Secure Communications of Chaotic Systems with Robust Performance via Fuzzy Observer-Based Design

Kuang-Yow Lian, Chian-Song Chiu, Tung-Sheng Chiang, and Peter Liu

Abstract—This paper presents a systematic design methodology for fuzzy observer-based secure communications of chaotic systems with guaranteed robust performance. The Takagi–Sugeno fuzzy models are given to exactly represent chaotic systems. Then, the general fuzzy model of many well-known chaotic systems is constructed with only one premise variable in fuzzy rules and the same premise variable in the system output. Based on this general model, the fuzzy observer of chaotic system is straightforwardly given and leads the stability condition to a linear-matrix inequality (LMI) problem. When taking the fuzzy observer-based design to applications on secure communications, the robust performance is presented by simultaneously considering the effects of parameter mismatch and external disturbances. Then, the error of the recovered message is stated in an $H_{\infty}$ criterion. In addition, if the communication system is free of external disturbances, the asymptotical recovering of the message is obtained in the same framework. The main results also hold for applications on chaotic synchronization. Numerical simulations illustrate that this proposed scheme yields robust performance.

Index Terms—Chaotic communication, fuzzy observer, $H_{\infty}$ criterion, linear-matrix inequality (LMI), T–S fuzzy model.

I. INTRODUCTION

In light of being broad-spectrum and noise-like, chaotic signals are particularly suitable for secure communications. In the pioneering works of [1] and [2], transmitting messages masked by chaotic signals or modulated by chaotic systems were studied as a form of secure communications. The idea of chaotic masking [1] is to directly add the message in a noise-like chaotic signal at the transmitter, while chaotic modulation [2]–[5] is by injecting the message into a chaotic system as spread-spectrum transmission. Then a coherent detector and some signal processing is thus employed to recover the message from the received signal at the receiver. Most approaches, either by control or observer based [6]–[9], have been developed as an application for chaotic synchronization. In a more systematic design, the control and synchronization of chaotic systems using the Takagi–Sugeno (T–S) fuzzy modeling [10] and their stability analysis have been investigated extensively [11]–[15]. In [12], the synchronization and model following control for chaotic systems are realized via exact linearization (EL) techniques and by solving linear-matrix inequalities (LMIs) problems [16]. However, the scheme is no longer suitable to secure communication due to the effects of signal masking and modulation. Recently, the authors [15] have given analysis without EL conditions to a better performing synchronization and secure communications of chaotic systems via LMIs and fuzzy model-based design.

Much research [17], [18] on controller and observer design for nonlinear systems are carried out based on T–S fuzzy models. The benefit of using a fuzzy model-based design is a straightforward manner to achieve the desired objective by using the parallel distributed compensation concept. Then the stability conditions are related to finding a common symmetric positive definite matrix from solving LMI problems. In recent years, fuzzy systems with guaranteed performances and robustness have been proposed using a quadratic Lyapunov function analysis. Thus optimization problems are stated to provide requirements in LMIs. For continuous-time fuzzy systems (CFS), the synthesis fuzzy controller and observer with desired performances and robustness are studied in [19] and [20]. For discrete-time fuzzy systems (DFS), similar researches of fuzzy control systems can be found in [21] and [22]. Now, the robustness of the fuzzy observer for CFS and DFS will be shown in an LMI manner in this paper.

Chaotic systems are first represented by T–S fuzzy models in this paper. It is shown that many well-known chaotic systems can be exactly represented by T–S fuzzy models with only one premise variable in fuzzy rules. In addition, these fuzzy models have the same premise variable and system output. Based on the general fuzzy model, the fuzzy observer of chaotic systems is related to LMI design problems. In the same framework, a robust observer is taken as an application to secure communications of chaotic systems. The parameter mismatch and external disturbances in a communication system are simultaneously presented, then the receiver gains are designed such that the error of the recovering message exhibits an $H_{\infty}$-type robust performance. Furthermore, the design gains for CFS and DFS can be found by using LMI techniques. When the communication system is free of external disturbances, the recovered message asymptotically converges to its true value. The main results also hold to applications on chaotic synchronization.

The rest of this paper is organized as follows. In Section II, we establish T–S fuzzy models that can exactly represent chaotic systems. In Section III, we construct the fuzzy observer of chaotic systems and derive the conditions via LMIs for CFS and DFS. In Section IV, a robust fuzzy observer is given as an application on secure communications of chaotic systems in an $H_{\infty}$ criterion. Some numerical examples are also illustrated and shown. Finally, some conclusions are made in Section V.
II. Fuzzy Representation of Chaotic Systems

