An Introduction to Proportional-Integral-Derivative (PID) Controllers

Stan Żak
School of Electrical and Computer Engineering
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Motivation

- Growing gap between “real world” control problems and the theory for analysis and design of linear control systems
- Design techniques based on linear system theory have difficulties with accommodating nonlinear effects and modeling uncertainties
- Increasing complexity of industrial process as well as household appliances

Effective control strategies are required to achieve high performance for uncertain dynamic systems
Usefulness of PID Controls

- Most useful when a mathematical model of the plant is not available
- Many different PID tuning rules available
- Our sources

Proportional-integral-derivative (PID) control framework is a method to control uncertain systems
Type A PID Control

- Transfer function of PID controller

\[ G_{PID}(s) = \frac{U(s)}{E(s)} = K_p \left( 1 + \frac{1}{T_i s} + T_d s \right) \]

- The three term control signal

\[ U(s) = K_p E(s) + K_i \frac{1}{s} E(s) + K_d s E(s) \]
PID-Controlled System

PID controller in forward path

\[ K_p (1 + \frac{1}{T_i s} + T_d s) \]

Plant
PID Tuning

- Controller tuning---the process of selecting the controller parameters to meet given performance specifications
- PID tuning rules---selecting controller parameter values based on experimental step responses of the controlled plant
- The first PID tuning rules proposed by Ziegler and Nichols in 1942
PID Tuning---First Method

Start with obtaining the step response
The S-shaped Step Response

Parameters of the S-shaped step response

\[ c(t) \]

\[ K \]

Tangent line at inflection point

\[ 0 \]

\[ L \]

\[ T \]
Transfer Function of System With S-Shaped Step Response

- The S-shaped curve may be characterized by two parameters: delay time $L$ and time constant $T$
- The transfer function of such a plant may be approximated by a first-order system with a transport delay

\[
\frac{C(s)}{U(s)} = \frac{Ke^{-Ls}}{Ts + 1}
\]
## PID Tuning---First Method

<table>
<thead>
<tr>
<th>Type of Controller</th>
<th>$K_p$</th>
<th>$T_i$</th>
<th>$T_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>$\frac{T}{L}$</td>
<td>$\infty$</td>
<td>0</td>
</tr>
<tr>
<td>PI</td>
<td>$0.9 \frac{T}{L}$</td>
<td>$\frac{L}{0.3}$</td>
<td>0</td>
</tr>
<tr>
<td>PID</td>
<td>$1.2 \frac{T}{L}$</td>
<td>$2L$</td>
<td>$0.5L$</td>
</tr>
</tbody>
</table>
Transfer Function of PID Controller
Tuned Using the First Method

\[ G_c(s) = K_p \left( 1 + \frac{1}{T_i s} + T_d s \right) \]

\[ = 1.2 \frac{T}{L} \left( 1 + \frac{1}{2L s} + 0.5L s \right) \]

\[ = 0.6T \frac{(s + \frac{1}{L})^2}{s} \]
Ziegler-Nichols PID Tuning---Second Method

Use the proportional controller to force sustained oscillations

\[ r(t) \rightarrow + - \rightarrow K_p \rightarrow u(t) \rightarrow \text{Plant} \rightarrow c(t) \]
PID Tuning---Second Method

Measure the period of sustained oscillation
# PID Tuning Rules---Second Method

## Table 10–2  Ziegler–Nichols Tuning Rule Based on Critical Gain $K_{cr}$ and Critical Period $P_{cr}$ (Second Method)

<table>
<thead>
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<th>$K_p$</th>
<th>$T_i$</th>
<th>$T_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>$0.5K_{cr}$</td>
<td>$\infty$</td>
<td>0</td>
</tr>
<tr>
<td>PI</td>
<td>$0.45K_{cr}$</td>
<td>$\frac{1}{1.2} P_{cr}$</td>
<td>0</td>
</tr>
<tr>
<td>PID</td>
<td>$0.6K_{cr}$</td>
<td>$0.5P_{cr}$</td>
<td>$0.125P_{cr}$</td>
</tr>
</tbody>
</table>
Transfer Function of PID Controller Tuned Using the Second Method

\[ G_c(s) = K_p \left( 1 + \frac{1}{T_i s} + T_d s \right) \]

\[ = 0.6 K_{cr} \left( 1 + \frac{1}{0.5 P_{cr} s} + 0.125 P_{cr} s \right) \]

\[ = 0.075 K_{cr} P_{cr} \frac{\left( s + \frac{4}{P_{cr}} \right)^2}{s} \]
Example 1---PID Controller for DC Motor

