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On decentralised control of non-linear interconnected systems

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On decentralised control of non-linear interconnected systems

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The asymptotic stabilisation problem of a class of large-scale interconnected systems is considered, where the non-linear interconnections between subsystems satisfy quadratic constraints that are functions of the whole system's state vector. A decentralised combined observer-controller compensator is proposed and analysed, where the subsystems' state vectors are estimated using local sliding mode observers. The closed-loop system driven by the proposed decentralised compensator is guaranteed to be asymptotically stable subject to two conditions that are easily verifiable. Simulation results illustrate the effectiveness of the proposed decentralised combined observer-controller compensator.

Keywords: non-linear interconnected systems; decentralised control; sliding mode observer; asymptotic stabilisation

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1. Introduction

Power systems, transportation systems, manufacturing processes, communication networks, economic systems are some of the examples of large-scale interconnected systems. Traditional centralised control methodology of such systems requires means of exchanging information between the subsystems for the controller implementation, and, therefore, sufficiently large communication bandwidth is needed for transferring information between the subsystems. This makes centralised controllers expensive and highly complex when applied to large-scale interconnected systems. In addition, often, there are no means for subsystem information exchange, which prevents the application of centralised control. To overcome this, decentralised control methodology has been developed, which only uses local information available at each subsystem level for the controller implementation. Hence, in real applications, decentralised controllers are simpler and more practical than centralised controllers. An exhaustive list of publications on the subject of decentralised control of large-scale interconnected systems is given in Šiljak (1991). A survey of early results can be found in Sandell, Varaiya, Athans, and Safanov (1978). Decentralised adaptive controllers for large-scale interconnected systems with unknown parameters, non-linearities and disturbances have been studied in Ioannou (1986), Reed and Ioannou (1988), Gavel and Šiljak (1989), Fu (1992), Datta (1993), Spooner and Passino (1996), Spooner and Passino (1999), Huang, Tan, and Lee (2002), Hovakimyan, Lavretsky, Yang, and Calise (2005). Decentralised robust controller design in the presence of uncertainties can be found in Gong (1995), Gong, Wen, and Mital (1996), Zhang, Mizukami, and Wu (1996).

Most of the proposed decentralised control strategies assume that subsystem states are available for feedback implementation (see, for example, Ioannou 1986; Gavel and Šiljak 1989; Datta 1993; Gong 1995; Gong et al. 1996; Spooner and Passino 1996; Spooner and Passino 1999; Huang et al. 2002). However, in real applications, the availability of the states of each subsystem cannot be guaranteed. This motivated the development of decentralised output feedback controllers, which incorporate local observers to estimate the states of the subsystems (see, for example, Viswannadham and Ramakrishna 1982; Šiljak 1991; Wen 1994; Aldeen and Marsh 1999; Jiang 2000; Šiljak and Stipanovic 2001; Narendra and Oleng 2002; Hovakimyan et al. 2005; Pagilla and Zhu 2005).

In this article, we consider the stabilisation problem of a class of large-scale interconnected systems with linear subsystems and unknown non-linear interconnections. The interconnection of each subsystem satisfies a quadratic bound as, for example, in Šiljak and Stipanovic (2001), Pagilla and Zhu (2005), and can be expressed as a product of a known matrix and an unknown vector. We first present a decentralised state feedback controller based on the decentralised controller proposed in Pagilla and Zhu (2005). Then we propose and analyse a decentralised combined observer-controller compensator. This compensator

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incorporates local sliding mode observer described by Hui and Żak (2005) to estimate the subsystems' states. The closed-loop system driven by the proposed decentralised compensator is guaranteed to be asymptotically stable under two sufficient conditions, which are computable for each subsystem.

The remainder of the article is organised as follows. The system model and the problem statement are given in § 2. In § 3, we present a decentralised state feedback based control strategy. A local sliding mode observer is designed for each subsystem in § 4. In § 5, we construct the decentralised combined observer-controller compensator, and present the design algorithm. Simulation results illustrating the operation of the proposed decentralised compensator are included in § 6. Conclusions are found in § 7.

