ECE 602 Fall 2011

A Glimpse at Linear Time-Varying (LTV) Systems

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1 Motivating Example

Of considerable importance in many applications are linear systems in which the elements of $\boldsymbol{A}$ and $\boldsymbol{B}$ are continuous functions of time. We present a linear time-varying model of a battery as a motivation for our discussion on linear time-varying systems. Our derivation follows that of Chiasson and Vairamohan [2, 3]. The simplest battery model is shown in Figure 1, which is a practical voltage source. This model consists of an ideal voltage source

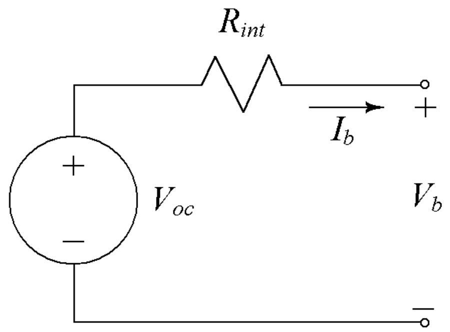

Figure 1: Practical voltage source as a simple battery model.

that produces voltage V_{oc}. We refer to V_{oc} as the open-circuit voltage. Thus the open-circuit voltage is obtained by measuring the voltage across the open-circuit, that is, when $I_b = 0$. The internal resistance of the practical voltage source is denoted R_{int}. The battery terminal voltage is labeled as V_b. This model, however, does not take into account the fact that the internal resistance is different under discharge and charge conditions. To account for this, the circuit of Figure 1 is modified by introducing two different internal resistances; R_c—the charging resistance and R_d—the discharging resistance. These resistances are connected to the diodes to model charging and discharging modes of the battery operation. The modified circuit is shown in Figure 2. This model, on the other hand, does not take into account

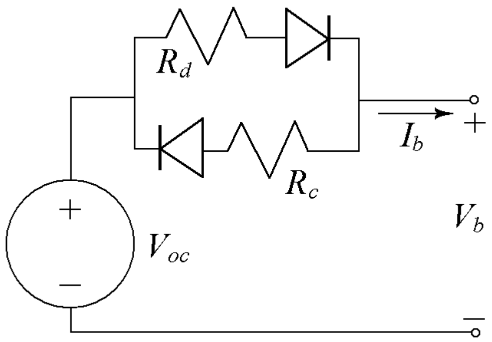

Figure 2: Battery model accounting for the different charging, R_c, and discharging, R_d, resistances.

the diffusion of the electrolytic through the battery. To model these effect, a polarization capacitor is added to model some electro-chemical reaction taking place in the battery. The resulting model is depicted in Figure 3.

In electric vehicles the state of charge (SOC) of the battery corresponds to a fuel gauge in a conventional, internal combustion, cars. To define the state of charge, consider a completely discharged battery. Let $I_b(t)$ be the charging current. Then, the charge delivered to the battery is

$$Q(t) = \int_{t_0}^{t} I_b(\tau)d\tau.$$

Let $Q_0 = \int_{t_0}^{\infty} I_b(\tau)d\tau$ be the total charge that battery can hold. Then the state of charge (SOC) of the battery is

$$\begin{aligned} \mathrm{SOC}(t) &= \frac{Q_0 - Q(t)}{Q_0} \times 100 \\ &= \frac{Q_0 - \int_{t_0}^{t} I_b(\tau)d\tau}{Q_0} \times 100. \end{aligned} \tag{1}$$

Direct application of (1) to estimate the battery SOC is not accurate. An alternative approach is to use an estimate of the open circuit voltage, V_{oc} of the battery. Then, the SOC can be evaluated from the relation,

$$\mathrm{SOC}(t) = \frac{V_{oc}(t) - a_0}{a_1}, \tag{2}$$

where a_0 is the battery terminal voltage when $\mathrm{SOC}(t) = 0\%$ and a_1 is obtained using a_0 and V_{oc} at $\mathrm{SOC}(t) = 100\%$. Thus estimating the SOC is equivalent to estimating the battery

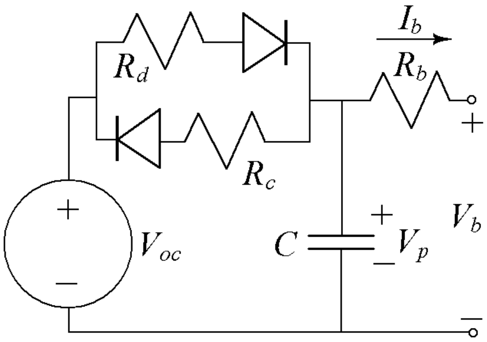

Figure 3: Battery model accounting for the different charging, R_c, and discharging, R_d, resistances and polarization capacitance, C.