In a fuzzy observer design, chaotic systems should first be exactly represented by T–S fuzzy models. Consider general chaotic systems as follows:

\begin{align}
\dot{x}(t) &= f(x(t)) \\
y(t) &= h(x(t))
\end{align}

where $x \in \mathbb{R}^n$ is the state vector; $\dot{x}(t)$ denotes $\frac{dx}{dt}$ and $x(t+1)$ in continuous-time and discrete-time systems, respectively; $y \in \mathbb{R}^m$ is the system’s output; $f(\cdot)$ and $h(\cdot)$ are nonlinear functions with appropriate dimensions. A fuzzy representation of (1) is composed of the following rules:

**Plant Rule i:**

\text{If } z_1(t) \text{ is } F_{1i} \text{ and } \ldots \text{ and } z_6(t) \text{ is } F_{6i} \text{ then } \begin{align}
\dot{x}(t) &= A_i x(t) + b_i \\
y(t) &= C_i x(t), \quad i = 1, 2, \ldots, r
\end{align}

where $z_1(t) \sim z_6(t)$ are the premise variables which consists of states of the system; $F_{ji}$ ($j = 1, 2, \ldots, 6$) are fuzzy sets; $r$ is the number of fuzzy rules; $A_i$ and $C_i$ are system and output matrices with appropriate dimensions; and $b_i \in \mathbb{R}^n$ denotes the constant bias term, which is generated by the exact fuzzy modeling procedure.

Using the singleton fuzzifier, product fuzzy inference and weighted average defuzzifier, the final output of the fuzzy system is inferred as follows:

\begin{align}
\dot{x}(t) &= \sum_{i=1}^{r} \mu_i(z(t))(A_i x(t) + b_i) \\
y(t) &= \sum_{i=1}^{r} \mu_i(z(t))(C_i x(t))
\end{align}

where $z(t) = [z_1(t) \ z_2(t) \ \ldots \ \ z_6(t)]^T$, and $\mu_i(z(t)) = \left(\omega_1(z_1(t)) \sum_{i=1}^{6} \omega_i(z_i(t))\right)$ with $\omega_i(z(t)) = \prod_{j=1}^{6} F_{ji}(z_j(t))$. Note that $\sum_{i=1}^{r} \mu_i(z(t)) = 1$ for all $t$, where $\mu_i(z(t)) \geq 0$ for $i = 1, 2, \ldots, r$.

Focus on (1) and (2), if we appropriately specify the fuzzy membership functions in premise parts and associated entries of matrices $A_i$, $C_i$, and $b_i$ in the consequent parts, the chaotic system can be represented by a T–S fuzzy model. From the investigation of a large class of continuous-time and discrete-time chaotic systems, we found that nonlinear terms have a common variable or depend only on one variable. If we take this as the premise variable of fuzzy rules, a simple fuzzy dynamic model can be obtained and will exactly represent the chaotic systems in the region of interest. The following chaotic systems in T–S fuzzy models are considered in this paper:

Lorenz’s equation:

\begin{align}
\dot{x}_1(t) &= -10x_1(t) + 10x_2(t) \\
\dot{x}_2(t) &= 28x_1(t) - x_2(t) - x_1(t)x_3(t) \\
\dot{x}_3(t) &= x_1(t)x_2(t) - \frac{\delta}{3} x_3(t) \\
y(t) &= x_1(t)
\end{align}

The premise variable of the fuzzy rules is $x_1(t)$, which satisfies $x_1(t) \in [-d \ d]$ with $d = 30$. Then, in this range, we derive the fuzzy dynamic model which exactly represents the Lorenz’s equation as follows:

\text{Rule i: If } x_1(t) \text{ is } F_i \text{ then } \begin{align}
\dot{x}(t) &= A_i x(t) + b_i \\
y(t) &= C_i x(t), \quad i = 1, 2
\end{align}

where $x(t) = [x_1(t) \ x_2(t) \ x_3(t)]^T$, the fuzzy sets are chosen as $F_1(x_1(t)) = 1/2(1+(x_1(t)/d))$, and $F_2(x_1(t)) = 1/2(1-(x_1(t)/d))$; and system matrices as

\begin{align}
A_1 &= \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & -d \\ 0 & d & -\frac{\delta}{3} \end{bmatrix} \\
A_2 &= \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & d \\ 0 & -d & -\frac{\delta}{3} \end{bmatrix}
\end{align}

$C_1 = C_2 = [1 \ 0 \ 0]$, and $b_1 = b_2 = 0$.

Chua’s Circuit:

\begin{align}
\dot{x}_1(t) &= \sigma_1(-x_1(t) + x_2(t) - f(x_1(t))) \\
\dot{x}_2(t) &= x_1(t) - x_2(t) + x_3(t) \\
\dot{x}_3(t) &= -\sigma_2 x_2(t) \\
y(t) &= x_1(t)
\end{align}

with $f(x_1(t)) = g_0 x_1(t) + 0.5(g_0 - g_6)(|x_1(t) + 1| - |x_1(t) - 1|)$, where $\sigma_1 = 10$, $\sigma_2 = 14.97$, $g_0 = -1.27$ and $g_6 = -0.68$.