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2. System model and problem statement

The class of large-scale non-linear interconnected systems we consider is modelled by

$$\dot{\boldsymbol{x}}_i = \boldsymbol{A}_i\boldsymbol{x}_i + \boldsymbol{B}_{i1}\boldsymbol{u}_{i1} + \boldsymbol{z}_i(\boldsymbol{x}), \tag{1}$$

$$\boldsymbol{y}_i = \boldsymbol{C}_i\boldsymbol{x}_i, \quad i = 1, \ldots, N \tag{2}$$

where $\boldsymbol{x}_i \in \mathbb{R}^{n_i}$, $\boldsymbol{u}_{i1} \in \mathbb{R}^{m_{i1}}$, $\boldsymbol{y}_i \in \mathbb{R}^{p_i}$ are the state, input and output vectors, respectively, of the ith subsystem, $\boldsymbol{x} = [\boldsymbol{x}_1^\top \cdots \boldsymbol{x}_N^\top]^\top \in \mathbb{R}^n$, $n = \sum_{i=1}^N n_i$, is the state vector of the whole system, and $\boldsymbol{z}_i(\boldsymbol{x})$ models the interconnection of the ith subsystem with other subsystems.

The following assumptions are imposed on the subsystems.

Assumption 1: The input matrix $\boldsymbol{B}_{i1} \in \mathbb{R}^{n_i \times m_{i1}}$ ($m_{i1} \le n_i$) has full rank, the pair $(\boldsymbol{A}_i, \boldsymbol{B}_{i1})$ is controllable, and the pair $(\boldsymbol{A}_i, \boldsymbol{C}_i)$ is observable.

Assumption 2: The interconnection $\boldsymbol{z}_i(\boldsymbol{x})$ satisfies the following quadratic constraint,

$$\boldsymbol{z}_i^\top(\boldsymbol{x})\boldsymbol{z}_i(\boldsymbol{x}) \le \alpha_i^2 \boldsymbol{x}^\top \boldsymbol{Z}_i^\top \boldsymbol{Z}_i \boldsymbol{x}, \tag{3}$$

where α_i is a known positive constant and $\boldsymbol{Z}_i \in \mathbb{R}^{n_i \times n}$ is a known interconnection matrix.

Assumption 3: The interconnection $\boldsymbol{z}_i(\boldsymbol{x})$ can be represented as

$$\boldsymbol{z}_i(\boldsymbol{x}) = \boldsymbol{B}_{i2}\boldsymbol{u}_{i2}(\boldsymbol{x}), \tag{4}$$

with some known matrix $\boldsymbol{B}_{i2} \in \mathbb{R}^{n_i \times m_{i2}}$ and bounded $\boldsymbol{u}_{i2} \in \mathbb{R}^{m_{i2}}$ satisfying

$$\|\boldsymbol{u}_{i2}\|_2 \le \rho_i, \tag{5}$$

where $\rho_i > 0$ and $\|\bullet\|_2$ is the standard Euclidean vector norm.

Substituting (4) into (1), we obtain

$$\dot{\boldsymbol{x}}_i = \boldsymbol{A}_i\boldsymbol{x}_i + \boldsymbol{B}_{i1}\boldsymbol{u}_{i1} + \boldsymbol{B}_{i2}\boldsymbol{u}_{i2}(\boldsymbol{x}), \tag{6}$$

where $\boldsymbol{u}_{i2}(\boldsymbol{x})$ can be viewed as an unknown input of the ith subsystem.

Assumption 4: For the triple $(\boldsymbol{A}_i, \boldsymbol{B}_{i2}, \boldsymbol{C}_i)$, we assume that

$$\operatorname{rank}(\boldsymbol{B}_{i2}) = \operatorname{rank}(\boldsymbol{C}_i\boldsymbol{B}_{i2}) = r_i, \tag{7}$$

where $r_i \le m_{i2} \le p_i$, and the system zeros of the system model given by the triple $(\boldsymbol{A}_i, \boldsymbol{B}_{i2}, \boldsymbol{C}_i)$ are in the open left-hand complex plane, that is,

$$\operatorname{rank}\begin{bmatrix} s\boldsymbol{I}_{n_i} - \boldsymbol{A}_i & \boldsymbol{B}_{i2} \\ \boldsymbol{C}_i & \boldsymbol{O} \end{bmatrix} = n_i + r_i \tag{8}$$

for all s such that $\Re(s) \ge 0$.

