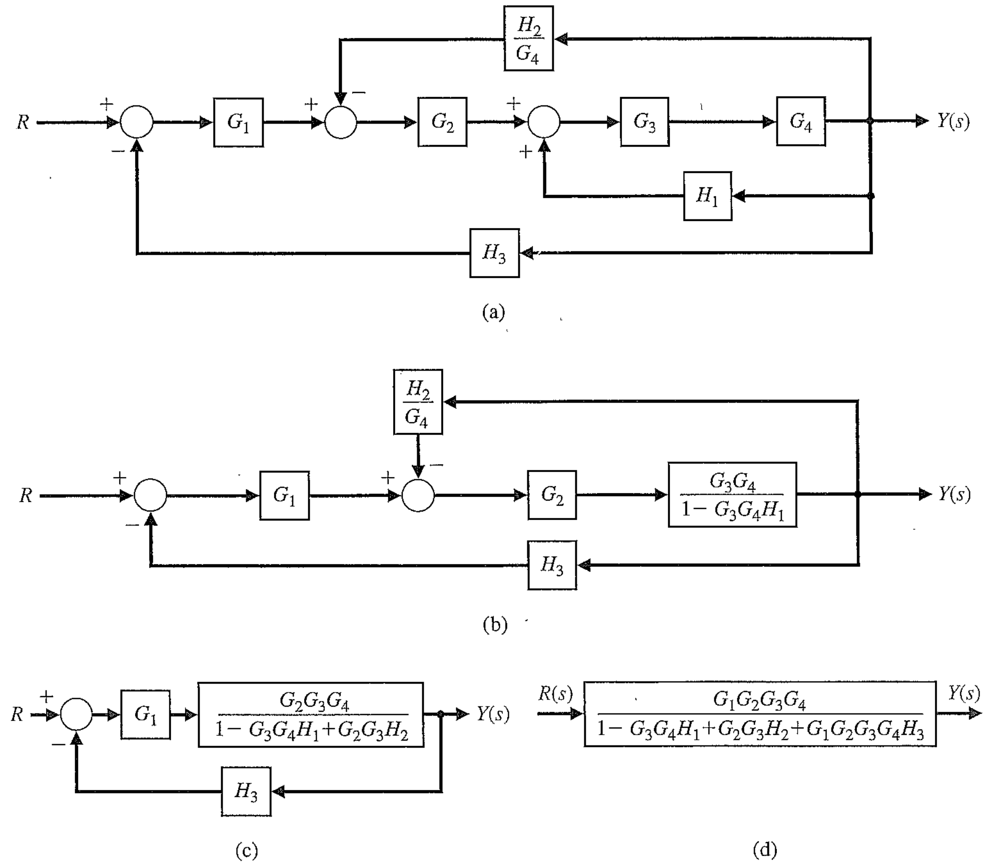

FIGURE 2.27 Block diagram reduction of the system of Figure 2.26.

2.7 SIGNAL-FLOW GRAPH MODELS

Block diagrams are adequate for the representation of the interrelationships of controlled and input variables. However, for a system with reasonably complex interrelationships, the block diagram reduction procedure is cumbersome and often quite difficult to complete. An alternative method for determining the relationship between system variables has been developed by Mason and is based on a representation of the system by line segments [4, 25]. The advantage of the line path method, called the signal-flow graph method, is the availability of a flow graph gain formula, which provides the relation between system variables without requiring any reduction procedure or manipulation of the flow graph.

The transition from a block diagram representation to a directed line segment representation is easy to accomplish by reconsidering the systems of the previous section. A **signal-flow graph** is a diagram consisting of nodes that are connected by several directed branches and is a graphical representation of a set of linear relations. Signal-flow graphs are particularly useful for feedback control systems because feedback theory is primarily concerned with the flow and processing of signals in systems. The basic element of a signal-flow graph is a unidirectional path segment called a **branch**, which relates the dependency of an input and an output variable in

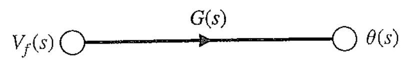

FIGURE 2.28 Signal-flow graph of the DC motor.