Let $x_1(t)$ as the premise variable of fuzzy rules, and define a function $\phi(x_1(t))$ as

$$\phi(x_1(t)) = \begin{cases} f(x_1(t))/x_1(t), & x_1(t) \neq 0 \\ g_0, & x_1(t) = 0 \end{cases}$$

Then, the fuzzy dynamic model which exactly represents the Chua’s circuit has:

$$A_1 = \begin{bmatrix} (d-1)\sigma_1 & \sigma_1 & 0 \\ 1 & -1 & 1 \\ 0 & -\sigma_2 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -(d+1)\sigma_1 & \sigma_1 & 0 \\ 1 & -1 & 1 \\ 0 & -\sigma_2 & 0 \end{bmatrix}$$

$x(t) = [x_1(t) \ x_2(t) \ x_3(t)]^T$, $C_1 = C_2 = [1 \ 0 \ 0]$, $b_1 = b_2 = 0$, and the fuzzy sets are $F_1(x_1(t)) = 1/2(1-(\phi(x_1(t))/d))$ and $F_2(x_1(t)) = 1 - F_1(x_1(t))$ with $d = \sup_{x_1(t) \in X_1} |\phi(x_1(t))| = 3$.

Henon map:

$$\begin{align}
x_1(t+1) &= -x_1^2(t) + 0.3x_2(t) + 1.4 \\
x_2(t+1) &= x_1(t) \\
y(t) &= x_1(t)
\end{align}$$

The premise variable of the fuzzy rules is $x_1(t)$, which satisfies $x_1(t) \in [-d \ d]$ with $d = 2$. The following equivalent fuzzy
dynamic model can be constructed as

\[
\text{Rule } i: \quad \text{IF } x_1(t) \text{ is } F_i \text{ THEN } \\
x(t+1) = A_i x(t) + b_i \quad y(t) = C_i x(t), \quad i = 1, 2
\]

where \( x(t) = [x_1(t) \ x_2(t)]^T \); the fuzzy sets are \( F_1(x_1(t)) = 1/2 (1 + (x_1(t)/d)) \) and \( F_2(x_1(t)) = 1/2 (1 - (x_1(t)/d)) \):

\[
A_1 = \begin{bmatrix} -d & 0.3 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} d & 0.3 \\ 1 & 0 \end{bmatrix} \\
b_1 = b_2 = \begin{bmatrix} 1.4 \\ 1 \end{bmatrix}, \quad C_1 = C_2 = [1 \ 0].
\]

Lozi map:

\[
\begin{align*}
x_1(t+1) &= -1.8|x_1(t)| + x_2(t) + 3 \\
x_2(t+1) &= 0.25x_1(t) \\
y(t) &= x_1(t).
\end{align*}
\]

(7)

Let us choose \( x_3(t) \) as the premise variable of fuzzy rules and define \( \phi(x_1(t)) = |x_1(t)| \). Then the equivalent fuzzy dynamic model can be constructed with

\[
A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ 0.25 & 0 \end{bmatrix}, \quad b_1 = \begin{bmatrix} -1.8d + 3 \\ 0 \end{bmatrix}
\]

and

\[
b_2 = \begin{bmatrix} 1.8d + 3 \\ 0 \end{bmatrix}
\]

\( x(t) = [x_1(t) \ x_2(t)]^T , \quad C_1 = C_2 = [1 \ 0] \), and the fuzzy sets are selected as \( F_1(x_1(t)) = 1/2 (1 + (\phi(x_1(t))/d)) \). \( F_2(x_1(t)) = 1/2 (1 - (\phi(x_1(t))/d)) \), with \( d = 3.5 \). Many well-known continuous and discrete chaotic systems can be exactly represented by T–S fuzzy models. It is worthy to note that these fuzzy representations have the common output matrix \( C \), also the same premise variable and system’s output. Then the general form of T–S fuzzy models for chaotic systems can be written as follows:

\[
\text{Rule } i: \quad \text{IF } y(t) \text{ is } F_i \text{ THEN } \\
x(t+1) = A_i x(t) + b_i \\
y(t) = C_i x(t), \quad i = 1, 2, \ldots, r
\]

(8)

in which each rule has only one premise variable. The following observer and its application on secure communications of chaotic systems will be proposed based on the fuzzy dynamic model (8).