The control objective of this article is to design the control inputs $\boldsymbol{u}_{i1}$ that stabilise the closed-loop system. In the following, we first develop a decentralised state feedback controller that guarantees the asymptotic stability of the closed-loop system. Then, local sliding mode state observers for subsystems are designed. Finally, a decentralised combined observer-controller compensator is constructed and analysed.

3. Decentralised state feedback controller development

In this section, we first assume that the system states $\boldsymbol{x}_i$ are available to us. We present a decentralised state feedback controller, of which the basic idea is from Pagilla and Zhu (2005). Consider the following local state feedback controller for the ith subsystem:

$$\boldsymbol{u}_{i1} = \boldsymbol{K}_i\boldsymbol{x}_i \tag{9}$$

with

$$\boldsymbol{K}_i = \boldsymbol{K}_{i1} + \boldsymbol{K}_{i2}, \tag{10}$$

where $\boldsymbol{K}_{i1}$ is a chosen pre-feedback gain matrix so that $\boldsymbol{A}_{ci} = \boldsymbol{A}_i + \boldsymbol{B}_{i1}\boldsymbol{K}_{i1}$ is Hurwitz, and $\boldsymbol{K}_{i2}$ is a feedback gain matrix to be determined later. Substituting (9) into (6) gives

$$\dot{\boldsymbol{x}}_i = \boldsymbol{A}_{ci}\boldsymbol{x}_i + \boldsymbol{B}_{i1}\boldsymbol{K}_{i2}\boldsymbol{x}_i + \boldsymbol{B}_{i2}\boldsymbol{u}_{i2}(\boldsymbol{x}). \tag{11}$$

It follows from the controllability of the pair $(\boldsymbol{A}_i, \boldsymbol{B}_{i1})$ that the pair $(\boldsymbol{A}_{ci}, \boldsymbol{B}_{i1})$ is controllable, because the state feedback does not change the controllability. In addition, because $\boldsymbol{B}_{i1}$ has full rank, it is easy to verify that the pair $(\boldsymbol{A}_{ci}, \boldsymbol{B}_{i1}(\boldsymbol{B}_{i1}^\top\boldsymbol{B}_{i1})^{-\frac{1}{2}})$ is also controllable.

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We define the distance, $\delta(\boldsymbol{A},\boldsymbol{B})$, between the pair $(\boldsymbol{A},\boldsymbol{B})$ and the set of pairs with an uncontrollable purely imaginary mode as

$$\delta(\boldsymbol{A},\boldsymbol{B}) = \min_{\omega\in\mathbb{R}} \sigma_{\min}\left(\begin{bmatrix} j\omega \boldsymbol{I} - \boldsymbol{A} & \boldsymbol{B}\end{bmatrix}\right),$$

where $\sigma_{\min}(\bullet)$ denotes the smallest singular value of $\bullet$. The above definition is an adaptation of the distance between the pair $(\boldsymbol{A},\boldsymbol{B})$ and the set of uncontrollable pairs introduced by Eising (1984). An efficient bisection algorithm for computing $\delta(\boldsymbol{A},\boldsymbol{B})$ can be obtained by substituting $(\boldsymbol{A}^\top,\boldsymbol{B}^\top)$ into the algorithm presented in Aboky, Sallet and Vivalda (2002), where the distance between the pair $(\boldsymbol{A},\boldsymbol{C})$ and the set of pairs with an unobservable purely imaginary mode is considered. In order to determine the feedback gain matrix $\boldsymbol{K}_{i2}$, we require an additional assumption on each subsystem.

Assumption 5: For the controllable pair $(\boldsymbol{A}_{ci},\boldsymbol{B}_{i1})$, we assume that

$$\delta\left(\boldsymbol{A}_{ci}, \sqrt{2\beta}\boldsymbol{B}_{i1}\left(\boldsymbol{B}_{i1}^\top\boldsymbol{B}_{i1}\right)^{-\frac{1}{2}}\right) > \sqrt{2\beta}, \tag{12}$$

where

$$\beta = \sum_{i=1}^{N} \alpha_i^2 \lambda_{\max}\left(\boldsymbol{Z}_i^\top \boldsymbol{Z}_i\right) \tag{13}$$

with α_i and $\boldsymbol{Z}_i$ defined in (3).

To proceed, we need the following two lemmas.

