta(t-a) = f(a)\delta(t-a) \quad \text{for all } t}$$

The above property is called the sampling property of the impulse function.

Examples

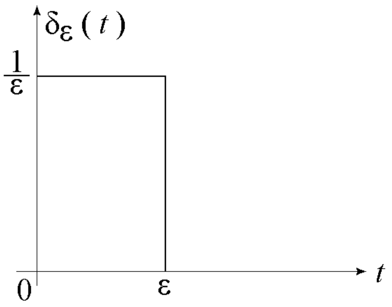

Figure 13: The derivative of $1_\varepsilon(t)$ shown in Figure 12. As $\varepsilon \to 0$, $\delta_\varepsilon(t)$ approches $\delta(t)$.

(i) $(\cos 3t)\delta(t-\pi) = (\cos 3\pi)\delta(t-\pi) = -\delta(t-\pi)$;

(ii) $e^{-2t}\delta(t) = e^{-2(0)}\delta(t) = \delta(t)$;

(iii) $(1-e^{-4t})\,\delta(t) = (1-e^{0})\,\delta(t) = 0\delta(t) = 0$.

3.2 Zero-State Response Description of Continuous-Time, Relaxed, Causal Systems

Suppose that the system under consideration is a single-input single-output (SISO) linear system. Let $\delta_\Delta(t-t_1)$ be the pulse depicted in Figure 14. The pulse has width Δ and height $1/\Delta$.

We can use the pulse $\delta_\Delta(t-t_1)$ to approximate an input signal $u(t)$ as illustrated in Figure 15. That is, we can approximate the control signal $u(t)$ as

$$u(t) \approx \sum_i u(t_i)\delta_\Delta(t-t_i)\Delta. \tag{1}$$

Let $y(t) = g_\Delta(t,t_i)$ be the system output at time t produced by the pulse $u(t) = \delta_\Delta(t-t_i)$ applied at time t_i. Then by the homogeneity property, the input $u(t) = \delta_\Delta(t-t_i)u(t_i)\Delta$ would produce the output, $y(t) = g_\Delta(t,t_i)u(t_i)\Delta$. Applying the additivity property, we conclude that the input given by (1) will produce the output

$$y(t) \approx \sum_i g_\Delta(t,t_i)u(t_i)\Delta. \tag{2}$$

Now if $\Delta \to 0$, then

$$\delta_\Delta(t-t_i) \to \delta(t-t_i)$$

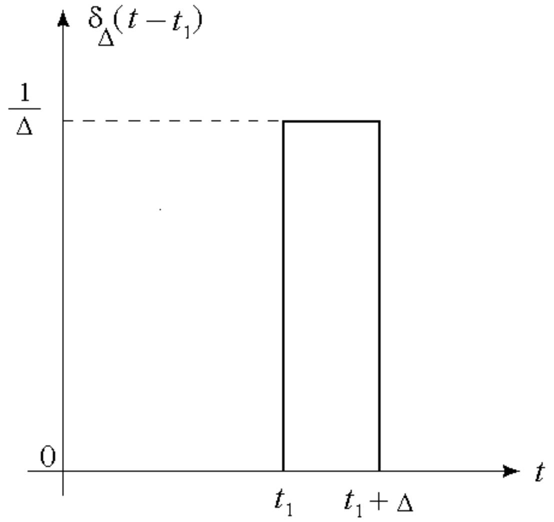

Figure 14: The pulse $\delta_\Delta(t - t_1)$ at $t = t_1$.

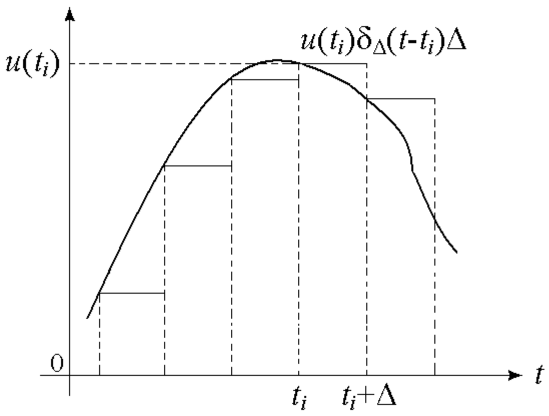

Figure 15: Approximating the control signal $u(t)$.

and the corresponding output of the system is denoted $g(t, t_i)$. Next, as $\Delta \to 0$ the approximation in (2) becomes an equality, the summation an integration, the discrete time t_i becomes a continuum that we denote using the symbol τ, while Δ will be represented as $d\tau$. Taking the above into account, (2) takes the form,

$$y(t) = \int_{-\infty}^{\infty} g(t, \tau) u(\tau) d\tau. \tag{3}$$

We note that $g(t, \tau)$ is a function of two variables. The first variable denotes the time t at which the output is observed, while the second variable, τ, denotes the time at which the impulse input is applied.

We call $g(t, \tau)$ the impulse response because it is the system's output when excited with the impulse input.

For the causal system, the output cannot appear before the input is applied. Therefore, for the causal systems,

$$g(t, \tau) = 0 \quad \text{for} \quad t < \tau$$

and we can replace the upper integration limit in (3) with t to obtain

$$y(t) = \int_{-\infty}^{t} g(t, \tau) u(\tau) d\tau. \tag{4}$$

For the relaxed system at t_0, its initial state, $\boldsymbol{x}(t_0) = \mathbf{0}$. Hence, the system output is produced exclusively by the input $u(t)$ for $t \geq t_0$. Thus we can replace the lower integration limit in (4) with t_0, and we obtain for the linear, relaxed at t_0, causal system the following expression for its impulse response;

$$y(t) = \int_{t_0}^{t} g(t, \tau) u(\tau) d\tau. \tag{5}$$

For the multi-input multi-output (MIMO) linear, relaxed at t_0, casual system with p output components and m inputs, the above expression generalizes to

$$\boldsymbol{y}(t) = \int_{t_0}^{t} \boldsymbol{G}(t, \tau) \boldsymbol{u}(\tau) d\tau,$$

where

$$\boldsymbol{G}(t, \tau) = \begin{bmatrix} g_{11}(t, \tau) & g_{12}(t, \tau) & \cdots & g_{1m}(t, \tau) \\ g_{21}(t, \tau) & g_{22}(t, \tau) & \cdots & g_{2m}(t, \tau) \\ \vdots & \vdots & \ddots & \vdots \\ g_{p1}(t, \tau) & g_{p2}(t, \tau) & \cdots & g_{pm}(t, \tau) \end{bmatrix}.$$

In the above, $g_{ij}(t, \tau)$ is the impulse response at time t at the i-th output due to an impulse applied at time τ at the j-th input, with other inputs set to zero.