III. FUZZY OBSERVER OF CHAOTIC SYSTEMS

Consider a chaotic system described by its T–S fuzzy model in (8), and assume \( (A_i, C) \) is a detectable pair for each local linear model. Then a fuzzy observer is derived to estimate the state \( \hat{x}(t) \) from system’s output. For simplicity, we let the premise variable of observer rules same as the fuzzy models. Accordingly, a fuzzy observer is given consisting of rules:

**Observer Rule i:**

\[
\text{IF } y(t) \text{ is } F_i \text{ THEN } \\
x(t+1) = A_i \hat{x}(t) + b_i + L_i (y(t) - \hat{y}(t)) \\
y(t) = C_i \hat{x}(t), \quad i = 1, 2, \ldots, r
\]

(9)

where \( L_i \) is an observer gain determined later. The overall observer is inferred in the following:

\[
\begin{align*}
\dot{s}(t) &= \sum_{i=1}^{r} \mu_i(y(t)) [A_i^T(t) \hat{x}(t) + b_i + L_i(y(t) - \hat{y}(t))] \\
\hat{y}(t) &= C_i \hat{x}(t)
\end{align*}
\]

where \( \mu_i(y(t)) = (\omega_i(y(t))/\sum_{i=1}^{r} \omega_i(y(t))) \) with \( \omega_i(y(t)) = F_i(y(t)) \geq 0 \). Define error signal \( \tilde{x}(t) = x(t) - \hat{x}(t) \). According to the inferred results of (8) and (9), the error dynamics of \( \tilde{x}(t) \) is expressed as

\[
\dot{s}(t) = G \tilde{x}(t)
\]

(10)

with

\[
G = \sum_{i=1}^{r} \mu_i(y(t))(A_i - L_i C).
\]

Now, the stability conditions for (10) are addressed here.

**Proposition 1—CFS:** Consider the continuous-time chaotic system described in (8) and its corresponding observer (9). If there exist a common positive definite matrix \( P \) and gains \( L_i \), for \( i = 1, 2, \ldots, r \), such that the following LMIs, with \( M_i \equiv PL_i \),

\[
- A_i^T P - PA_i + C_i^T M_i^T + M_i C > 0,
\]

(11)

have feasible solutions, the error \( \tilde{x}(t) \) asymptotically converges to zero as \( t \to \infty \).

**Proof:** Define the Lyapunov function candidate as

\[
V(\tilde{x}(t)) = \tilde{x}^T(t)(G^T P + PG) \tilde{x}(t)
\]

If \( G^T P + PG < 0 \), then \( \dot{V}(\tilde{x}(t)) < 0 \). Also, substituting the expression of \( G \) into \( -G^T P - PG > 0 \), and denoting \( M_i = PL_i \), the stability conditions (11) are obtained in LMIs. Therefore, if (11) has feasible solutions, the overall system has

\[
\lim_{t \to \infty} \tilde{x}(t) = 0.
\]

**Proposition 2—DFS:** Consider the discrete-time chaotic system described in (8) and its corresponding observer (9). If there exist a common positive definite matrix \( P \) and gains \( L_i \), for \( i = 1, 2, \ldots, r \), such that the following LMIs, with \( M_i \equiv PL_i \),

\[
\begin{bmatrix} P & (PA_i - M_i C)^T \\
PA_i - M_i C & P \end{bmatrix} > 0,
\]

(12)

have feasible solutions. Then the error \( \tilde{x}(t) \) asymptotically converges to zero as \( t \to \infty \).
Proof: Given a Lyapunov function candidate for DFS as
\[ V(\hat{x}(t)) = \hat{x}^T(t)P\hat{x}(t) \] with \( P > 0 \), then we have
\[ \Delta V(\hat{x}(t)) = \hat{x}^T(t)(G^TPG - P)\hat{x}(t). \]
If \( G^TPG - P < 0 \), then \( \Delta V(\hat{x}(t)) < 0 \). Since this condition can be expressed as an equivalent LMI
\[
\begin{bmatrix}
P & (PG)^T \\
PG & P
\end{bmatrix} > 0.
\]
The remaining proof is similar to Proposition 1. If there are \( P \) and \( L_k \) such that the conditions (12) are held, then \( \Delta V(\hat{x}(t)) < 0 \). Therefore, the estimation error \( \hat{x}(t) \) asymptotically converges to zero as \( t \to \infty \). □

If the LMI problems have feasible solutions, the gain \( L_i \) can be determined by \( L_i = P^{-1}M_i \) and a common \( P \) can be found such that stability is ensured. Moreover, the asymptotic estimation is guaranteed in semiglobal convergence, which depends on the prescribed region of exact fuzzy representation.