Lemma 1: *For a controllable pair* $(\boldsymbol{A},\boldsymbol{B})$, *if* $\delta(\boldsymbol{A},\sqrt{2\beta}\boldsymbol{B}) > \sqrt{2\beta}$, *then there exists a* $\gamma_i^* > 0$ *such that* $\delta(\boldsymbol{A},\sqrt{2(\beta+\gamma_i)}\boldsymbol{B}) > \sqrt{2(\beta+\gamma_i)}$ for $\gamma_i \in [0\ \gamma_i^*)$.

proof: See Appendix 1 □

Lemma 2: *For the following quadratic matrix equation,*

$$\boldsymbol{A}^\top\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{P}\boldsymbol{R}\boldsymbol{P} + \boldsymbol{Q} = \boldsymbol{O}, \tag{14}$$

if $\boldsymbol{R}=\boldsymbol{R}^\top \geq 0$, $\boldsymbol{Q}=\boldsymbol{Q}^\top > 0$, $\boldsymbol{A}$ is Hurwitz and the associated Hamiltonian matrix

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{R} \\ -\boldsymbol{Q} & -\boldsymbol{A}^\top \end{bmatrix}$$

has no eigenvalues on the imaginary axis (i.e. $\boldsymbol{H}$ is hyperbolic), then there exist symmetric positive definite solutions $\boldsymbol{P}$ to the quadratic matrix equation (14).

Proof: If $\boldsymbol{R}=\boldsymbol{R}^\top \geq 0$, $\boldsymbol{A}$ is Hurwitz and the Hamiltonian matrix $\boldsymbol{H}$ has no eigenvalues on the imaginary axis, there exist symmetric matrices $\boldsymbol{P}=\boldsymbol{P}^\top$ to the quadratic matrix equation (14). A proof of this fact is given in Francis (1987). Then we can rearrange (14) to obtain

$$\boldsymbol{A}^\top\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{P}\boldsymbol{R}\boldsymbol{P} - \boldsymbol{Q}.$$

Because $\boldsymbol{R}=\boldsymbol{R}^\top \geq 0$, $\boldsymbol{P}=\boldsymbol{P}^\top$ and $\boldsymbol{Q}=\boldsymbol{Q}^\top > 0$, therefore $-\boldsymbol{P}\boldsymbol{R}\boldsymbol{P} - \boldsymbol{Q}$ is negative definite. In addition, $\boldsymbol{A}$ is Hurwitz, so $\boldsymbol{P}$ is positive definite, which concludes the proof of the lemma. □

Remark 1: Pagilla and Zhu (Pagilla and Zhu 2005) stated in Lemma 1 that there exists a unique symmetric positive definite solution $\boldsymbol{P}$ to the quadratic matrix equation (14). They attributed their lemma to (Aboky et al. 2002). However, the statement of Pagilla and Zhu is incorrect, because Aboky et al. (2002) only claimed that the solution $\boldsymbol{P}$ to the quadratic matrix equation with $-(\boldsymbol{A}+\boldsymbol{R}\boldsymbol{P})$ stable is unique. Actually, there are at least two positive definite solutions $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ such that $\boldsymbol{A}+\boldsymbol{R}\boldsymbol{P}_1$ and $-(\boldsymbol{A}+\boldsymbol{R}\boldsymbol{P}_2)$ are stable, respectively. The following numerical counterexample illustrates this fact.

Let

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -1.5 & -1.25 \end{bmatrix}, \quad \boldsymbol{R} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and}$$

$$\boldsymbol{Q} = \begin{bmatrix} 0.18 & 0 \\ 0 & 0.18 \end{bmatrix}.$$

Then we can verify that the associated Hamiltonian matrix with the quadratic matrix equation (14) is hyperbolic, and there are two solutions to (14),

$$\begin{bmatrix} 0.4872 & 0.2182 \\ 0.2182 & 0.2847 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0.7628 & 0.4479 \\ 0.4479 & 0.5909 \end{bmatrix},$$

which are both symmetric positive definite.

