ECE 382 Fall 2018

The Internal Model Principle[1]

In this handout we acquaint ourselves with the concept of internal models. "The concept of internal models plays a crucial role in regulator problems. The internal model principle can intuitively be expressed as: 'Any good regulator must create a model of the dynamic structure of the environment in the closed loop system' " [1, page 333].

To proceed, we consider a block diagram of a closed-loop system shown in Figure 1. Using basic rules of the block diagram algebra yields

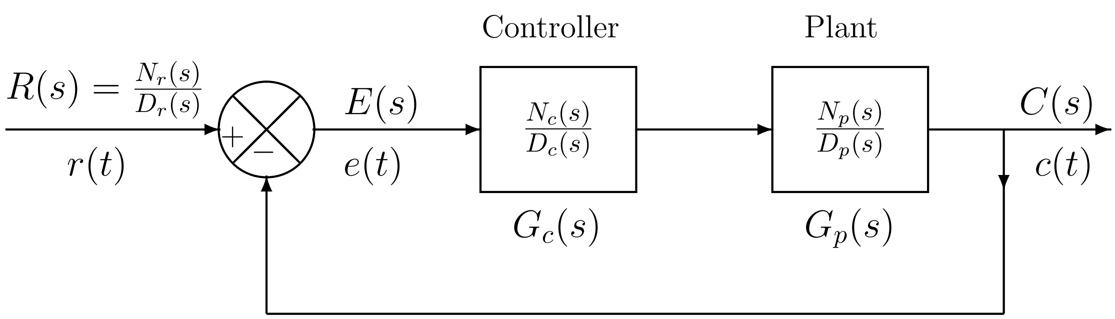

Figure 1: Block diagram of a closed-loop control system.

$$\begin{aligned} E(s) &= R(s) - G_p(s)G_c(s)E(s) \\ &= \frac{1}{1+G_p(s)G_c(s)}R(s). \end{aligned}$$

Performing simple manipulations, we obtain

$$\boxed{E(s) = \frac{D_p(s)D_c(s)}{D_p(s)D_c(s)+N_p(s)N_c(s)}\frac{N_r(s)}{D_r(s)}} \tag{1}$$

Our objective is to design a controller $G_c(s) = \frac{N_c(s)}{D_c(s)}$ so that

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty}(r(t)-c(t)) = 0,$$

[1] handout prepared by S. H. Żak

where $e(t) = \mathcal{L}^{-1}(E(s))$.

Suppose that the poles of the Laplace transform of the reference signal $r(t)$ are in the right half plane, that is, they belong to the set $\{s : \Re(s) \geq 0\}$. The polynomial

$$P_c(s) = D_p(s)D_c(s) + N_p(s)N_c(s)$$

is the closed-loop characteristic polynomial (CLCP) of the closed-loop system depicted in Figure 1 and the polynomial zeros are the closed-loop poles. Using (1), we can easily prove the following result that we call the Internal Model Principle (IMP).

Theorem 1 (Internal Model Principle) *In the configuration depicted in Figure 1, where the poles of $R(s)$ are in the right half plane,*

$$\lim_{t\to\infty} e(t) = 0$$

if and only if

1. *the closed-loop poles are in the open left-half plane;*
2. *$D_r(s)$ is a factor of the open-loop characteristic polynomial $D_p(s)D_c(s)$, that is, there is a polynomial, say $Q(s)$, such that $D_p(s)D_c(s) = Q(s)D_r(s)$.*

The second condition of the IMP means that the tracking controller must be chosen in such way that the open-loop transfer function, $G_p(s)G_c(s)$, contains a model of the reference signal to be tracked. If none of the pole of $R(s)$ is a pole of the plant's transfer function, $G_p(s)$, then we can restate the IMP as follows:

Any good tracking controller must stabilize the closed-loop system and must contain a model of the reference signal.

Let us now solve two numerical examples to get better insight into the IMP.

Example 1 For the closed-loop system shown in Figure 2, our objective is to construct a transfer function $G_c(s)$ such that $\lim_{t\to\infty} e(t) = 0$. The Laplace transform of the error is

$$E(s) = \frac{1}{1 + \frac{1}{s+2}\frac{N_c(s)}{D_c(s)}}\frac{1}{s} = \frac{(s+2)D_c(s)}{(s+2)D_c(s) + N_c(s)}\frac{1}{s}.$$