We refer to the matrix $\boldsymbol{G}$ as the *impulse response matrix.*

3.3 Input-Output Description of Continuous Time-Invariant Systems

Suppose that we have a system with the initial state $\boldsymbol{x}(t_0)$, the input $\boldsymbol{u}(t)$ applied from t_0 that produced the output $y(t)$ for $t \geq t_0$. We compactly describe the above mental experiment as

$$L(\boldsymbol{u}(t), \boldsymbol{x}(t_0)) = \boldsymbol{y}(t), \quad \text{for} \quad t \geq t_0.$$

The above system is said to be time-invariant if for any T, if the initial state is shifted to time $t_0 + T$, the same input is applied from $t_0 + T$ rather than at t_0, then the output is the same except that now it starts to appear at $t_0 + T$, that is,

$$L(\boldsymbol{u}(t-T), \boldsymbol{x}(t_0+T)) = \boldsymbol{y}(t-T), \quad \text{for} \quad t \geq t_0 + T.$$

Therefore, the impulse response of a linear, relaxed at t_0, causal, time invariant system can be described as

$$g(t, \tau) = g(t+T, \tau+T) = g(t-\tau, 0), \tag{6}$$

which we compactly denote as $g(t-\tau)$. Note that the impulse response of an LTI system, $g(t-\tau)$, is a function of one variable, where $g(t) = g(t-0)$ is the system output at time t to an impulse applied at $\tau = 0$. The LTI system is causal if $g(t) = 0$ for $t < 0$.

Taking into account (6), we represent the expression for the output (5) of a linear, relaxed at t_0, causal, LTI system as

$$y(t) = \int_0^t g(t-\tau) u(\tau) d\tau, \tag{7}$$

where, without loss of generality, we replaced t_0 with 0. The integration in (7) is known as the *convolution* integral. The above integral relation is also written in a shorthand notation as

$$y(t) = g(t) * u(t),$$

where the asterisk signifies the integral relation between $g(t)$ and $u(t)$. The integral relation $g(t) * u(t)$ is read as "$g(t)$ is convolved with $u(t)$."

Note that the convolution is a commutative operation, that is,

$$y(t) = \int_0^t g(t-\tau) u(\tau) d\tau = \int_0^t g(\tau) u(t-\tau) d\tau, \tag{8}$$

which can be represented as

$$y(t) = g(t) * u(t) = u(t) * g(t).$$

A very useful tool in the analysis and design of LTI dynamic systems is the Laplace transform. Applying the Laplace transform to (7) gives

$$\begin{aligned} Y(s) &= \mathcal{L}(y(t)) \\ &= \int_{0^-}^{\infty} \left(\int_{\tau=0}^{t} g(t-\tau)u(\tau)d\tau \right) e^{-st}dt. \end{aligned}$$

For causal systems, $g(t) = 0$ for $t < 0^-$; equivalently, $g(t-\tau) = 0$ for $\tau > t^-$. This means that we can replace the upper integration limit t in the above with ∞, and performing some simple manipulations, we obtain

$$\begin{aligned} Y(s) &= \int_{0^-}^{\infty} \left(\int_{\tau=0^-}^{\infty} g(t-\tau)u(\tau)d\tau \right) e^{-st}dt \\ &= \int_{0^-}^{\infty} \left(\int_{\tau=0^-}^{\infty} g(t-\tau)u(\tau)d\tau \right) e^{-s(t-\tau)}e^{-s\tau}dt \\ &= \int_{0^-}^{\infty} \left(\int_{\tau=0^-}^{\infty} g(t-\tau)e^{-s(t-\tau)}dt \right) u(\tau)e^{-s\tau}d\tau \\ &= \int_{0^-}^{\infty} g(t-\tau)e^{-s(t-\tau)}dt \int_{\tau=0^-}^{\infty} u(\tau)e^{-s\tau}d\tau \\ &= G(s)U(s), \end{aligned} \tag{9}$$

where $G(s)$ is called the transfer function of the system and $U(s)$ is the Laplace transform of the input signal.

The Laplace transform of the convolution of two functions of time is equal to the products of their Laplace transforms, that is,

$$\mathcal{L}(g(t) * u(t)) = \mathcal{L}(g(t))\mathcal{L}(u(t)) = G(s)U(s).$$

That is, to the product in the s-domain there corresponds the the convolution in the time domain.

If a linear time-invariant system is lumped, its transfer function, $G(s)$, is a rational function of s, that is,

$$G(s) = \frac{N(s)}{D(s)},$$

where $N(s)$ and $D(s)$ are polynomials in s.

For the MIMO LTI dynamic system, (9) takes the form,

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \\ \vdots \\ Y_p(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1m}(s) \\ G_{21}(s) & G_{22}(s) & \cdots & G_{2m}(s) \\ \vdots & \vdots & \ddots & \vdots \\ G_{p1}(s) & G_{p2}(s) & \cdots & G_{pm}(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \\ \vdots \\ U_m(s) \end{bmatrix}.$$

We represent the above in a vector-matrix notation as

$$\boldsymbol{Y}(s) = \boldsymbol{G}(s)\boldsymbol{U}(s),$$

where $\boldsymbol{G}(s)$ is called the *transfer function matrix.*

Example 2 If we apply the impulse input in the unit-time delay system of Example 1, then the output is $\delta(t-1)$. Hence the impulse response of this system is

$$g(t) = \delta(t-1). \tag{10}$$

The transfer function of the unit-time delay system of Example 1 is

$$G(s) = e^{-s}. \tag{11}$$

Example 3 The integrator can be described as

$$y(t) = k_I \int_{-\infty}^{t} u(\tau)d\tau, \tag{12}$$

where k_I is the gain. We can equivalently describe the integrator as

$$\frac{dy(t)}{dt} = k_I u(t). \tag{13}$$

The impulse response of the integrator is the unit step function also known as the Heaviside step function.

The transfer function of the integrator described by (12) or by (13) is

$$G(s) = \frac{k_I}{s}.$$

Example 4 The ideal differentiator is described as

$$y(t) = k_D \frac{du(t)}{dt},$$

where k_D is the differentiator's gain. Its impulse response is

$$
\begin{aligned}
g(t) &= k_D \frac{d\delta(t)}{dt} \\
&= \begin{cases} 0 & \text{for} \quad t \neq 0 \\ +\infty & \text{for} \quad t = 0^- \\ -\infty & \text{for} \quad t = 0^+, \end{cases}
\end{aligned}
\tag{14}
$$

The transfer function of the ideal differentiator is

$$G(s) = k_D s.$$

3.4 State-Space Description of Continuous Time-Invariant Systems

Every linear time-invariant (LTI) lumped causal system can be described by a set of equations of the form,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{15}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t), \tag{16}$$

where

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

is the state vector. The matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ are constant real matrices. A block diagram representation of the state-space description given by (15) and (16) is depicted in Figure 16.