Proposition 1: *For the controllable pair* $(\boldsymbol{A}_{ci},\boldsymbol{B}_{i1})$, *if Assumption* 5 *holds, there exists a* $\gamma_i^* > 0$ *such that there exist symmetric positive definite solutions* $\boldsymbol{P}_i^c$ *to the following quadratic matrix equation,*

$$\boldsymbol{A}_{ci}^\top\boldsymbol{P}_i^c + \boldsymbol{P}_i^c\boldsymbol{A}_{ci} + 2\boldsymbol{P}_i^c\left(\boldsymbol{I}_{n_i} - \boldsymbol{B}_{i1}\left(\boldsymbol{B}_{i1}^\top\boldsymbol{B}_{i1}\right)^{-1}\boldsymbol{B}_{i1}^\top\right)\boldsymbol{P}_i^c + (\beta+\gamma_i)\boldsymbol{I}_{n_i} = \boldsymbol{O} \tag{15}$$

for $\gamma_i \in [0\ \gamma_i^*)$.

Proof: Let $\boldsymbol{\Pi}_i = \boldsymbol{I}_{n_i} - \boldsymbol{B}_{i1}(\boldsymbol{B}_{i1}^\top\boldsymbol{B}_{i1})^{-1}\boldsymbol{B}_{i1}^\top$. It is easy to verify that $\boldsymbol{\Pi}_i$ is symmetric and idempotent, that is, $\boldsymbol{\Pi}_i^\top = \boldsymbol{\Pi}_i$ and $\boldsymbol{\Pi}_i^2 = \boldsymbol{\Pi}_i$. It follows from Lemma 1 in Hui and Żak (2005) that the eigenvalues of $\boldsymbol{\Pi}_i$ are 0s and 1s, which implies that $\boldsymbol{\Pi}_i \geq 0$. For the quadratic matrix equation (15), the associated Hamiltonian matrix is

$$\boldsymbol{H}_i = \begin{bmatrix} \boldsymbol{A}_{ci} & \boldsymbol{R}_i \\ -\boldsymbol{Q}_i^c & -\boldsymbol{A}_{ci}^\top \end{bmatrix}, \tag{16}$$

where $\boldsymbol{R}_i = 2\boldsymbol{\Pi}_i \geq 0$ and $\boldsymbol{Q}_i^c = (\beta + \gamma_i)\boldsymbol{I}_{n_i} > 0$. If Assumption 5 holds, then it follows from Lemma 1 that there exists a $\gamma_i^* > 0$ such that

$$\delta\left(\boldsymbol{A}_{ci}, \sqrt{2(\beta+\gamma_i)}\boldsymbol{B}_{i1}\left(\boldsymbol{B}_{i1}^\top \boldsymbol{B}_{i1}\right)^{-\frac{1}{2}}\right) > \sqrt{2(\beta+\gamma_i)} \quad (17)$$

for $\gamma_i \in [0\ \gamma_i^*)$, which implies that the above Hamiltonian matrix $\boldsymbol{H}_i$ has no eigenvalues on the imaginary axis; a proof is given in Appendix 2. Then it follows from Lemma 2 that there exist symmetric positive definite solutions $\boldsymbol{P}_i^c$ to the quadratic matrix equation (15) for $\gamma_i \in [0\ \gamma_i^*)$, which concludes the proof of the proposition. □

Theorem 1: *For the interconnected system with the ith subsystem modelled by* (11), *if Assumptions* 1, 2, 3 *and* 5 *hold, there exists a* $\gamma_i^* > 0$ *such that the closed-loop system driven by the decentralised state feedback controller* (9) *is asymptotically stable, where*

$$\boldsymbol{K}_{i2} = -\left(\boldsymbol{B}_{i1}^\top \boldsymbol{B}_{i1}\right)^{-1} \boldsymbol{B}_{i1}^\top \boldsymbol{P}_i^c \quad (18)$$

and $\boldsymbol{P}_i^c$ *is a positive definite solution to the quadratic matrix equation* (15) *for* $\gamma_i \in [0\ \gamma_i^*)$.