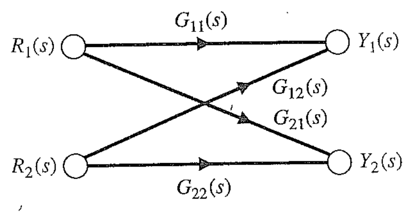

FIGURE 2.29 Signal-flow graph of interconnected system.

a manner equivalent to a block of a block diagram. Therefore, the branch relating the output $\theta(s)$ of a DC motor to the field voltage $V_f(s)$ is similar to the block diagram of Figure 2.22 and is shown in Figure 2.28. The input and output points or junctions are called **nodes**. Similarly, the signal-flow graph representing Equations (2.77) and (2.78), as well as Figure 2.24, is shown in Figure 2.29. The relation between each variable is written next to the directional arrow. All branches leaving a node will pass the nodal signal to the output node of each branch (unidirectionally). The summation of all signals entering a node is equal to the node variable. A **path** is a branch or a continuous sequence of branches that can be traversed from one signal (node) to another signal (node). A **loop** is a closed path that originates and terminates on the same node, with no node being met twice along the path. Two loops are said to be **nontouching** if they do not have a common node. Two touching loops share one or more common nodes. Therefore, considering Figure 2.29 again, we obtain

$$Y_1(s) = G_{11}(s)R_1(s) + G_{12}(s)R_2(s), \tag{2.87}$$

and

$$Y_2(s) = G_{21}(s)R_1(s) + G_{22}(s)R_2(s). \tag{2.88}$$

The flow graph is simply a pictorial method of writing a system of algebraic equations that indicates the interdependencies of the variables. As another example, consider the following set of simultaneous algebraic equations:

$$a_{11}x_1 + a_{12}x_2 + r_1 = x_1 \tag{2.89}$$

$$a_{21}x_1 + a_{22}x_2 + r_2 = x_2. \tag{2.90}$$

The two input variables are r_1 and r_2, and the output variables are x_1 and x_2. A signal-flow graph representing Equations (2.89) and (2.90) is shown in Figure 2.30. Equations (2.89) and (2.90) may be rewritten as

$$x_1(1 - a_{11}) + x_2(-a_{12}) = r_1, \tag{2.91}$$

and

$$x_1(-a_{21}) + x_2(1 - a_{22}) = r_2. \tag{2.92}$$

The simultaneous solution of Equations (2.91) and (2.92) using Cramer's rule results in the solutions

$$x_1 = \frac{(1 - a_{22})r_1 + a_{12}r_2}{(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}} = \frac{1 - a_{22}}{\Delta}r_1 + \frac{a_{12}}{\Delta}r_2, \tag{2.93}$$

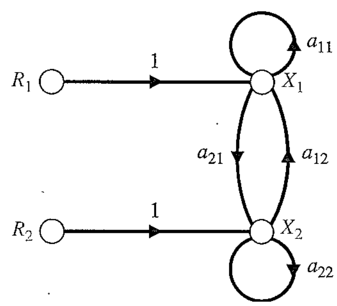

FIGURE 2.30 Signal-flow graph of two algebraic equations.

and

$$x_2 = \frac{(1 - a_{11})r_2 + a_{21}r_1}{(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}} = \frac{1 - a_{11}}{\Delta}r_2 + \frac{a_{21}}{\Delta}r_1. \tag{2.94}$$

The denominator of the solution is the determinant Δ of the set of equations and is rewritten as

$$\Delta = (1 - a_{11})(1 - a_{22}) - a_{12}a_{21} = 1 - a_{11} - a_{22} + a_{11}a_{22} - a_{12}a_{21}. \tag{2.95}$$

In this case, the denominator is equal to 1 minus each self-loop a_{11}, a_{22}, and $a_{12}a_{21}$, plus the product of the two nontouching loops a_{11} and a_{22}. The loops a_{22} and $a_{21}a_{12}$ are touching, as are a_{11} and $a_{21}a_{12}$.