Taking the Laplace transform of (15) and (16) gives

$$s\boldsymbol{X}(s) - \boldsymbol{x}(0^-) = \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{B}\boldsymbol{U}(s) \tag{17}$$

$$\boldsymbol{Y}(s) = \boldsymbol{C}\boldsymbol{X}(s) + \boldsymbol{D}\boldsymbol{U}(s). \tag{18}$$

We compute $\boldsymbol{X}(s)$ from (17),

$$\boldsymbol{X}(s) = (s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{x}(0^-) + (s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s)$$

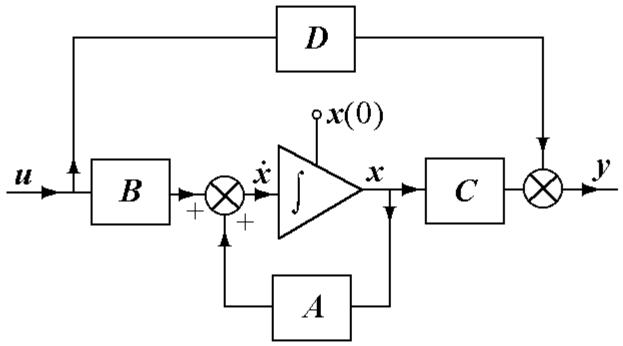

Figure 16: Block diagram representation of the state-space description given by (15) and (16).

Substituting the above into (18) yields

$$\boldsymbol{Y}(s) = \boldsymbol{C}\left(s\boldsymbol{I}_n - \boldsymbol{A}\right)^{-1}\boldsymbol{x}(0^-) + \boldsymbol{C}\left(s\boldsymbol{I}_n - \boldsymbol{A}\right)^{-1}\boldsymbol{B}\boldsymbol{U}(s) + \boldsymbol{D}\boldsymbol{U}(s).$$

If the initial condition is zero, then we obtain the expression for the transfer function matrix,

$$\boldsymbol{G}(s) = \boldsymbol{C}\left(s\boldsymbol{I}_n - \boldsymbol{A}\right)^{-1}\boldsymbol{B} + \boldsymbol{D}. \tag{19}$$

3.5 Input-Output Description of Discrete Time-Invariant Systems

Recall that a discrete-time system accepts discrete-time signals and produces discrete-time signals as its output.

As in with a continuous-time system, a discrete-time system is linear if it possesses the superposition property.

The response of a discrete-time linear system is the sum of the zero-state and zero-input responses. Both the zero-state and zero-input responses possess the superposition property.

A discrete-time system is causal if its current output depends on the current and past inputs.

The discrete system state, $\boldsymbol{x}[k_0]$, is the information at time k_0 that together with $\boldsymbol{u}[k]$ for $k \geq k_0$ uniquely determines the system output $\boldsymbol{y}[k]$ for $k \geq k_0$.

If the number of the components in the state vector $\boldsymbol{x}$ is finite, then this discrete system is lumped; otherwise it is distributed. If there is a time delay in a continuous-time system,

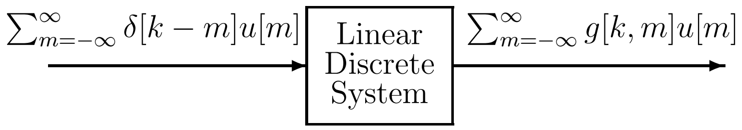

Figure 17: Output of linear discrete-time system in response to the input sequence (20).

then this system is distributed. On the other hand, if in a discrete-time system the time delay is an integer multiple of the sampling period h, then such a system is a lumped system.

To proceed, we define the impulse sequence denoted $\delta[k-m]$ as

$$\delta[k-m] = \begin{cases} 1 & \text{if} \quad k = m \\ 0 & \text{if} \quad k \neq m, \end{cases}$$

where both k and m are integers that denote sampling instances.

Suppose now that we have an input sequence denoted $u[k]$. We express this input sequence as

$$u[k] = \sum_{m=-\infty}^{\infty} u[k]\delta[k-m] \tag{20}$$

Let now $g[k,m]$ denote the discrete-time system output at time k when excited with the impulse sequence, $\delta[k-m]$, at the time instant m. Because by assumption the discrete system under consideration is linear, hence by the homogeneity property the input $\delta[k-m]u[m]$ will yield the output, $g[k,m]u[m]$. Combining the above with the additivity property, we conclude that the input, $\sum_{m=-\infty}^{\infty} \delta[k-m]u[m]$ will yield the output, $\sum_{m=-\infty}^{\infty} g[k,m]u[m]$. We illustrate the above in Figure 17.

In a causal discrete-time linear system, no output can appear before an input is applied. Hence,

$$g[k,m] = 0 \quad \text{for} \quad k < m.$$

If, in addition, the system is relaxed at k_0, then the output will take the form,

$$y[k] = \sum_{m=k_0}^{k} g[k,m]u[m].$$

Note that for time-invariant linear causal system we can always set $k_0 = 0$ to obtain

$$y[k] = \sum_{m=0}^{k} g[k-m]u[m]. \tag{21}$$

The above is referred to as a *discrete convolution.* It is easy to check that the discrete convolution possesses the commutativity property, that is,

$$y[k] = \sum_{m=0}^{k} g[k-m]u[m] = \sum_{m=0}^{k} g[m]u[k-m].$$

Applying the z-transform to (21) gives

$$\begin{aligned} Y(z) &= \mathcal{Z}(y[k]) \\ &= \sum_{k=0}^{\infty} \left(\sum_{m=0}^{k} g[k-m]u[m] \right) z^{-k}. \end{aligned}$$

Taking into account the causality assumption, we can replace the upper integration limit k with ∞. Then performing simple manipulations, we obtain

$$\begin{aligned} Y(z) &= \sum_{k=0}^{\infty} \left(\sum_{m=0}^{\infty} g[k-m]u[m] \right) z^{-(k-m)} z^{-m} \\ &= \sum_{m=0}^{\infty} \left(\sum_{m=0}^{\infty} g[k-m]z^{-(k-m)} \right) u[m]z^{-m} \\ &= \left(\sum_{l=0}^{\infty} g[l]z^{-l} \right) \left(\sum_{m=0}^{\infty} u[m]z^{-m} \right) \\ &= G(z)U(z), \end{aligned}$$

where we interchanged the order of summations, introduced the new variable $l = k - m$ and took into account the fact that $g[l] = 0$ for $l < 0$.

The z-transform of the impulse response, denoted $G(z)$, is called the *discrete transfer function.* Note that the transfer functions describe only the zero-state responses.