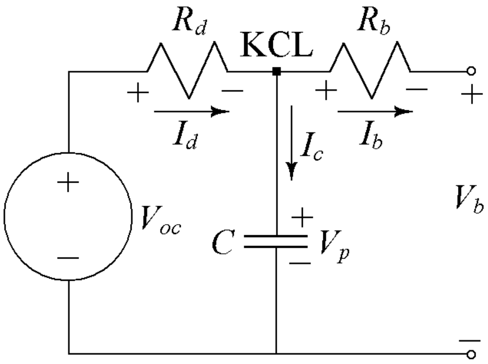

Figure 4: A battery model in the discharging mode of operation.

open-circuit voltage $V_{oc}(t)$. Now, in order to measure V_{oc}, the battery must be disconnected from its load, which is not possible during the electric vehicle operation that uses the battery whose SOC we wish to estimate. The approach proposed by Chiasson and Vairamohan [2, 3] is to construct the open-circuit voltage estimator. This estimator construction is based on the battery model that we develop next.

We analyze the condition when the battery is in the discharging mode, that is, when $I_b \geq 0$. The charging mode is analyzed in a similar way. The battery model in the discharging mode is depicted in Figure 4. To proceed, we apply Kirchhoff's current law (KCL) at the node between the resistances R_d and R_b to obtain

$$I_c + I_b - I_d = 0,$$

that is,

$$I_c = I_d - I_b.$$

Applying Ohm's law gives

$$I_d = \frac{V_{oc} - V_p}{R_d} \quad \text{and} \quad I_b = \frac{V_p - V_b}{R_b}. \tag{3}$$

We also have

$$I_c = C\dot{V}_p.$$

Combining the above equations yields

$$\begin{aligned} C\dot{V}_p &= I_d - I_b \\ &= \frac{V_{oc} - V_p}{R_d} - I_b \\ &= -\frac{V_p}{R_d} + \frac{V_{oc}}{R_d} - I_b. \end{aligned} \tag{4}$$

Dividing both sides by C, we obtain

$$\dot{V}_p = -\frac{V_p}{R_d C} + \frac{V_{oc}}{R_d C} - \frac{1}{C} I_b. \tag{5}$$

The output equation is obtained from the second of the equations in (3),

$$V_b = V_p - R_b I_b \tag{6}$$

We define the state variables,

$$x_1 = V_p, \quad x_2 = \frac{1}{R_d C}, \quad x_3 = \frac{V_{oc}}{R_d C}, \quad x_4 = \frac{1}{C}, \quad x_5 = R_b,$$

to obtain a nonlinear time-varying state-space model,

$$\begin{aligned} \dot{x}_1 &= -x_1 x_2 + x_3 - I_b(t) x_4 \\ \dot{x}_2 &= 0 \\ \dot{x}_3 &= 0 \\ \dot{x}_4 &= 0 \\ \dot{x}_5 &= 0 \\ y &= V_b = x_1 - I_b(t) x_5. \end{aligned}$$

The initial state vector is unknown to us. The state variable x_3 depends on V_{oc}. The state variables, x_2, x_4 and x_5 are constant. Our objective is to compute the open-circuit voltage V_{oc}, which is a constant. We note that

$$x_3 = \frac{V_{oc}}{R_d C} = V_{oc} x_2.$$

Therefore

$$V_{oc} = \frac{x_3}{x_2} = \frac{x_{30}}{x_{20}}$$

because V_{oc} is constant. Hence, we can determine V_{oc} if we can estimate the initial values of the state variables x_{20} and x_{30}. We proceed by redefining the state variables,

$$z_1 = x_1, \quad z_2 = x_3, \quad z_3 = x_4, \quad z_4 = x_5.$$

Using the above we write,

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} -x_{20} & 1 & -I_b(t) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \tag{7}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & -I_b(t) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}. \tag{8}$$