The numerator for x_1 with the input r_1 is 1 times $1 - a_{22}$, which is the value of Δ excluding terms that touch the path 1 from r_1 to x_1. Therefore the numerator from r_2 to x_1 is simply a_{12} because the path through a_{12} touches all the loops. The numerator for x_2 is symmetrical to that of x_1.

In general, the linear dependence T_{ij} between the independent variable x_i (often called the input variable) and a dependent variable x_j is given by Mason's signal-flow gain formula [11, 12],

$$T_{ij} = \frac{\sum_k P_{ijk}\,\Delta_{ijk}}{\Delta}, \tag{2.96}$$

P_{ijk} = gain of kth path from variable x_i to variable x_j,

Δ = determinant of the graph,

Δ_{ijk} = cofactor of the path P_{ijk},

and the summation is taken over all possible k paths from x_i to x_j. The path gain or transmittance P_{ijk} is defined as the product of the gains of the branches of the path, traversed in the direction of the arrows with no node encountered more than once. The cofactor Δ_{ijk} is the determinant with the loops touching the kth path removed. The determinant Δ is

$$\Delta = 1 - \sum_{n=1}^{N} L_n + \sum_{\substack{n,\,m \\ \text{nontouching}}} L_n L_m - \sum_{\substack{n,\,m,\,p \\ \text{nontouching}}} L_n L_m L_p + \cdots, \tag{2.97}$$

where L_q equals the value of the qth loop transmittance. Therefore the rule for evaluating Δ in terms of loops $L_1, L_2, L_3, \ldots, L_N$ is

$\Delta = 1 -$ (sum of all different loop gains)
$+$ (sum of the gain products of all combinations of two nontouching loops)
$-$ (sum of the gain products of all combinations of three nontouching loops)
$+ \cdots.$

The gain formula is often used to relate the output variable $Y(s)$ to the input variable $R(s)$ and is given in somewhat simplified form as

$$\boxed{T = \frac{\Sigma_k P_k \Delta_k}{\Delta},} \tag{2.98}$$

where $T(s) = Y(s)/R(s)$.

Several examples will illustrate the utility and ease of this method. Although the gain Equation (2.96) appears to be formidable, one must remember that it represents a summation process, not a complicated solution process.

EXAMPLE 2.8 **Transfer function of an interacting system**

A two-path signal-flow graph is shown in Figure 2.31(a) and the corresponding block diagram is shown in Figure 2.31(b). An example of a control system with multiple signal paths is a multilegged robot. The paths connecting the input $R(s)$ and output $Y(s)$ are

$$P_1 = G_1G_2G_3G_4 \text{ (path 1)} \quad \text{and} \quad P_2 = G_5G_6G_7G_8 \text{ (path 2)}.$$