Example 5 [2, p. 33] Consider the unit-sampling-time delay system model of the form,

$$y[k] = u[k-1].$$

The impulse response of the above system is

$$g[k] = \delta[k-1].$$

The discrete transfer function is obtained by taking the z-transform of the impulse response. We obtain

$$\begin{aligned} G(z) &= \mathcal{Z}(\delta[k-1]) \\ &= \frac{1}{z}. \end{aligned}$$

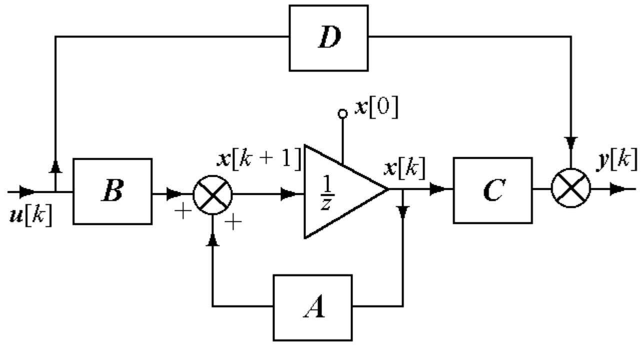

Figure 18: Block diagram representation of the state-space description given by (22) and (23).

The above transfer function is a rational function of z. Thus the above system is a lumped system. Note that in the continuous-time case, systems involving time delays are distributed system. As the above example shows, this is not necessarily the case in discrete-time systems.

3.6 State-Space Description of Discrete Time-Invariant Systems

Every discrete linear time-invariant (LTI) lumped causal system can be described by a set of equations of the form,

$$\boldsymbol{x}[k+1] = \boldsymbol{A}\boldsymbol{x}[k] + \boldsymbol{B}\boldsymbol{u}[k], \quad \boldsymbol{x}[0] = \boldsymbol{x}_0 \tag{22}$$

$$\boldsymbol{y}[k] = \boldsymbol{C}\boldsymbol{x}[k] + \boldsymbol{D}\boldsymbol{u}[k], \tag{23}$$

where

$$\boldsymbol{x}[k] = \begin{bmatrix} x_1[k] \\ \vdots \\ x_n[k] \end{bmatrix}$$

is the state vector. The matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ are constant real matrices. A block diagram representation of the state-space description given by (22) and (23) is depicted in Figure 18.

Let $\boldsymbol{X}(z)$ denote the z-transform of $\boldsymbol{x}[k]$, that is,

$$\boldsymbol{X}(z) = \mathcal{Z}(\boldsymbol{x}[k]).$$

Then, the z-transform of $\boldsymbol{x}[k+1]$ is

$$\begin{aligned}
\mathcal{Z}(\boldsymbol{x}[k+1]) &= \sum_{k=0}^{\infty} \boldsymbol{x}[k+1] z^{-k} \\
&= z \sum_{k=0}^{\infty} \boldsymbol{x}[k+1] z^{-(k+1)} \\
&= z \left(\sum_{j=1}^{\infty} \boldsymbol{x}[j] z^{-j} + \boldsymbol{x}[0] - \boldsymbol{x}[0] \right) \\
&= z \left(\boldsymbol{X}(z) - \boldsymbol{x}[0] \right). \qquad (24)
\end{aligned}$$

Applying the z-transform to (22) and (23) and taking into account (24) gives

$$\begin{aligned}
z\boldsymbol{X}(z) - z\boldsymbol{x}[0] &= \boldsymbol{A}\boldsymbol{X}(z) + \boldsymbol{B}\boldsymbol{U}(z), \quad \boldsymbol{x}[0] = \boldsymbol{x}_0 \qquad (25) \\
\boldsymbol{Y}(z) &= \boldsymbol{C}\boldsymbol{X}(z) + \boldsymbol{D}\boldsymbol{U}(z). \qquad (26)
\end{aligned}$$

We calculate $\boldsymbol{X}(z)$ from (25) to obtain

$$\boldsymbol{X}(z) = (z\boldsymbol{I}_n - \boldsymbol{A})^{-1} z\boldsymbol{x}[0] + (z\boldsymbol{I}_n - \boldsymbol{A})^{-1} \boldsymbol{B}\boldsymbol{U}(z).$$

Substituting the above into (26) yields

$$\boldsymbol{Y}(z) = \boldsymbol{C}(z\boldsymbol{I}_n - \boldsymbol{A})^{-1} z\boldsymbol{x}[0] + \boldsymbol{C}(z\boldsymbol{I}_n - \boldsymbol{A})^{-1} \boldsymbol{B}\boldsymbol{U}(z) + \boldsymbol{D}\boldsymbol{U}(z).$$

If set the initial condition to zero, that is, $\boldsymbol{x}[0] = \boldsymbol{0}$, then we have

$$\boldsymbol{Y}(z) = \left(\boldsymbol{C}(z\boldsymbol{I}_n - \boldsymbol{A})^{-1} \boldsymbol{B} + \boldsymbol{D} \right) \boldsymbol{U}(z).$$

Thus the transfer function of the discrete time-invariant system described by (22) and (23) is

$$\boldsymbol{G}(z) = \boldsymbol{C}(z\boldsymbol{I}_n - \boldsymbol{A})^{-1} \boldsymbol{B} + \boldsymbol{D}. \qquad (27)$$

Note that the above is the discrete equivalent of (19).

We add that to a model represented in a state-space format, there corresponds a unique transfer function matrix. On the hand, there are infinitely many state-space representations that have the same transfer function matrix.

4 From Transfer Function to State-Space Description

If the Laplace variable s is replaced in (19) with the z-transform variable z then we obtain (27), and vice versa. Thus a method of converting a transfer function into a state-space representation for continuous linear time-invariant lumped systems is applicable also for discrete linear time-invariant lumped systems.

We say that a transfer function $\boldsymbol{G}(s)$ is realizable if there exists a quadruple of constant matrices $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ such that $\boldsymbol{G}(s) = \boldsymbol{C}\left(s\boldsymbol{I}_n - \boldsymbol{A}\right)^{-1}\boldsymbol{B} + \boldsymbol{D}$. We call such a quadruple $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ a realization of $\boldsymbol{G}(s)$.

We begin our discussion with a single-input single-output (SISO) system modeled by a transfer function,

$$\begin{aligned} \frac{Y(s)}{U(s)} &= G(s) \\ &= \frac{N(s)}{D(s)} \\ &= \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \end{aligned} \tag{28}$$

that is a proper rational function, which means that

$$\deg_s N(s) \le \deg_s D(s).$$

Note that in (28) the highest coefficient of the denominator polynomial is unity. If this was not the case, we would divide the numerator and the denominator by the highest coefficient of the denominator thus forcing $a_n = 1$.

If $G(s)$ is proper but not strictly, that is, $\deg_s N(s) = \deg_s D(s)$, equivalently, $m = n$, then we divide the numerator $N(s)$ by the denominator $D(s)$ to obtain

$$G(s) = G(s)_{\mathrm{sp}} + G(\infty), \tag{29}$$

where $G(s)_{\mathrm{sp}}$ denotes the strictly proper part of $G(s)$ and $G(\infty) = b_m$.