Let $N_c(s) = 1$ and $D_c(s) = s$, that is, let

$$G_c(s) = \frac{1}{s},$$

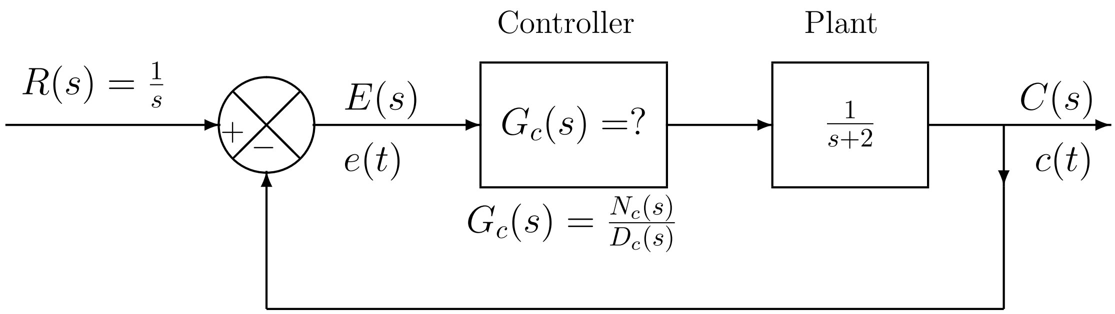

Figure 2: A closed-loop control system of Example 1.

Then,

$$E(s) = \frac{(s+2)s}{s^2+2s+1}\frac{1}{s} = \frac{s+2}{s^2+2s+1}.$$

It is easy to check using, for example, the final value theorem that $\lim_{t\to\infty} e(t) = 0$. Thus, in this case a simple integral controller does the job.

Example 2 Consider the closed-loop system shown in Figure 3. This example looks very similar to the previous one. The only difference is that now the reference signal is the unit ramp function rather than the unit step. Our objective is the same as before, that is, we wish to construct a controller $G_c(s) = \frac{N_c(s)}{D_c(s)}$ so that $\lim_{t\to\infty} e(t) = 0$. In Example 1, the simple integrator was all we needed. Let us try the same controller here. Then, the Laplace transform of the error signal is

$$\begin{aligned} E(s) &= \frac{1}{1+\frac{1}{s+2}\frac{N_c(s)}{D_c(s)}}\frac{1}{s^2} \\ &= \frac{(s+2)D_c(s)}{(s+2)D_c(s)+N_c(s)}\frac{1}{s^2} \\ &= \frac{s+2}{s^2+2s+1}\frac{1}{s}. \end{aligned}$$

Note that now $e(\infty) = 2$. Thus the simple integrator is not enough now to force the steady-state error to zero. Note that in this case the polynomial $D_r(s)$ was not a factor of $D_c(s)$.

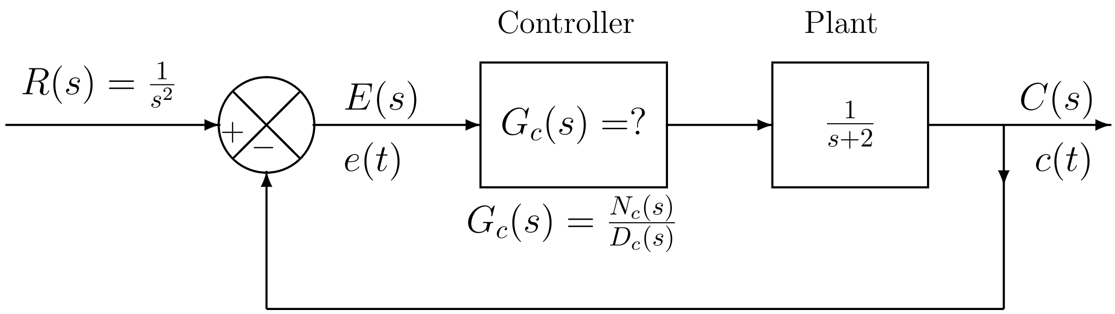

Figure 3: A closed-loop control system of Example 2.

Let us then try a controller that would satisfy the divisibility property of the IMP. If we take the double integrator than the divisibility condition will be satisfied and we obtain

$$E(s) = \frac{s+2}{s^3 + 2s^2 + 1}.$$

However, now $e(t)$ is divergent because the poles of $E(s)$ are not stable. The three poles of $E(s)$ are located at -2.2056, and $0.1028 \pm j0.6655$. This time the first condition of the Internal Model Principle is not satisfied.

For more numerical examples illustrating the internal model principle, we refer to Dorf and Bishop [2, Sections 11.8 and 12.7]. For a generalization of the internal model principle to multi-input multi-output (MIMO) systems, see Bengtsson [1].

References

[1] G. Bengtsson. Output regulation and internal models—a frequency domain approach. *Automatica*, 13(4):333–345, 1977.

[2] R. C. Dorf and R. H. Bishop. *Modern Control Systems*. Prentice Hall, Upper Saddle River, New Jersey 07458, twelfth edition, 2011.