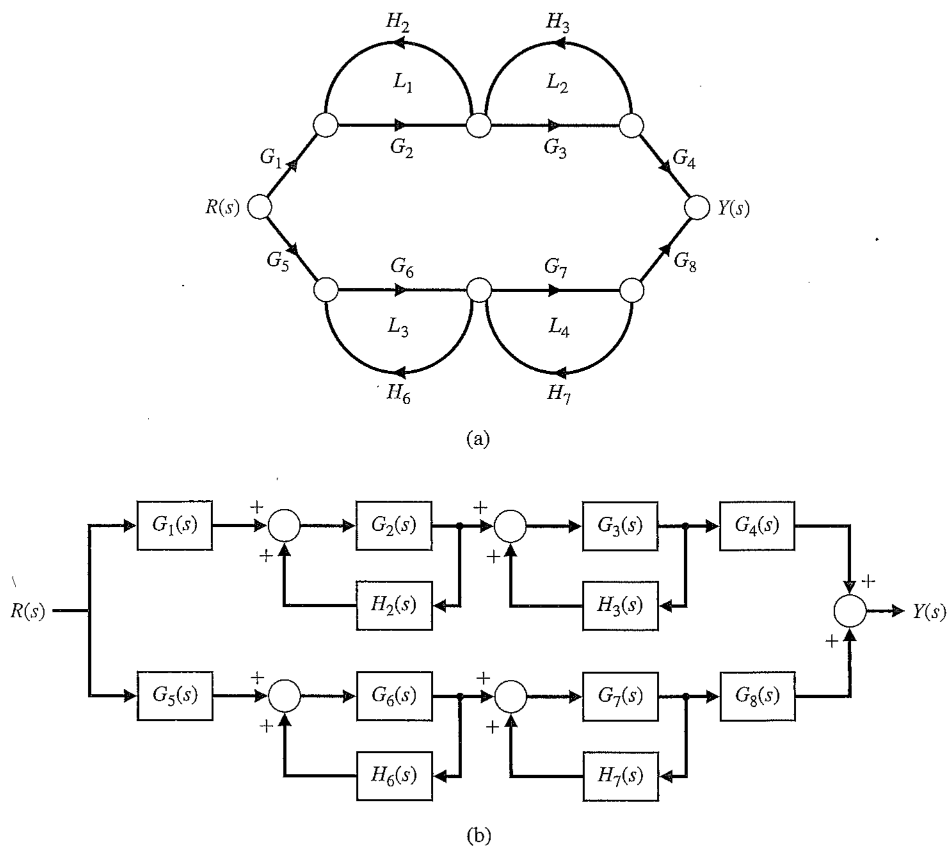

FIGURE 2.31 Two-path interacting system. (a) Signal-flow graph. (b) Block diagram.

There are four self-loops:

$$L_1 = G_2H_2, \quad L_2 = H_3G_3, \quad L_3 = G_6H_6, \quad \text{and} \quad L_4 = G_7H_7.$$

Loops L_1 and L_2 do not touch L_3 and L_4. Therefore, the determinant is

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4). \tag{2.99}$$

The cofactor of the determinant along path 1 is evaluated by removing the loops that touch path 1 from Δ. Hence, we have

$$L_1 = L_2 = 0 \quad \text{and} \quad \Delta_1 = 1 - (L_3 + L_4).$$

Similarly, the cofactor for path 2 is

$$\Delta_2 = 1 - (L_1 + L_2).$$

Therefore, the transfer function of the system is

$$\begin{aligned}\frac{Y(s)}{R(s)} = T(s) &= \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} \\ &= \frac{G_1G_2G_3G_4(1 - L_3 - L_4) + G_5G_6G_7G_8(1 - L_1 - L_2)}{1 - L_1 - L_2 - L_3 - L_4 + L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4}.\end{aligned} \tag{2.100}$$

A similar analysis can be accomplished using block diagram reduction techniques. The block diagram shown in Figure 2.31(b) has four inner feedback loops within the overall block diagram. The block diagram reduction is simplified by first reducing the four inner feedback loops and then placing the resulting systems in series. Along the top path, the transfer function is

$$\begin{aligned}Y_1(s) &= G_1(s)\left[\frac{G_2(s)}{1 - G_2(s)H_2(s)}\right]\left[\frac{G_3(s)}{1 - G_3(s)H_3(s)}\right]G_4(s)R(s) \\ &= \left[\frac{G_1(s)G_2(s)G_3(s)G_4(s)}{(1 - G_2(s)H_2(s))(1 - G_3(s)H_3(s))}\right]R(s).\end{aligned}$$

Similarly across the bottom path, the transfer function is

$$\begin{aligned}Y_2(s) &= G_5(s)\left[\frac{G_6(s)}{1 - G_6(s)H_6(s)}\right]\left[\frac{G_7(s)}{1 - G_7(s)H_7(s)}\right]G_8(s)R(s) \\ &= \left[\frac{G_5(s)G_6(s)G_7(s)G_8(s)}{(1 - G_6(s)H_6(s))(1 - G_7(s)H_7(s))}\right]R(s).\end{aligned}$$

The total transfer function is then given by

$$\begin{aligned}Y(s) = Y_1(s) + Y_2(s) &= \left[\frac{G_1(s)G_2(s)G_3(s)G_4(s)}{(1 - G_2(s)H_2(s))(1 - G_3(s)H_3(s))}\right. \\ &\quad \left. + \frac{G_5(s)G_6(s)G_7(s)G_8(s)}{(1 - G_6(s)H_6(s))(1 - G_7(s)H_7(s))}\right]R(s).\end{aligned}$$ ■

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