Our goal is to find a realization of $G(s)$, that is, a quadruple, $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, d)$ such that

$$G(s) = \boldsymbol{c}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} + d. \tag{30}$$

Note that $\boldsymbol{c}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}$ is a strictly proper rational function and d is a scalar. Therefore, in our construction of a realization of $G(s)$, we first extract the strictly proper part as in (29). We then find a triple $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ such that

$$G(s)_{\mathrm{sp}} = \boldsymbol{c}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}. \tag{31}$$

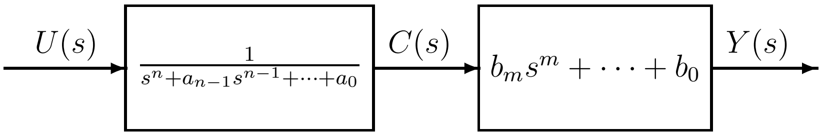

Figure 19: Decomposition of $G(s)$ to construct its state-space realization.

A realization of $G(s)$ will have the form $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, G(\infty))$, that is, $d = G(\infty)$.

To proceed, we assume for simplicity that $G(s) = G(s)_{\rm sp}$. We split our procedure of finding a triple $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ such that (31) holds into two steps. In the first step, we introduce an intermediate Laplace variable, $C(s)$, such that

$$\begin{aligned} \frac{Y(s)}{U(s)} &= \frac{Y(s)}{C(s)}\frac{C(s)}{U(s)} \\ &= \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}\left(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0\right). \end{aligned} \tag{32}$$

We illustrate the operation given by (32) in Figure 19. We first concern ourselves with the transfer function

$$\frac{Y(s)}{C(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}.$$

Performing cross-multiplication gives

$$\left(s^n + a_{n-1}s^{n-1} + \cdots + a_0\right) Y(s) = C(s).$$

Taking the inverse Laplace transform and assuming zero initial conditions yields

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = c.$$

We define the state variables:

$$\left.\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} = \dot{x}_1 \\ &\vdots \\ x_{n-1} &= y^{(n-2)} = \dot{x}_{n-2} \\ x_n &= y^{(n-1)} = \dot{x}_{n-1} \end{aligned}\right\} \tag{33}$$

Note that

$$\begin{aligned} \dot{x}_n &= y^{(n)} \\ &= -a_0 y - a_1\dot{y} - \cdots - a_{n-1}y^{(n-1)} + c \\ &= -a_0 x_1 - a_1 x_2 - \cdots - a_{n-1}x_n + c. \end{aligned} \tag{34}$$

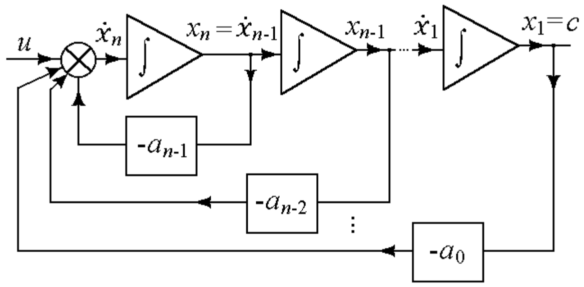

Figure 20: Implementation of (35).

We represent (33) and (34) in the matrix-vector format as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} c \tag{35}$$

The above set of n first-order ordinary differential equations can be simulated using a circuit shown in Figure 20.

We next concern ourselves with the transfer function

$$\frac{Y(s)}{C(s)} = b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0,$$

which we represent as

$$Y(s) = b_m s^m C(s) + b_{m-1} s^{m-1} C(s) + \cdots + b_1 s C(s) + b_0 C(s).$$

Taking the inverse Laplace transform, we obtain

$$y = b_m c^{(m)} + b_{m-1} c^{(m-1)} + \cdots + b_1 \dot{c} + b_0 c \tag{36}$$

Note that

$$\begin{aligned}
c &= x_1 \\
\dot{c} &= \dot{x}_1 = x_2 \\
&\vdots \\
c^{(m-1)} &= \dot{x}_{m-1} = x_m \\
c^{(m)} &= \dot{x}_m = x_{m+1}.
\end{aligned}$$

Hence, we can represent y in terms of the state variables,

$$y = b_m x_{m+1} + b_{m-1} x_m + \cdots + b_1 x_2 + b_0 x_1,$$

or, equivalently, as

$$y = \begin{bmatrix} b_0 & b_1 & \cdots & b_{m-1} & b_m & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}. \tag{37}$$

Combining (35) and (37), we obtain a realization of a strictly proper transfer function (28). This state-space realization has the form,

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \tag{38}$$

$$y = \begin{bmatrix} b_0 & \cdots & b_m & \cdots & 0 \end{bmatrix} \boldsymbol{x}. \tag{39}$$

We depict the above realization in Figure 21.

The above realization is only one of the infinitely possible realization of a given rational function $G(s)$. Using (39) and (39), we can easily obtain another realization of $G(s)$. First note that $G(s)$ can be viewed as an 1×1 matrix; therefore, its transpose equals itself, that is,

$$G(s)^\top = G(s).$$

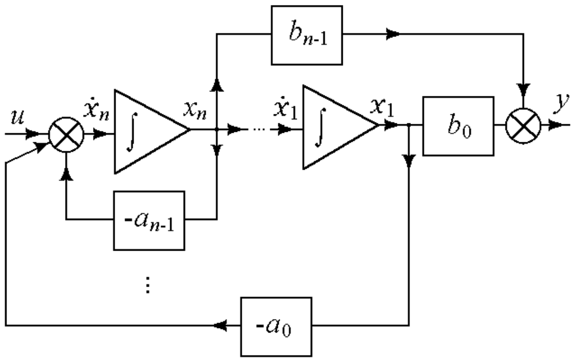

Figure 21: Implementation of (21).

Applying the transposition operation to both sides of (30) and using the property of the transpose of a product of matrices, we obtain

$$\begin{aligned}
G(s) &= G(s)^\top \\
&= \left(\boldsymbol{c}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{b}+d\right)^\top \\
&= \boldsymbol{b}^\top\left((s\boldsymbol{I}-\boldsymbol{A})^{-1}\right)^\top \boldsymbol{c}^\top + d^\top \\
&= \boldsymbol{b}^\top\left(s\boldsymbol{I}-\boldsymbol{A}^\top\right)^{-1} \boldsymbol{c}^\top + d.
\end{aligned}$$

Thus, we have another realization of $G(s)$ of the form

$$\dot{\tilde{\boldsymbol{x}}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \tilde{\boldsymbol{x}} + \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-2} \\ b_{n-1} \end{bmatrix} u \tag{40}$$

$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix} \tilde{\boldsymbol{x}} + G(\infty). \tag{41}$$

We now summarize and generalize the above discussion for multi-input multi-output transfer functions in the form of the following theorem.