The above is a linear LTV systemn because the current $I_b(t)$ is time-varying. In summary, we have a homogenous, or free, or unforced, LTV system model

$$\begin{aligned} \dot{\boldsymbol{z}} &= \boldsymbol{A}(t, x_{20})\boldsymbol{z}, \quad \boldsymbol{z}(t_0) \\ y &= \boldsymbol{c}(t)\boldsymbol{z}. \end{aligned}$$

Our objective is to estimate $\boldsymbol{z}(t_0)$ from knowledge of the output $y(t)$ and the current $I_b(t)$ on some time interval $[t_0,\ t_f]$.

We now outline a solution to the problem of estimating $\boldsymbol{z}(t_0)$. The first step is to compute the state-transition matrix, $\boldsymbol{\Phi}(t, t_0)$, which relates the state at time t to the initial state as

$$\boldsymbol{z}(t) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{z}(t_0).$$

We then use the output equation to obtain

$$\begin{aligned} y(t) &= \boldsymbol{c}\boldsymbol{z}(t) \\ &= \boldsymbol{c}\boldsymbol{\Phi}(t, t_0)\boldsymbol{z}(t_0). \end{aligned}$$

Premultiply both sides of the above by $\boldsymbol{\Phi}(t,t_0)^\top \boldsymbol{c}^\top$ and then integrate both sides to get

$$\int_{t_0}^{t_f} \boldsymbol{\Phi}(t,t_0)^\top \boldsymbol{c}^\top y(t)dt = \int_{t_0}^{t_f} \boldsymbol{\Phi}(t,t_0)^\top \boldsymbol{c}^\top \boldsymbol{c}\boldsymbol{\Phi}(t,t_0)dt\ \boldsymbol{z}(t_0).$$

Let

$$\boldsymbol{W}_o(t_0,t_f) = \int_{t_0}^{t_f} \boldsymbol{\Phi}(t,t_0)^\top \boldsymbol{c}^\top \boldsymbol{c}\boldsymbol{\Phi}(t,t_0)dt,$$

which as in the LTI case, we refer to as the observability gramian. If $\boldsymbol{W}_o(t_0,t_f)$ is invertible then

$$\boldsymbol{z}(t_0) = \boldsymbol{W}_o(t_0,t_f)^{-1} \int_{t_0}^{t_f} \boldsymbol{\Phi}(t,t_0)^\top \boldsymbol{c}^\top y(t)dt.$$

Having obtained $\boldsymbol{z}(t_0)$ we can compute V_{oc} and then the SOC. We refer to Chiasson and Vairamohan [2, 3] for a complete solution to this problem; see also Pang et al. [6], and McIntyre et al. [5] for alternative solutions.

To be able to solve this problem, we need to acquaint ourselves with the state transition matrix for linear LTV systems. We start with the LTV uncontrolled system.

2 Solving the Uncontrolled LTV System

In general, it is not possible to give explicit expressions for solutions of time varying linear systems. We obtain some general properties that for some cases will be used to obtain explicit solutions. We first consider the uncontrolled system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0. \tag{9}$$

Theorem 1 *If $\boldsymbol{A}(t)$ is continuous for $t \geq t_0$ then the uncontrolled system modeled by* (9) *has a unique solution for $t \geq t_0$ given by*

$$\boldsymbol{x}(t) = \boldsymbol{X}(t)\boldsymbol{x}_0,$$

where $\boldsymbol{X}(t)$ is the unique $n \times n$ matrix satisfying

$$\frac{d\boldsymbol{X}}{dt} = \boldsymbol{A}(t)\boldsymbol{X}, \quad \boldsymbol{X}(t_0) = \boldsymbol{I}_n.$$

Proof See Rugh [7, Chp. 3] or Antsaklis and Michel [1, Section 1.8]. See also Exercise 2.