IV. ROBUST FUZZY CHAOTIC COMMUNICATIONS

In this section, a robust fuzzy observer is taken as an application to secure communications. In practical communications, the transmitter-receiver pair system may face an unknown environment. Therefore, assume the communication system has a parameter mismatch that is exactly known. Inspired by previous works of modulated chaotic communications [3]–[5], the modulated chaotic transmitter with external disturbances are considered and described by fuzzy rules

Transmitter Rule \( \hat{i} \):
\[
\text{If } \overline{y}(t) \text{ is } F_i \text{ THEN } \]
\[ s\hat{x}(t) = A_i\hat{x}(t) + b_i + L_i m(t) + E_iu(t) \]
\[ y(t) = C\hat{x}(t) + m(t), \quad i = 1, 2, \ldots, r \]  \hspace{1cm} (13)
where the message \( m(t) \) is modulated into the chaotic system via vector \( L_i \) designed later; the transmitted signal \( y(t) \) is embedded with the message \( m(t) \); \( u(t) \in R^p \) is an unknown external disturbance with an upper bound; and \( E_i \in R^{m \times p} \) denotes the disturbance injection matrix. The final output of the fuzzy chaotic transmitter is inferred as
\[ s\hat{x}(t) = \sum_{i=1}^{r} \mu_i(\overline{y}(t)) \{ (A_i - L_iC)\hat{x}(t) + b_i + L_i m(t) + E_iu(t) \} \]  \hspace{1cm} (14)
\[ y(t) = C\hat{x}(t) + m(t) \]  \hspace{1cm} (15)
where \( \mu_i(\overline{y}(t)) = (\omega_i(\overline{y}(t))/\sum_{i=1}^{r} \omega_i(\overline{y}(t))) \) with \( \omega_i(\overline{y}(t)) = F_i(\overline{y}(t)) \geq 0 \). According to the fuzzy observer of chaotic systems proposed in Section III, the fuzzy receiver is designed in the form

Receiver Rule \( \hat{i} \):
\[
\text{If } \overline{y}(t) \text{ is } F_i \text{ THEN } \]
\[ s\hat{\hat{x}}(t) = D_i\hat{\hat{x}}(t) + b_i + Hu(t) + L_i(\overline{y}(t) - \hat{\overline{y}}(t)) \]
\[ \hat{y}(t) = C\hat{x}(t), \quad i = 1, 2, \ldots, r \]  \hspace{1cm} (16)

The error of the recovered message can be expressed as
\[ \hat{m}(t) = \hat{m}(t) - m(t) = C\hat{x}(t), \]
Define \( s\hat{\hat{x}}(t) = x(t) - \hat{x}(t) \) and apply (14) and (17), then the following equation can be obtained:
\[ s\hat{\hat{x}}(t) = G\hat{x}(t) + Eu(t) + (G - \overline{G}) \hat{x}(t) \]  \hspace{1cm} (19)
where
\[
G = \sum_{i=1}^{r} \mu_i(\overline{y}(t)) (A_i - L_iC)
\]
\[ \overline{G} = \sum_{i=1}^{r} \mu_i(\overline{y}(t)) (D_i - L_iC - HK_i) \]
\[ E = \sum_{i=1}^{r} \mu_i(\overline{y}(t)) E_i. \]

Theorem 1: Suppose there exist feedback gains \( K_i \), for \( i = 1, 2, \ldots, r \), such that
\[ (G - \overline{G})^T (G - \overline{G}) = 0 \]  \hspace{1cm} (20)
and its corresponding stability conditions as stated in Propositions 1 and 2 are satisfied. Then the error system expressed in (19) for CFS or DFS is uniformly stable if the external disturbance is bounded. Meanwhile, the error of the recovered message is uniformly bounded.
Proof: It is clear that \( G - \overline{G} = 0 \) once the condition (20) is held. Then the error dynamics (19) becomes
\[ s\hat{\hat{x}}(t) = G\hat{x}(t) + Eu(t) \]  \hspace{1cm} (21)
the stability conditions of the system are reduced as same as Propositions 1 and 2 for CFS and DFS, respectively. Thus, if \( w(t) \in L_\infty \), the errors \( \hat{x}(t) \) and \( \hat{n}(t) \) are uniformly bounded according to Lyapunov’s direct method.

It is noted that (20) can be realized by the following method. If \( H \) is nonsingular, the feedback gain \( K_i \) is determined by \( K_i = -H^{-1}(A_i - D_i) \), due to the same premise variable and \( A_i = D_i - HK_i \). This assumption on \( H \) is acceptable since it is not dependent on the model of the drive system. If given \( H \) is a nonsquare matrix, we convert (20) to find a positive definite matrix \( X \) and a small constant \( \varepsilon > 0 \) such that \( \varepsilon X^{-2} \approx 0 \) and

\[
\varepsilon I - (GX - \mathcal{G}X)^T (GX - \mathcal{G}X) > 0. \tag{22}
\]

By premultiplying and postmultiplying above inequality by \( X^{-1} \), we have that if all elements in \( \varepsilon X^{-2} \) are near zero, then (20) is achieved. Furthermore, (22) can be equivalently expressed as an LMI [12]

\[
\min_{X > 0} \varepsilon \\
\text{subject to} \\
X > 0, \enspace \varepsilon > 0 \tag{23}
\]

\[
\varepsilon I - \begin{bmatrix} (GX - \mathcal{G}X)^T & \varepsilon \end{bmatrix} > 0
\]

to find gains \( K_i \) for \( i = 1, 2, \ldots, r \). The error system with the gain \( L_i \) can be also designed to guarantee the desired performance for (21). By LMIs technique, the robust performances of chaotic communications are stated as follows:

**Theorem 2—CFS:** Consider the communication systems (13) and (16) for CFS. If the gains \( L_i \) and \( K_i \) can be determined by solving the following design problem:

Given \( \alpha > 0, \rho > 0 \)

\[
\min_{M_i, N_i, P} \varepsilon \\
\text{subject to} \\
P > 0, \varepsilon > 0 \tag{24}
\]

\[
\begin{bmatrix} -A^T P - PA_i + CT M_i^T + M_i C - 2\alpha P & PE_i \\
E^T P \end{bmatrix} \geq 0
\]

\[
\begin{bmatrix} \varepsilon I & \{A_1 P - (D_1 P - H N_1)\}^T \\
A_i P - (D_i P - H N_i) \end{bmatrix} > 0
\]

for all \( i \) \tag{25}

where \( M_i = P L_i \), and \( N_i = K_i P \), then the error system has \( H_\infty \) performance as

\[
2\alpha \int_0^T \dot{x}(t)^T P \dot{x}(t) \; dt \leq \dot{x}(0)^T P \dot{x}(0) + \frac{1}{\rho^2} \int_0^T ||w(t)||^2 \; dt. \tag{26}
\]

Moreover, the error of recovered message is ultimately bounded by \( q/\rho \sqrt{2\alpha} \) for some \( q > 0 \).

**Proof:** The condition (20) can be equivalently converted to minimize \( \|x^{P-2}\| \approx 0 \) in (25), i.e., let \( X = P \), in (23). Given a Lyapunov function candidate \( V(\hat{x}(t)) = \hat{x}^T(t) P \hat{x}(t) \) with a matrix \( P \) satisfying (24), the time derivative of \( V(\hat{x}(t)) \) is yielded as

\[
\dot{V}(t) = [\hat{x}(t)^T \; w(t)]^T \begin{bmatrix} G^T P + PG + 2\alpha P & PE \\
E^T P & \frac{1}{\rho^2} I \end{bmatrix} [\hat{x}(t)^T \; w(t)]
- 2\alpha \hat{x}^T(t) P \hat{x}(t) + \frac{1}{\rho^2} w^T(t) w(t)
\leq -2\alpha V(\hat{x}(t)) + \frac{1}{\rho^2} ||w(t)||^2. \tag{27}
\]

Therefore, the \( H_\infty \) performance as (26) is obtained by integrating both sides of (27). This means that the effect of disturbance is attenuated to a prescribed level \( 1/\rho^2 \).

From the fact that the Lyapunov function \( V(t) \) satisfies (27) and \( \lambda_m ||\hat{x}(t)||^2 \leq V(\hat{x}(t)) \leq \lambda_M ||\hat{x}(t)||^2 \), for \( \lambda_m, \lambda_M > 0 \), the error trajectory of \( \hat{x}(t) \) is shaped by

\[
||\hat{x}(t)||^2 \leq \frac{\lambda_M}{\lambda_m} ||\hat{x}(0)||^2 \exp(-2\alpha t)
+ \frac{1}{2\alpha \rho^2 \lambda_m} [1 - \exp(-2\alpha t)] \omega^2
\]

where \( \omega = \sup_{t} ||w(t)|| \). Accordingly, the error \( \hat{x}(t) \) is uniformly ultimately bounded. The error of the recovered message is also within a uniform ultimate bound, that is

\[
||\hat{n}(t)|| \leq \frac{\|C\|}{\rho} \sqrt{\frac{1}{2\alpha \lambda_m}}, \text{ for } t > 0.
\]

**Theorem 3—DFS:** Consider the communication systems (13) and (16) for DFS. If the gains \( L_i \) and \( K_i \) can be determined by solving the following design problem:

Given \( 1 > \beta > 0, \rho > 0 \)

\[
\min_{M_i, N_i, P} \varepsilon \\
\text{subject to} \\
P > 0, \varepsilon > 0 \tag{28}
\]

\[
\begin{bmatrix} \beta P & (PA_i - M_i C)^T \\
PA_i - M_i C & P \\
0 & PE_i \\
E^T P & \frac{1}{\rho^2} I \end{bmatrix} \geq 0
\]

\[
\begin{bmatrix} \varepsilon I & \{A_1 P - (D_1 P - H N_1)\}^T \\
A_i P - (D_i P - H N_i) \end{bmatrix} > 0
\]

for all \( i \) \tag{29}

where \( M_i = P L_i \), and \( N_i = K_i P \), then the error system has \( H_\infty \) performance as

\[
(1 - \beta) \sum_{t=0}^{T} \hat{x}^T(t) P \hat{x}(t)
\leq \hat{x}(0)^T P \hat{x}(0) + \frac{1}{\rho^2} \sum_{t=0}^{T} ||w(t)||^2. \tag{30}
\]
Moreover, the error of recovered message is ultimately bounded by $q/\rho \sqrt{1-\beta}$, for some $q > 0$.