Theorem 1 *A transfer function $\boldsymbol{G}(s)$ is realizable if and only if $\boldsymbol{G}(s)$ is a proper rational matrix.*

Proof ($\Longleftarrow$) We first prove that for a given a rational proper transfer function $\boldsymbol{G}(s) \in \mathbb{R}^{p\times m}(s)$ there is a realization. The proof is constructive. We begin by decomposing $\boldsymbol{G}(s)$ as

$$\boldsymbol{G}(s) = \boldsymbol{G}(s)_{\text{sp}} + \boldsymbol{G}(\infty),$$

where $\boldsymbol{G}(s)_{\text{sp}}$ denotes the strictly proper part of $\boldsymbol{G}(s)$. Let

$$d(s) = s^r + \alpha_{r-1}s^{r-1} + \cdots + \alpha_1 s + \alpha_0$$

be the least common denominator of all rational function components of $\boldsymbol{G}(s)_{\text{sp}}$. Note that the highest coefficient of $d(s)$ is unity. We can always make $\alpha_r = 1$ by dividing, if necessary, the corresponding numerator polynomials and $d(s)$ by α_r. We then express $\boldsymbol{G}(s)_{\text{sp}}$ as

$$\begin{aligned}\boldsymbol{G}(s)_{\text{sp}} &= \frac{\boldsymbol{N}(s)}{d(s)}\\ &= \frac{\boldsymbol{N}_{r-1}s^{r-1} + \cdots + \boldsymbol{N}_1 s + \boldsymbol{N}_0}{d(s)},\end{aligned}$$

where each $\boldsymbol{N}_i$ is an $p\times m$ constant matrix. We now show that the following state-space representation yields a realization of $\boldsymbol{G}(s)$:

$$\left.\begin{aligned}\dot{\boldsymbol{x}} &= \begin{bmatrix} \boldsymbol{O} & \boldsymbol{I}_m & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{I}_m & \cdots & \boldsymbol{O} & \boldsymbol{O}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{I}_m\\ -\alpha_0\boldsymbol{I}_m & -\alpha_1\boldsymbol{I}_m & -\alpha_2\boldsymbol{I}_m & \cdots & -\alpha_{r-2}\boldsymbol{I}_m & -\alpha_{r-1}\boldsymbol{I}_m \end{bmatrix}\boldsymbol{x} + \begin{bmatrix}\boldsymbol{O}\\ \boldsymbol{O}\\ \vdots\\ \boldsymbol{O}\\ \boldsymbol{I}_m\end{bmatrix}\boldsymbol{u}\\ \boldsymbol{y} &= \begin{bmatrix}\boldsymbol{N}_0 & \boldsymbol{N}_1 & \boldsymbol{N}_2 & \cdots & \boldsymbol{N}_{r-2} & \boldsymbol{N}_{r-1}\end{bmatrix}\boldsymbol{x} + \boldsymbol{G}(\infty)\boldsymbol{u},\end{aligned}\right\} \tag{42}$$

where $\boldsymbol{D} = \boldsymbol{G}(\infty)$. Note that the matrix $\boldsymbol{A}$ in the above realization is a block matrix; its blocks are $m\times m$ matrices.

To show that the above state-space representation is indeed a realization of $\boldsymbol{G}(s)$, we define

$$\boldsymbol{Z} = \begin{bmatrix}\boldsymbol{Z}_0\\ \boldsymbol{Z}_1\\ \vdots\\ \boldsymbol{Z}_{r-1}\end{bmatrix} = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B}, \tag{43}$$

where $\boldsymbol{Z}_i$ is an $m\times m$ sub-matrix of the $rm\times m$ block matrix $\boldsymbol{Z}$. Then the transfer function of (42) is

$$\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{G}(\infty) = \boldsymbol{N}_0\boldsymbol{Z}_0 + \boldsymbol{N}_1\boldsymbol{Z}_1 + \cdots + \boldsymbol{N}_{r-1}\boldsymbol{Z}_{r-1} + \boldsymbol{G}(\infty). \tag{44}$$

We pre-multiply both sides of (43) by $(s\boldsymbol{I}-\boldsymbol{A})$ and represent the result as

$$s\boldsymbol{Z}=\boldsymbol{A}\boldsymbol{Z}+\boldsymbol{B}.$$

Taking into account the structure of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ in (42), we obtain

$$s\boldsymbol{Z}_0=\boldsymbol{Z}_1,\quad s\boldsymbol{Z}_1=\boldsymbol{Z}_2,\quad s\boldsymbol{Z}_{r-2}=\boldsymbol{Z}_{r-1}, \tag{45}$$

and

$$s\boldsymbol{Z}_{r-1}=-\alpha_0\boldsymbol{Z}_0-\alpha_1\boldsymbol{Z}_1-\cdots-\alpha_{r-1}\boldsymbol{Z}_{r-1}+\boldsymbol{I}_m. \tag{46}$$

Note that $\boldsymbol{Z}_2=s\boldsymbol{Z}_1=s^2\boldsymbol{Z}_0$ and in general,

$$\boldsymbol{Z}_i=s^i\boldsymbol{Z}_0,\quad i=1,\ldots,r-1 \tag{47}$$

Substituting (47) into the above expression and performing simple manipulations gives

$$\left(s^r+\alpha_{r-1}s^{r-1}+\cdots+\alpha_1 s+\alpha_0\right)\boldsymbol{Z}_0=\boldsymbol{I}_m,$$

that is,

$$\boldsymbol{Z}_0=\frac{1}{d(s)}\boldsymbol{I}_m. \tag{48}$$

Using (45) or (47) and the above yields

$$\boldsymbol{Z}_1=\frac{s}{d(s)}\boldsymbol{I}_m,\quad \ldots\quad \boldsymbol{Z}_{r-1}=\frac{s^{r-1}}{d(s)}\boldsymbol{I}_m. \tag{49}$$

Substituting (48) and (49) into (44), we obtain

$$\begin{aligned}\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}+\boldsymbol{G}(\infty) &= \frac{1}{d(s)}\left(\boldsymbol{N}_0+\boldsymbol{N}_1 s+\cdots+\boldsymbol{N}_{r-1}s^{r-1}+\boldsymbol{G}(\infty)\right)\\ &= \boldsymbol{G}(s),\end{aligned} \tag{50}$$

which shows that (42) is a realization of $\boldsymbol{G}(s)$. The proof of this part of the theorem is complete.
($\Longrightarrow$) If the transfer function $\boldsymbol{G}(s)$ is realizable then there exists a quadruple $(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C},\boldsymbol{D})$ such that

$$\boldsymbol{G}(s)=\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}+\boldsymbol{D}.$$

Note that $\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}$ is a strictly proper rational matrix and because $\boldsymbol{D}$ is a constant matrix, $\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}+\boldsymbol{D}$ is a proper rational matrix, which completes the proof of this part of the theorem.