□

Note that it follows from the above theorem that for every initial state $\boldsymbol{x}^{(i)}(t_0)$, there exists a unique solution $\boldsymbol{x}^{(i)}(t)$, $i = 1, 2, \ldots, n$. We arrange these n solutions as

$$\boldsymbol{X}(t) = \left[\begin{array}{cccc} \boldsymbol{x}^{(1)}(t) & \boldsymbol{x}^{(2)}(t) & \cdots & \boldsymbol{x}^{(n)}(t) \end{array}\right].$$

Because every $\boldsymbol{x}^{(i)}(t)$ satisfies (9), we have

$$\dot{\boldsymbol{X}}(t) = \boldsymbol{A}(t)\boldsymbol{X}(t).$$

If $\boldsymbol{X}(t_0)$ is nonsingular, then $\boldsymbol{X}(t)$ is called a *fundamental matrix* of $\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}$. Because the initial states are linearly independent, the fundamental matrix is nonsingular—see Exercise 1. The fundamental matrix is not unique because we can choose initial states arbitrarily.

We now analyze a scalar LTV case,

$$\dot{x} = a(t)x, \quad x(0) = x_0. \tag{10}$$

Recall that when $a(t) = a$ =constant, then the solution has the form

$$x(t) = e^{at}x(0) = e^{\int_0^t a d\tau}x(0).$$

The above formula works for the scalar LTV case, that is,

$$x(t) = e^{\int_0^t a(\tau)d\tau}x(0). \tag{11}$$

It is easy to verify that the above is the solution of (10) because

$$\begin{aligned} \frac{d}{dt}e^{\int_0^t a(\tau)d\tau} &= e^{\int_0^t a(\tau)d\tau}a(t) \\ &= a(t)e^{\int_0^t a(\tau)d\tau}. \end{aligned}$$

That is, $a(t)$ and $\int_0^t a(\tau)d\tau$ commute.

Example 1 We use the solution formula (11) to solve the equation,

$$\dot{x} = (\sin t)x, \quad x(0) = x_0.$$

We have

$$\begin{aligned} x(t) &= e^{\int_0^t a(\tau)d\tau}x_0 \\ &= x_0 e^{\int_0^t \sin(\tau)d\tau} \\ &= x_0\, e^{-\cos t}\Big|_0^t \\ &= e^{-\cos t+1}x_0 \\ &= e \times e^{-\cos t}x_0. \end{aligned}$$

The solution formula (11) does not generalize to the multi-dimensional case because, in general,

$$\frac{d}{dt}e^{\int_0^t \boldsymbol{A}(\tau)d\tau} \neq \boldsymbol{A}(t)e^{\int_0^t \boldsymbol{A}(\tau)d\tau}.$$

If, however, $\boldsymbol{A}(t)$ and $\int_0^t \boldsymbol{A}(\tau)d\tau$ commute, then the solution to (9) can be written as

$$\boldsymbol{x}(t) = e^{\int_0^t \boldsymbol{A}(\tau)d\tau}\boldsymbol{x}(0).$$

We note that $\boldsymbol{A}(t)$ and $\int_0^t \boldsymbol{A}(\tau)d\tau$ commute if

$$\boldsymbol{A}(t)\boldsymbol{A}(s) = \boldsymbol{A}(s)\boldsymbol{A}(t).$$

We now analyze a general case when t_0 is not necessarily equal to 0. If $\boldsymbol{A}(t) = \boldsymbol{A}$ in (9), then the solution is

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0).$$

We re-write the above as

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}e^{-\boldsymbol{A}t_0}\boldsymbol{x}(t_0).$$

Let $\boldsymbol{X}(t) = e^{\boldsymbol{A}t}$, then

$$\boldsymbol{X}(t_0)^{-1} = e^{-\boldsymbol{A}t_0}$$

and we represent the solution as

$$\begin{aligned} \boldsymbol{x}(t) &= \boldsymbol{X}(t)\boldsymbol{X}(t_0)^{-1}\boldsymbol{x}(t_0) \\ &= \boldsymbol{\Phi}(t,t_0)\boldsymbol{x}(t_0), \end{aligned} \tag{12}$$

where $\boldsymbol{\Phi}(t,t_0)$ is the state transition matrix.