- Plant---Armature-controlled DC motor; MOTOMATIC system produced by Electro-Craft Corporation
- Design a Type A PID controller and simulate the behavior of the closed-loop system; plot the closed-loop system step response
- Fine tune the controller parameters so that the max overshoot is 25% or less
Armature-Controlled DC Motor Modeling

\[ L_a \quad i_a \quad R_a \]

\[ u \quad e_b = \text{back emf} \]

\[ \theta_m = \text{motor shaft position} \]

\[ i_f = \text{constant} \]

armature circuit

field circuit
Oersted (1820): A current in a wire can produce magnetic effects; it can change the orientation of a compass needle.
Force Acting on a Moving Charge in a Magnetic Field

- **Force**
  \[ \vec{F} = q_0 \vec{v} \times \vec{B} \]

- **Magnitude**
  \[ F = q_0 v B \sin \theta \]

- The unit of B (flux density)---1 Tesla, where
  \[ 1 \text{ Tesla} = \frac{1 \text{ Weber}}{1 \text{ m}^2} = 10^4 \text{ Gauss} \]
Torque on a Current Loop

The force $F_4$ has the same magnitude as $F_2$ but points in the opposite direction.
An End View of the Current Loop

The common magnitude of $F_1$ and $F_3$ is $iaB$
Building a Motor From a Current Loop
DC Motor Construction

To keep the torque in the same direction as the loop rotates, change the direction of the current in the loop---do this using slip rings at 0 and \( \pi \) (pi) or - \( \pi \).

The brushes are fixed and the slip rings are connected to the current loop with electrical contact made by the loop’s slip rings sliding against the brushes.
Modeling Equations

- Kirchhoff’s Voltage Law to the armature circuit
  \[ U(s) = (L_a s + R_a) I_a(s) + E_b(s) \]

- Back-emf (equivalent to an “electrical friction”)
  \[ E_b(s) = K_b \omega_m(s) \]

- Torque developed by the motor
  \[ T_m(s) = (J_m s^2 + B_m s) \Theta_m(s) = (J_m s + B_m) \omega_m(s) \]

- Electromechanical coupling
  \[ T_m(s) = K_t I_a(s) \]
Relationship between $K_t$ and $K_b$

- Mechanical power developed in the motor armature (in watts)
  \[ p = e_b(t)i_a(t) \]

- Mechanical power can also be expressed as
  \[ p = T_m(t)\omega_m(t) \]

- Combine
  \[ p = T_m\omega_m = e_b i_a = K_b \omega_m \frac{T_m}{K_t} \]
In SI Units \( K_t = K_b \)

- The back-emf and the motor torque constants are equal in the SI unit system

\[
K_t \left( \frac{V}{\text{rad/sec}} \right) = K_b \left( \text{N} \cdot \text{m/A} \right)
\]
Transfer Function of the DC Motor System

- Transfer function of the DC motor

\[ G_p(s) = \frac{Y(s)}{U(s)} = \frac{0.1464}{7.89 \times 10^{-7}s^3 + 8.25 \times 10^{-4}s^2 + 0.00172s} \]

where \( Y(s) \) is the angular displacement of the motor shaft and \( U(s) \) is the armature voltage.
Tuning the Controller Using the Second Method of Ziegler and Nichols

- Use the Routh-Hurwitz stability test; see e.g. Section 5-6 of Ogata (2010)

- Determine $K_{cr}$

- Determine $P_{cr}$

- Compute the controller parameters
Generating the Step Response

t=0:0.00005:.017;
K_cr=12.28; P_cr=135;
K=0.075*K_cr*P_cr; a=4/P_cr;
num1=K*[1 2*a a^2]; den1=[0 1 0];
tf1=tf(num1,den1);
num2=[0 0 0 0.1464];
den2=[7.89e-007 8.25e-004 0.00172 0];
tf2=tf(num2,den2);
tf3=tf1*tf2;
sys=feedback(tf3,1);
y=step(sys,t); m=max(y);
Closed-Loop System Performance

Unit-step response for Type A PID

Closed-loop system output

$m = 1.7083$

$K = 124.335$

$a = 0.02963$
Example 2 (Based on Ex. 10-3 in Ogata, 2002)

- Use a computational approach to generate an optimal set of the DC motor PID controller’s parameters

\[ G_c(s) = K \frac{(s + a)^2}{s} \]