□

We can obtain another realization of $\boldsymbol{G}(s)$ from (42). We begin by constructing a realization for $\boldsymbol{G}(s)^\top$ using (42) to obtain

$$\boldsymbol{G}(s)^\top = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}. \tag{51}$$

We next transpose both sides of (51) to obtain

$$\begin{aligned}\left(\boldsymbol{G}(s)^\top\right)^\top = \boldsymbol{G}(s) & \\ &= \left(\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}\right)^\top \\ &= \boldsymbol{B}^\top\left(s\boldsymbol{I} - \boldsymbol{A}^\top\right)^{-1}\boldsymbol{C}^\top + \boldsymbol{D}^\top.\end{aligned}$$

The resulting realization has the form

$$\left.\begin{aligned}\dot{\tilde{\boldsymbol{x}}} &= \begin{bmatrix} \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{O} & -\alpha_0\boldsymbol{I}_p \\ \boldsymbol{I}_p & \boldsymbol{O} & \boldsymbol{I}_m & \cdots & \boldsymbol{O} & -\alpha_1\boldsymbol{I}_p \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{O} & -\alpha_{r-2}\boldsymbol{I}_p \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{I}_p & -\alpha_{r-1}\boldsymbol{I}_m \end{bmatrix}\tilde{\boldsymbol{x}} + \begin{bmatrix} \boldsymbol{N}_0^\top \\ \boldsymbol{N}_1^\top \\ \vdots \\ \boldsymbol{N}_{r-2}^\top \\ \boldsymbol{N}_{r-1}^\top \end{bmatrix}\boldsymbol{u} \\ \boldsymbol{y} &= \begin{bmatrix} \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{I}_p \end{bmatrix}\tilde{\boldsymbol{x}} + \boldsymbol{G}(\infty)\boldsymbol{u}.\end{aligned}\right\} \tag{52}$$

Note that although the realization (52) is based on the realization (42), their dimensions will be different if $p \neq m$.

5 Summary and Notes

Classification linear systems discussed by us in this chapter is shown in Table 1.

A landmark paper on mathematical description of linear dynamic systems from control point of view is by Kalman [5]. The first chapter of Sontag's book [10] is a nice, easy to read, and comprehensive introduction to the subject of mathematical control theory.

Mayr writes on page 109 in [8], "It is still widely believed that the steam-engine governor is the oldest feedback device, and that James Watt had not only invented but also patented it. While both errors are easily refuted, we are still not able to reconstruct the history of this invention in all desired completeness." Watt did not patent the governor. He did not invent it either. On page 112 of his book [8], Mayr adds the following: "But the application of the centrifugal pendulum in a system of speed regulation of steam engines was a new breakthrough for which the firm of Boulton & Watt, if not James Watt himself, clearly deserves the credit."

Table 1: Table summarizing linear system classification; adapted from Chen [2, p. 37].

System	Internal Description	External Description
Distributed, linear		$\boldsymbol{y}(t) = \int_{t_0}^{t} \boldsymbol{G}(t,\tau)\boldsymbol{u}(\tau)d\tau$
Lumped, linear	$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{B}(t)\boldsymbol{u}\\ \boldsymbol{y} &= \boldsymbol{C}(t)\boldsymbol{x} + \boldsymbol{D}(t)\boldsymbol{u}\end{aligned}$	$\boldsymbol{y}(t) = \int_{t_0}^{t} \boldsymbol{G}(t,\tau)\boldsymbol{u}(\tau)d\tau$
Distributed, linear, time-invariant		$\begin{aligned}\boldsymbol{y}(t) &= \int_{t_0}^{t} \boldsymbol{G}(t-\tau)\boldsymbol{u}(\tau)d\tau\\ \boldsymbol{Y}(s) &= \boldsymbol{G}(s)\boldsymbol{U}(s),\\ & \quad \boldsymbol{G}(s) \text{ irrational}\end{aligned}$
Lumped, linear, time-invariant	$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}\\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u}\end{aligned}$	$\begin{aligned}\boldsymbol{y}(t) &= \int_{t_0}^{t} \boldsymbol{G}(t-\tau)\boldsymbol{u}(\tau)d\tau\\ \boldsymbol{Y}(s) &= \boldsymbol{G}(s)\boldsymbol{U}(s),\\ & \quad \boldsymbol{G}(s) \text{ rational}\end{aligned}$

6 Exercises

Exercise 1 Draw a block diagram illustrating the operation of the Watt's governor shown in Figure 6.

Exercise 2 An example of a closed-loop feedback system is a toilet flushing device connected to a water tank. A toilet system is shown in Figure 22. The control objective is to maintain the level of water in the tank at a constant level. Draw a block diagram of the system.

Exercise 3 (based on Chen [2, p. 33])
Compute the transfer function of a discrete-time system whose impulse response is

$$g[k] = \begin{cases} 0 & \text{for} \quad k \le 0 \\ \frac{1}{k} & \text{for} \quad k > 1. \end{cases}$$

Is this system lumped or distributed?

Exercise 4 Suppose that $\boldsymbol{G}(s) \in \mathbb{R}^{p\times m}$ is a proper rational matrix such that

$$\boldsymbol{G}(s) = \boldsymbol{G}(s)_{\mathrm{sp}} + \boldsymbol{G}(\infty),$$

where

$$\frac{\boldsymbol{N}_{r-1}s^{r-1} + \cdots + \boldsymbol{N}_1 s + \boldsymbol{N}_0}{d(s)}$$

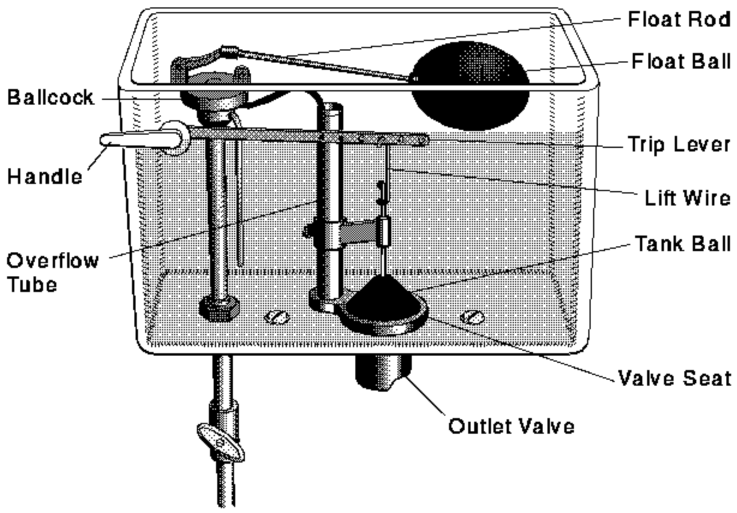

Figure 22: Toilet-flushing system for Exercise 2.