Definition 1 *Let $\boldsymbol{X}(t)$ be a fundamental matrix of* (9)*. Then*

$$\boldsymbol{\Phi}(t,t_0) = \boldsymbol{X}(t)\boldsymbol{X}(t_0)^{-1}$$

is the state transition matrix of (9)*. The state transition matrix is the unique solution of*

$$\frac{\partial}{\partial t}\boldsymbol{\Phi}(t,t_0) = \boldsymbol{A}(t)\boldsymbol{\Phi}(t,t_0), \quad \boldsymbol{\Phi}(t_0,t_0) = \boldsymbol{I}_n.$$

As in the LTI case, the state transition matrix in the LTV case has the following properties (see Exercise 5):

$$\boldsymbol{\Phi}(\tau,\tau) = \boldsymbol{I}_n \tag{13}$$

$$\boldsymbol{\Phi}(t,\tau)^{-1} = \boldsymbol{\Phi}(\tau,t) \tag{14}$$

$$\boldsymbol{\Phi}(t,t_0) = \boldsymbol{\Phi}(t,t_1)\boldsymbol{\Phi}(t_1,t_0) \tag{15}$$

The third of the above properties is referred to as the *semigroup property.*

Example 2 The state transition matrix of

$$\dot{x} = (\sin t)x, \quad x(t_0) = x_0,$$

has the form

$$\begin{aligned}\Phi(t, t_0) &= X(t)X(t_0)^{-1}\\ &= \frac{e^{-\cos t+1}}{e^{-\cos t_0+1}}.\end{aligned}$$

Example 3 We construct the state transition matrix for the LTV system,

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & -t \\ 0 & 0 \end{bmatrix}\boldsymbol{x}, \quad \boldsymbol{x}(0) = \boldsymbol{x}(0).$$

First, we represent the above system as

$$\dot{x}_1 = -tx_2, \quad \text{and} \quad \dot{x}_2 = 0.$$

We can easily obtain the solution to the second of the above equations,

$$x_2(t) = x_2(0).$$

With the above we can obtain the solution to $\dot{x}_1 = -tx_2$ by integrating its both sides,

$$\begin{aligned}x_1(t) &= x_1(0) - \int_0^t \tau x_2(0)d\tau\\ &= x_1(0) - \frac{t^2 x_2(0)}{2}.\end{aligned}$$

We construct a fundamental matrix now. Let

$$\boldsymbol{x}^{(1)}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x}^{(2)}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then the fundamental matrix corresponding to the above selection of the initial conditions is

$$\begin{aligned}\boldsymbol{X}(t) &= \begin{bmatrix} \boldsymbol{x}^{(1)}(t) & \boldsymbol{x}^{(2)}(t) \end{bmatrix}\\ &= \begin{bmatrix} 1 - \frac{t^2}{2} & -\frac{t^2}{2} \\ 1 & 1 \end{bmatrix}.\end{aligned}$$

If we select the initial conditions

$$\boldsymbol{x}^{(1)}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x}^{(2)}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

then we obtain a different fundamental matrix,

$$\boldsymbol{X}(t) = \begin{bmatrix} 1 & -\frac{t^2}{2} \\ 0 & 1 \end{bmatrix}.$$

The state transition matrix, though, is unique and has the form

$$\begin{aligned}
\boldsymbol{\Phi}(t, t_0) &= \boldsymbol{X}(t)\boldsymbol{X}(t_0)^{-1} \\
&= \begin{bmatrix} 1 - \frac{t^2}{2} & -\frac{t^2}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 - \frac{t_0^2}{2} & -\frac{t_0^2}{2} \\ 1 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 - \frac{t^2}{2} & -\frac{t^2}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{t_0^2}{2} \\ -1 & 1 - \frac{t_0^2}{2} \end{bmatrix} \\
&= \begin{bmatrix} 1 & \frac{1}{2}(t_0^2 - t^2) \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

Example 4 We construct the state-transition matrix for the LTV system model found in Luenberger [4, p. 115],

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 \\ t & \frac{1}{t} \end{bmatrix} \boldsymbol{x}, \quad t \geq 1.$$

Then, for the given $\boldsymbol{x}(2) = \begin{bmatrix} 2 & -1 \end{bmatrix}^\top$, we compute $\boldsymbol{x}(1)$.