**Proof:** Similar as the proof of Theorem 2, the overall error dynamics for DFS can be represented as (21) when matrix inequalities (29) with $X = P$ are satisfied. Define a Lyapunov function candidate $V(\tilde{z}(t)) = \tilde{z}^T(t)P\tilde{z}(t)$ with a matrix $P$ and gains $L_i$, for $i = 1, 2, \ldots, r$, satisfying (28), the difference of $V(\tilde{z}(t))$ leads to

$$
\Delta V(t) = \left[ \begin{array}{c} \tilde{z}(t) \\ u(t) \end{array} \right]^T \begin{bmatrix} G^T P G - \beta P & G^T P E \\ E^T P G & E^T P E - \frac{1}{\rho^2} I \end{bmatrix} \left[ \begin{array}{c} \tilde{z}(t) \\ u(t) \end{array} \right] - (1-\beta)\tilde{z}^T(t)P\tilde{z}(t) + \frac{1}{\rho^2} u^T(t)u(t)
\leq - (1-\beta)V(\tilde{z}(t)) + \frac{1}{\rho^2} ||u(t)||^2.
$$

(31)

In the above derivation, the fact is applied that the matrix inequality

$$
\begin{bmatrix} G^T P G - \beta P & G^T P E \\ E^T P G & E^T P E - \frac{1}{\rho^2} I \end{bmatrix} \leq 0
$$

is equivalent to an LMI

$$
\begin{bmatrix} \beta P & (PG)^T \\ PG & P & PE \\ 0 & E^T P & \frac{1}{\rho^2} I \end{bmatrix} \geq 0.
$$

By using expressions of $G$ and $E$, and denoting $M_i = PL_i$, the above LMI succeeds by (28). Thus, the $H^\infty$ performance is obtained as (30) by summing both sides of (31). The external disturbance is attenuated to a prescribed level $1/\rho^2$.

According to (31) and $\lambda_m||\tilde{x}(t)||^2 \leq V(\tilde{x}(t)) \leq \lambda_M||\tilde{x}(t)||^2$ with some $\lambda_m$, $\lambda_M > 0$, the error trajectory of $\tilde{x}(t)$ is shaped by

$$
||\tilde{x}(t+1)||^2 \leq \frac{\lambda_m}{\lambda_m} ||\tilde{x}(0)||^2 \beta^{t+1} + \frac{1}{\rho^2 \lambda_m} \left( \frac{1-\beta+1}{1-\beta} \right) \overline{\sigma}^2
$$

where $\overline{\sigma} = \sup_t ||u(t)||$. Due to $1 > \beta > 0$, the error $\tilde{x}(t)$ is thus uniformly ultimately bounded, in turn, the error of recovered message is with a uniform ultimate bound as follows:

$$
||\tilde{m}(t)|| \leq \frac{\overline{\sigma}||C||}{\rho} \sqrt{\frac{1}{(1-\beta)\lambda_m}}, \quad \text{for } t > 0.
$$

It is worthy to note that when the embedded message is set as zero, the system design can be regarded as a synchronization problem of chaotic systems. The results can provide with $H^\infty$ performance of (26) and (30) for CFS and DFS, respectively. However, these conditions seem strict for some secure communications applying chaotic systems. In secure communications, the focus is to minimize the external disturbance to message error ratio. Thus, to be more feasible for the purpose, the robust performance can be further relaxed into minimizing $\sup_{||u(t)||_2 \neq 0} ||\tilde{m}(t)||_2/||u(t)||_2$ by designing observer gain $L_i$. This is equivalent to minimizing $\sup_{||u(t)||_2 \neq 0} (||C\tilde{z}(t)||_2/||u(t)||_2)$. Then an optimal disturbance rejection is expressed by the following LMI design problems.

**Secure Communication with Robust Performance for CFS:**

$$
\begin{array}{ll}
\min_{M_i, N_i, P} & \varepsilon \\
\min_{M_i, N_i, P} & \kappa \\
\text{subject to} & P > 0, \quad \varepsilon > 0, \quad \kappa > 0 \\
\begin{bmatrix}
-A_i^T P & PA_i - C^T P E_i + M_i C - C^T C PE_i \\
E_i^T P & \kappa I
\end{bmatrix} & \geq 0 \\
\begin{bmatrix}
\varepsilon I & \{A_i P - (D_i P - H N_i)\}^T \\
A_i P - (D_i P - H N_i) & I
\end{bmatrix} & > 0,
\end{array}
$$

for all $i$

where $\kappa = 1/\rho^2$, $M_i = PL_i$, and $N_i = K_i P$.