and $d(s) = s^r + \alpha_{r-1}s^{r-1} + \cdots + \alpha_1 s + \alpha_0$. Show that the following state-space representation is a realization of $\boldsymbol{G}(s)$:

$$\left.\begin{array}{rcl} \dot{\hat{\boldsymbol{x}}} & = & \begin{bmatrix} -\alpha_0\boldsymbol{I}_m & -\alpha_1\boldsymbol{I}_m & -\alpha_2\boldsymbol{I}_m & \cdots & -\alpha_{r-2}\boldsymbol{I}_m & -\alpha_{r-1}\boldsymbol{I}_m \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{I}_m & \boldsymbol{O} & \boldsymbol{O} & \cdots & \boldsymbol{I}_m & \boldsymbol{O} \end{bmatrix} \hat{\boldsymbol{x}} + \begin{bmatrix} \boldsymbol{I}_m \\ \boldsymbol{O} \\ \vdots \\ \boldsymbol{O} \\ \boldsymbol{O} \end{bmatrix} \boldsymbol{u} \\ \boldsymbol{y} & = & \begin{bmatrix} \boldsymbol{N}_{r-1} & \boldsymbol{N}_{r-2} & \boldsymbol{N}_{r-3} & \cdots & \boldsymbol{N}_1 & \boldsymbol{N}_0 \end{bmatrix} \hat{\boldsymbol{x}} + \boldsymbol{G}(\infty)\boldsymbol{u}, \end{array}\right\} \tag{53}$$

Exercise 5 For a proper rational matrix

$$\boldsymbol{G}(s) = \begin{bmatrix} \frac{s+1}{(s+2)^2} & \frac{1}{(2s+1)(s+2)} & \frac{3s-1}{s} \\ \frac{3}{s+2} & \frac{4s-10}{2s+1} & \frac{2s+1}{s^2+1} \end{bmatrix}, \tag{54}$$

(i) construct its two realizations;

(ii) use MATLAB's `tf2ss` command to find a realization of $\boldsymbol{G}(s)$.

Exercise 6 Compute the impulse response of a dynamic system whose transfer function is given by (54).

Exercise 7 Can you construct a state-space model of a linear system whose output is zero in response to any input?

Exercise 8 For the circuit shown in Figure 23 draw a block diagram with the current $i(t)$ as the system's input and the voltage $v(t)$ as the system's output. Then, find the transfer function, $V(s)/I(s)$.

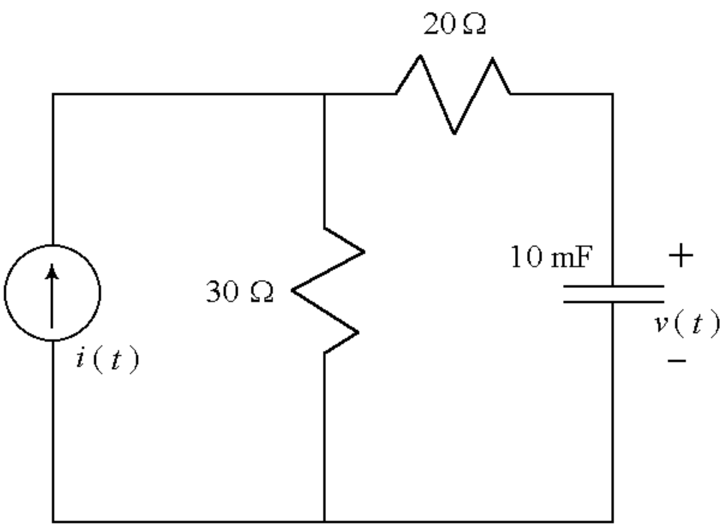

Figure 23: Circuit for Exercise 8.

Exercise 9 Find the transfer function $V_{out}(s)/V_{in}(s)$ of the circuit shown in Figure 9 that conatins an ideal operational amplifier.

Exercise 10 The purpose of this Exercise is to show some benefits that the state-space description can offer in the analysis of control systems. Begin by verifying that the transfer function $Y(s)/R(s)$ for both systems shown in Figure 25 and 26 is the same. Obtain the step response for this transfer function.

Next obtain state-space realizations of the systems shown in Figure 25 and 26. That is, construct state-space realizations of $G_c(s) = \frac{s-1}{s+2}$ and $G_p(s) = \frac{3}{s-1}$ and then construct the overall state-space model of the cascade systems. The state-space models should of of second-order. Verify that the transfer functions of the models are the same as in the part above. Implement the obtained state-space models in SIMULINK. Obtain plots of the

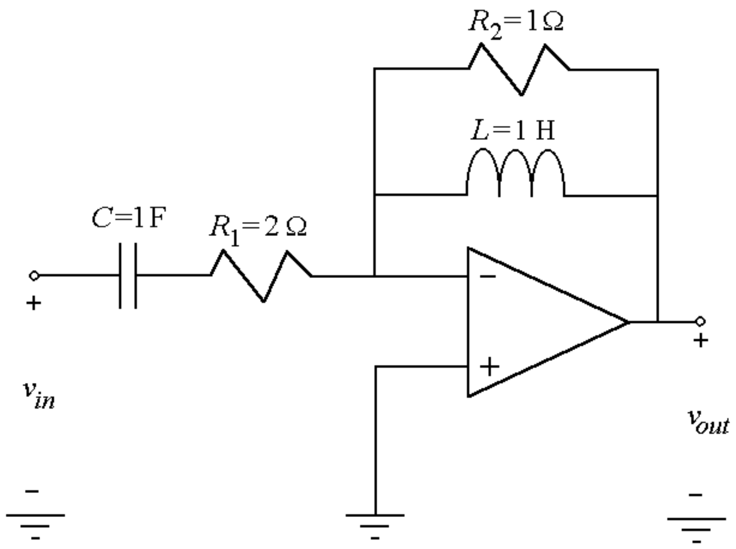

Figure 24: Circuit for Exercise 9.

states and the outputs in response to a unit step input. Perform your simulations for different initial conditions that you can impose on the integrators. Compare the external, that is, input-output system behavior against the internal behavior. Can you give an explanation for an unexpected behavior of the state variables even when the external behavior seems to be acceptable.

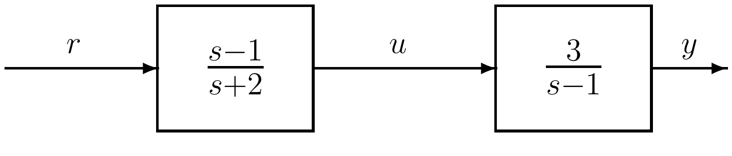

Figure 25: Cascade compensation for Exercise 10: compensator preceding the plant.

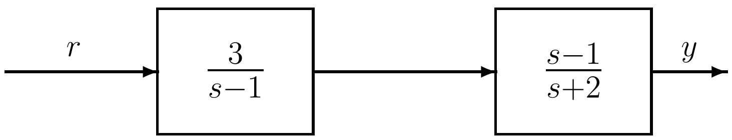

Figure 26: Compensator following the plant.

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