First, we solve for x_1 to obtain,

$$x_1(t) = x_1(1), \quad t \geq 1$$

Substituting the above into the equation for $\dot{x}_2$ gives

$$\dot{x}_2 = t x_1(t) + \frac{x_2(t)}{t}.$$

Next, we make the substitution

$$z(t) = \frac{x_2(t)}{t}.$$

Computing the derivative of $z(t)$ yields

$$\begin{aligned}\dot{z} &= \frac{\dot{x}_2 t - x_2}{t^2} \\ &= \frac{\dot{x}_2}{t} - \frac{x_2}{t^2}.\end{aligned}$$

Multiplying both sides of the above, we obtain

$$\begin{aligned}t\dot{z} &= \dot{x}_2 - \frac{x_2}{t^2} \\ &= \dot{x}_2 - z.\end{aligned}$$

We substitute into the above $\dot{x}_2 = tx_1(1) + z$ to obtain

$$t\dot{z} + z = tx_1(1) + z,$$

that is,

$$\dot{z} = x_1(1).$$

Integrating the above gives

$$z(t) = z(1) + x_1(1)(t-1).$$

Therefore,

$$\begin{aligned}z &= \frac{x_2}{t} \\ &= z(1) + x_1(1)(t-1),\end{aligned}$$

and multiplying both sides by t, we obtain

$$x_2(t) = tx_2(1) + tx_1(1)(t-1).$$

We are now ready to generate a fundamental matrix. Let

$$\boldsymbol{x}^{(1)}(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x}^{(2)}(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

then we have

$$\boldsymbol{x}^{(1)}(t) = \begin{bmatrix} 1 \\ t^2 - t \end{bmatrix} \quad \text{and} \quad \boldsymbol{x}^{(2)}(t) = \begin{bmatrix} 0 \\ t \end{bmatrix}.$$

The fundamental matrix corresponding to the above initial conditions is

$$\boldsymbol{X}(t) = \begin{bmatrix} \boldsymbol{x}^{(1)}(t) & \boldsymbol{x}^{(2)}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t^2 - t & t \end{bmatrix}.$$

The state transition matrix is

$$\begin{aligned}
\boldsymbol{\Phi}(t,\tau) &= \boldsymbol{X}(t)\boldsymbol{X}(\tau)^{-1} \\
&= \begin{bmatrix} 1 & 0 \\ t^2-t & t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \tau^2-\tau & \tau \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 0 \\ t^2-t & t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\tau+1 & \frac{1}{\tau} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ t(t-\tau) & \frac{t}{\tau} \end{bmatrix}.
\end{aligned}$$

Note that

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t,\tau)\boldsymbol{x}(\tau).$$

Thus, for example, if we are given $\boldsymbol{x}(2) = \begin{bmatrix} 2 & -1 \end{bmatrix}^\top$ and we wish to find $\boldsymbol{x}(1)$, we have,

$$\begin{aligned}
\boldsymbol{x}(1) &= \boldsymbol{\Phi}(1,2)\boldsymbol{x}(2) \\
&= \begin{bmatrix} 1 & 0 \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ -\frac{5}{2} \end{bmatrix}.
\end{aligned}$$

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3 Solving the LTV Controlled System

We now discuss the problem of obtaining the solution to the LTV controlled system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{B}(t)\boldsymbol{u}, \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0. \tag{16}$$

We claim that, by analogy with the LTI controlled system, the solution to (16) is

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t,t_0)\boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{\Phi}(t,\tau)\boldsymbol{B}(\tau)\boldsymbol{u}(\tau)d\tau. \tag{17}$$

We can prove our claim by verification utilizing Leibniz's rule

$$\frac{\partial}{\partial t}\int_{t_0}^{t} f(t,\tau)d\tau = \int_{t_0}^{t}\left(\frac{\partial}{\partial t}f(t,\tau)\right)d\tau + f(t,\tau)|_{\tau=t} \tag{18}$$

Example 5 (R. A. DeCarlo) The LTV state model,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \begin{bmatrix} \pi e^{-\alpha(t)^2} \\ 0 \end{bmatrix} u(t),$$

has associated state transition matrix

$$\boldsymbol{\Phi}(t,\tau) = e^{\alpha(\tau)^2-\alpha(t)^2} \begin{bmatrix} 1 & 0 \\ \cos(\pi t) - \cos(\pi\tau) & 1 \end{bmatrix},$$

where $\alpha(\cdot)$ is a continuous function.