**Secure Communication with Robust Performance for DFS:**

$$
\begin{array}{ll}
\min_{M_i, N_i, P} & \varepsilon \\
\min_{M_i, N_i, P} & \kappa \\
\text{subject to} & P > 0, \quad \varepsilon > 0, \quad \kappa > 0 \\
\begin{bmatrix}
-P - C^T C & (PA_i - M_i C)^T \\
PA_i - M_i C & P & PE_i
\end{bmatrix} & \geq 0 \\
\begin{bmatrix}
\varepsilon I & \{A_i P - (D_i P - H N_i)\}^T \\
A_i P - (D_i P - H N_i) & I
\end{bmatrix} & > 0,
\end{array}
$$

for all $i$

where $\kappa = 1/\rho^2$, $M_i = PL_i$, and $N_i = K_i P$.

When the system is with $\tilde{x}(0) = 0$, then the robust capability of fuzzy secure communications in (26) and (30) is modified as

$$
\begin{array}{ll}
\frac{(||\tilde{m}(t)||_2^T}{||u(t)||_2^T} \leq \frac{1}{\rho^2}.
\end{array}
$$

The satisfactory performance is guaranteed in an $L_2$-gain manner.

**Example 1:** For CFS, the Lorenz’s equation and Chua’s circuit will be applied to secure communications. Let the message $m(t)$ be a square wave with the amplitude as 0.01 and frequency as 1 Hz. Assume that the fuzzy transmitter (13) has an external disturbance with zero mean, uniform distribution, and magnitude of 0.1. When the fuzzy receiver (16) is activated (connected) at $t = 10$, the robust performance is shown. For Lorenz’s equation (3), the system parameters are set as

$$
E_1 = E_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \quad H = I_{3 \times 3}
$$

$$
D_1 = \begin{bmatrix} -5 & 5 & 0 \\
36 & -1 & -d \\
0 & d & -\frac{1}{3} \end{bmatrix}, \quad D_2 = \begin{bmatrix} -5 & 5 & 0 \\
36 & -1 & d \\
0 & -d & -\frac{1}{3} \end{bmatrix}.
$$

Notice that some entries of $D_1$ and $D_2$ are different to the entries of $A_1$ and $A_2$ in (4). The results are shown in Fig. 1. These subfigures are: (a) transmitted signal; (b) message (dotted line); and recovered message (solid line).
For Chua’s circuit (5), the system parameters are set as

\[
E_1 = E_2 = [1 \ 0 \ 0]^T, \quad H = I_{3 \times 3}
\]

\[
D_1 = \begin{bmatrix}
(d - 1)\sigma_1/2 & \sigma_1/2 & 0 \\
1 & -1 & 1 \\
0 & -2\sigma_2 & 0
\end{bmatrix}
\]

\[
D_2 = \begin{bmatrix}
-(d + 1)\sigma_1/2 & \sigma_1/2 & 0 \\
1 & -1 & 1 \\
0 & -2\sigma_2 & 0
\end{bmatrix}
\]

The results are shown in Fig. 2. These subfigures are: (a) transmitted signal; (b) message (dotted line) and recovered message (solid line).

Example 2: For DFS, the Henon and Lozi map will be applied to secure communications. Assume the magnitude of disturbance is 0.01, and other simulation conditions are the same as Example 1. For Henon map (6), the system parameters are chosen as

\[
E_1 = E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

\[
D_1 = \begin{bmatrix} -d & 0.15 \\ 1 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} d & 0.15 \\ 1 & 0 \end{bmatrix}
\]

The results are shown in Fig. 3. These subfigures are: (a) transmitted signal; (b) message; (c) error of recovered message.
Fig. 3. This illustrates the (a) transmitted signal; (b) message; and (c) error of recovered message using Hénon map as an example.

For Lozi map (7), the system parameters are

\[ E_1 = H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E_2 = 0, \quad D_1 = D_2 = \begin{bmatrix} 0 & 1 \\ 0.125 & 0 \end{bmatrix} \, . \]

The results are shown in Fig. 4. These subfigures are: (a) transmitted signal; (b) message; (c) error of recovered message.

V. CONCLUSION

In this paper, an LMI-based fuzzy observer for chaotic systems and its application to communications with robust performance are proposed. The general fuzzy models of chaotic systems were found to accomplish the design. It was shown that the general fuzzy model of many well-known chaotic systems have the same premise variable in fuzzy rules and system output. This leads to a unified way to design a fuzzy observer with guaranteed stability in the form of LMIs. When extended to application of chaotic communications, the parameter mismatch can be compensated by PDC and the effects of disturbances are attenuated to a prescribed level. The performance is expressed to be \( L_2 \)-gain from disturbances to recovered message errors. In addition, if the transmitter is free of disturbances, the recovered message will asymptotically converge to its true value. Simulation results have been illustrated and shown that the fuzzy observer-based communications have the robust capability for both continuous and discrete chaotic systems.
REFERENCES