- Generate the step response of the closed-loop system
Optimizing PID Parameters

t=0:0.0002:0.02;
font=14;
for K=5:-0.2:2
  for a=1:-0.01:0.01
    num1=K*[1 2*a a^2]; den1=[0 1 0];
    tf1=tf(num1,den1);
    num2=[0 0 0 0.1464];
    den2=[7.89e-007 8.25e-004 0.00172 0];
    tf2=tf(num2,den2);
    tf3=tf1*tf2;
    sys=feedback(tf3,1);
    y=step(sys,t);
    m=max(y);

Finishing the Optimizing Program

if m<1.1 & m>1.05;
    plot(t,y); grid; set(gca,'Fontsize',font)
sol=[K;a;m]
    break % Breaks the inner loop
end
end

if m<1.1 & m>1.05;
    break; % Breaks the outer loop
end
end
Closed-Loop System Performance

Unit-step response

- Closed-loop system output
- Time (sec)

Graph parameters:
- \( m = 1.0999 \)
- \( K = 4.2 \)
- \( a = 1 \)
Modified PID Control Schemes

- If the reference input is a step, then because of the presence of the derivative term, the controller output will involve an impulse function.
- The derivative term also amplifies higher frequency sensor noise.
- Replace the pure derivative term with a derivative filter---PIDF controller.
- Set-Point Kick---for step reference the PIDF output will involve a sharp pulse function rather than an impulse function.
The Derivative Term

- Derivative action is useful for providing a phase lead, to offset phase lag caused by integration term.
- Differentiation increases the high-frequency gain.
- Pure differentiator is not proper or causal.
- 80% of PID controllers in use have the derivative part switched off.
- Proper use of the derivative action can increase stability and help maximize the integral gain for better performance.
Remedies for Derivative Action---PIDF Controller

- Pure differentiator approximation

\[
\frac{T_d s}{\gamma T_d s + 1}
\]

where \( \gamma \) is a small parameter, for example, 0.1

- Pure differentiator cascaded with a first-order low-pass filter
The Set-Point Kick Phenomenon

- If the reference input is a step function, the derivative term will produce an impulse (delta) function in the controller action.
- Possible remedy---operate the derivative action only in the feedback path; thus differentiation occurs only on the feedback signal and not on the reference signal.
Eliminating the Set-Point Kick

PID controller revisited
Eliminating the Set-Point Kick---
Finding the source of trouble

More detailed view of the PID controller
Eliminating the Set-Point Kick---PI-D Control or Type B PID

Operate derivative action only in the feedback
I-PD---Moving Proportional and Derivative Action to the Feedback

I-PD control or Type C PID
I-PD Equivalent to PID With Input Filter (No Noise)

Closed-loop transfer function $Y(s)/R(s)$ of the I-PD-controlled system

$$
Y(s) = \frac{K_p}{T_i s} \frac{G_p(s)}{1 + K_p \left(1 + \frac{1}{T_i s} + T_d s\right) G_p(s)}
$$
PID-Controlled System

- Closed-loop transfer function $Y(s)/R(s)$ of the PID-controlled system with input filter

$$
\frac{Y(s)}{R(s)} = \frac{1}{1 + T_i s + T_i T_d s^2} \frac{K_p \left( 1 + \frac{1}{T_i s} + T_d s \right) G_p(s)}{1 + K_p \left( 1 + \frac{1}{T_i s} + T_d s \right) G_p(s)}
$$

- After manipulations it is the same as the transfer function of the I-PD-controlled closed-loop system
In the absence of the reference input and noise signals, the closed-loop transfer function between the disturbance input and the system output is the same for the three types of PID control

\[
\frac{Y(s)}{D(s)} = \frac{G_p(s)}{1 + K_p G_p(s) \left( 1 + \frac{1}{T_i s} + T_d s \right)}
\]
The Three Terms of Proportional-Integral-Derivative (PID) Control

- Proportional term responds immediately to the current tracking error; it cannot achieve the desired setpoint accuracy without an unacceptably large gain. Needs the other terms.
- Derivative action reduces transient errors.
- Integral term yields zero steady-state error in tracking a constant setpoint. It also rejects constant disturbances.

Proportional-Integral-Derivative (PID) control provides an efficient solution to many real-world control problems.
Summary

- PID control---most widely used control strategy today
- Over 90% of control loops employ PID control, often the derivative gain set to zero (PI control)
- The three terms are intuitive---a non-specialist can grasp the essentials of the PID controller’s action. It does not require the operator to be familiar with advanced math to use PID controllers
- Engineers prefer PID controls over untested solutions