1. Compute $\boldsymbol{A}(t)$ using the given state transition matrix.
2. Find the zero-state response when the input $u(t)$ is the unit step.

We first compute $\boldsymbol{A}(t)$ using the given state transition matrix. We use the state transition matrix properties to accomplish this. Recall that the state transition matrix satisfies the equation

$$\frac{\partial}{\partial t}\boldsymbol{\Phi}(t,\tau) = \boldsymbol{A}(t)\boldsymbol{\Phi}(t,\tau), \quad \boldsymbol{\Phi}(\tau,\tau) = \boldsymbol{I}_n.$$

Note that, we can also write.

$$\boldsymbol{\Phi}(t,\tau)\big|_{\tau=t} = \boldsymbol{I}_n$$

Taking this into account, we have

$$\boldsymbol{A}(t) = \left.\frac{\partial}{\partial t}\boldsymbol{\Phi}(t,\tau)\right|_{\tau=t}.$$

We first evaluate

$$\begin{aligned} \left.\frac{\partial}{\partial t} e^{\alpha(\tau)^2-\alpha(t)^2}\right|_{\tau=t} &= \left. e^{\alpha(\tau)^2-\alpha(t)^2}(-2\alpha(t)\dot{\alpha}(t))\right|_{\tau=t} \\ &= -2\alpha(t)\dot{\alpha}(t). \end{aligned}$$

We next evaluate the derivative

$$\begin{aligned} \left.\frac{\partial}{\partial t}\left(e^{\alpha(\tau)^2-\alpha(t)^2}\left(\cos(\pi t)-\cos(\pi\tau)\right)\right)\right|_{\tau=t} &= -2\alpha(t)\dot{\alpha}(t)(\cos(\pi t)-\cos(\pi t)) - \pi\sin(\pi t) \\ &= -\pi\sin(\pi t). \end{aligned}$$

Hence

$$\begin{aligned} \boldsymbol{A}(t) &= \left.\frac{\partial}{\partial t}\boldsymbol{\Phi}(t,\tau)\right|_{\tau=t} \\ &= \begin{bmatrix} -2\alpha(t)\dot{\alpha}(t) & 0 \\ -\pi\sin(\pi t) & -2\alpha(t)\dot{\alpha}(t) \end{bmatrix}. \end{aligned}$$

We now compute the zero-state response. The formula for the zero-state response is obtained from (17) by setting $\boldsymbol{x}_0 = \boldsymbol{0}$,

$$\boldsymbol{x}(t) = \int_{t_0}^{t} \boldsymbol{\Phi}(t,\tau)\boldsymbol{B}(\tau)\boldsymbol{u}(\tau)d\tau.$$

We first compute the integrand

$$\begin{aligned}
\boldsymbol{\Phi}(t,\tau)\boldsymbol{B}(\tau) &= e^{\alpha(\tau)^2-\alpha(t)^2}\begin{bmatrix} 1 & 0 \\ \cos(\pi t)-\cos(\pi\tau) & 1 \end{bmatrix}\begin{bmatrix} \pi e^{-\alpha(\tau)^2} \\ 0 \end{bmatrix} \\
&= \pi e^{-\alpha(t)^2}\begin{bmatrix} 1 \\ \cos(\pi t)-\cos(\pi\tau) \end{bmatrix}.
\end{aligned}$$

Therefore, the zero-state response when $u(t) = 1(t)$ is

$$\begin{aligned}
\boldsymbol{x}(t) &= \int_{t_0}^{t} \boldsymbol{\Phi}(t,\tau)\boldsymbol{B}(\tau)\boldsymbol{u}(\tau)d\tau \\
&= \pi e^{-\alpha(t)^2}\int_{0}^{t}\begin{bmatrix} 1 \\ \cos(\pi t)-\cos(\pi\tau) \end{bmatrix} d\tau \\
&= \pi e^{-\alpha(t)^2}\begin{bmatrix} t \\ t\cos(\pi t)-\frac{\sin(\pi\tau)}{\pi} \end{bmatrix}.
\end{aligned}$$

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4 Notes

5 Exercises

Exercise 1 Prove that a fundamental matrix is nonsingular for any t.

Exercise 2 Use the Contraction Map Theorem **??** to prove that (9) has a unique solution going through $\boldsymbol{x}_0$ at $t = t_0$, where t_0 is an element of a finite time interval $[a,\ b]$. Start with the Lipschitz condition.

Exercise 3 **(a)** Find the state-transition matrix $\boldsymbol{\Phi}(t,\tau)$ associated with a homogenous LTI system with

$$\boldsymbol{A} = \boldsymbol{T}\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}\boldsymbol{T}^{-1},$$

where

$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}.$$

(b) Find the state-transition matrix $\boldsymbol{\Phi}(t, \tau)$ associated with a homogenous LTV system with

$$\boldsymbol{A}(t) = \boldsymbol{T} \begin{bmatrix} -2 & 0 \\ 0 & \cos t \end{bmatrix} \boldsymbol{T}^{-1},$$

where

$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}.$$

Exercise 4 Compute the state transition matrix $\boldsymbol{\Phi}(t, \tau)$ for the linear time-varying (LTV) system model,

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & 0 \\ 1 & \frac{1}{t} \end{bmatrix} \boldsymbol{x}(t)$$

defined for $t \geq 1$. Then, for the given $\boldsymbol{x}(2) = \begin{bmatrix} 0 & 4 \end{bmatrix}^\top$, compute $\boldsymbol{x}(1)$.

Exercise 5 Use the definition of the state transition matrix to prove its properties (13), (14), and (15).

Exercise 6 **(a)** The value of the state-transition matrix for a homogeneous linear system is

$$\boldsymbol{\Phi}(1, 3) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Find $\boldsymbol{x}(3)$ when

$$\boldsymbol{x}(1) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

(b) The values of the state-transition matrix for a homogeneous linear system is

$$\boldsymbol{\Phi}(1, 3) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Phi}(2, 3) = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

If

$$\boldsymbol{x}(1) = \begin{bmatrix} 2 \\ -3 \end{bmatrix},$$

find $\boldsymbol{x}(2)$.

Exercise 7 Derive Leibniz's rule given by (18).

Exercise 8 **(a)** The state-transition matrix of a homogenous LTV system is

$$\boldsymbol{\Phi}(t,\tau) = \begin{bmatrix} e^{\cos t - \cos\tau} & 0 \\ 0 & e^{-\sin t + \sin\tau} \end{bmatrix}.$$

Find the associated $\boldsymbol{A}(t)$-matrix.

(b) A fundamental matrix associated with a homogenous continuous-time LTV state model has the form,

$$\boldsymbol{X}(t) = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t^2 - t & t \end{bmatrix}.$$

Compute the state-transition matrix $\boldsymbol{\Phi}(t,\tau)$.

Exercise 9 (J. Hu) For the linear time-invariant (LTV) system model,

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} -e^{-t} & \frac{1}{t+1} \\ 0 & -e^{-t} \end{bmatrix} \boldsymbol{x}(t),$$

- Find the state transition matrix $\boldsymbol{\Phi}(t,\tau)$,
- Is the system asymptotically stable?
- Let $\boldsymbol{x}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$. Will the system trajectory $\boldsymbol{x}(t)$ stay bounded as $t \to \infty$?

Exercise 10 (R. Balakrishnan) For the linear time-varying (LTV) system model,

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 0 & -2t \\ 0 & -t \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ t \end{bmatrix} u(t),$$

- Find the state transition matrix $\boldsymbol{\Phi}(t,\tau)$;
- Let $\boldsymbol{x}(1) = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ and $u(t) = 0$ for $t \geq 0$. Find $\boldsymbol{x}(0)$;
- Let $\boldsymbol{x}(0) = \boldsymbol{0}$ and $u(t) = 1$ for $t \geq 0$. Find $\boldsymbol{x}(t)$ for $t \geq 0$.

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