Proof: If Assumptions 1, 2, 3 and 5 hold, then it follows from Proposition 1 that there exists a $\gamma_i^* > 0$ such that there exist symmetric positive definite solutions $\boldsymbol{P}_i^c$ to the following quadratic matrix equation,

$$\boldsymbol{A}_{ci}^\top \boldsymbol{P}_i^c + \boldsymbol{P}_i^c \boldsymbol{A}_{ci} + 2\boldsymbol{P}_i^c\left(\boldsymbol{I} - \boldsymbol{B}_{i1}\left(\boldsymbol{B}_{i1}^\top \boldsymbol{B}_{i1}\right)^{-1}\boldsymbol{B}_{i1}^\top\right)\boldsymbol{P}_i^c + (\beta+\gamma_i)\boldsymbol{I}_{n_i} = \boldsymbol{O}$$

for $\gamma_i \in [0\ \gamma_i^*)$. Then we consider the following Lyapunov function candidate,

$$V = \boldsymbol{x}^\top \boldsymbol{P}^c \boldsymbol{x} = \sum_{i=1}^N \boldsymbol{x}_i^\top \boldsymbol{P}_i^c \boldsymbol{x}_i,$$

where $\boldsymbol{P}^c = \text{diag}[\boldsymbol{P}_1^c \ \cdots \ \boldsymbol{P}_N^c]$. Evaluating the time derivative of V on the solutions of (11) yields

$$\dot{V} = \sum_{i=1}^N 2\boldsymbol{x}_i^\top \boldsymbol{P}_i^c \dot{\boldsymbol{x}}_i = \sum_{i=1}^N \left[2\boldsymbol{x}_i^\top \boldsymbol{P}_i^c\left(\boldsymbol{A}_{c_i} + \boldsymbol{B}_{i1}\boldsymbol{K}_{i2}\right)\boldsymbol{x}_i + 2\boldsymbol{x}_i^\top \boldsymbol{P}_i^c(\boldsymbol{B}_{i2}\boldsymbol{u}_{i2})\right]. \quad (19)$$

Using the well-known result, $2\boldsymbol{a}^\top \boldsymbol{b} \leq \boldsymbol{a}^\top \boldsymbol{a} + \boldsymbol{b}^\top \boldsymbol{b}$, for any vector $\boldsymbol{a}$ and $\boldsymbol{b}$, we have

$$2\boldsymbol{x}_i^\top \boldsymbol{P}_i^c(\boldsymbol{B}_{i2}\boldsymbol{u}_{i2}) \leq \boldsymbol{x}_i^\top \boldsymbol{P}_i^c \boldsymbol{P}_i^c \boldsymbol{x}_i + (\boldsymbol{B}_{i2}\boldsymbol{u}_{i2})^\top(\boldsymbol{B}_{i2}\boldsymbol{u}_{i2}) \quad (20)$$

$$\leq 2\boldsymbol{x}_i^\top \boldsymbol{P}_i^c \boldsymbol{P}_i^c \boldsymbol{x}_i + (\boldsymbol{B}_{i2}\boldsymbol{u}_{i2})^\top(\boldsymbol{B}_{i2}\boldsymbol{u}_{i2}). \quad (21)$$

It follows from (3) that

$$(\boldsymbol{B}_{i2}\boldsymbol{u}_{i2})^\top(\boldsymbol{B}_{i2}\boldsymbol{u}_{i2}) \leq \alpha_i^2 \boldsymbol{x}^\top \boldsymbol{Z}_i^\top \boldsymbol{Z}_i \boldsymbol{x} \leq \alpha_i^2 \lambda_{\max}\left(\boldsymbol{Z}_i^\top \boldsymbol{Z}_i\right)\boldsymbol{x}^\top \boldsymbol{x} = \beta_i \sum_{j=1}^N \boldsymbol{x}_j^\top \boldsymbol{x}_j, \quad (22)$$

where $\beta_i = \alpha_i^2 \lambda_{\max}(\boldsymbol{Z}_i^\top \boldsymbol{Z}_i)$. Substituting (22) into (21), we obtain

$$2\boldsymbol{x}_i^\top \boldsymbol{P}_i^c(\boldsymbol{B}_{i2}\boldsymbol{u}_{i2}) \leq \beta_i \sum_{j=1}^N \boldsymbol{x}_j^\top \boldsymbol{x}_j + 2\boldsymbol{x}_i^\top \boldsymbol{P}_i^c \boldsymbol{P}_i^c \boldsymbol{x}_i. \quad (23)$$

It follows from (19) and (23) that

$$\begin{aligned}\dot{V} &= \sum_{i=1}^N \left[2\boldsymbol{x}_i^\top \boldsymbol{P}_i^c\left(\boldsymbol{A}_{c_i} + \boldsymbol{B}_{i1}\boldsymbol{K}_{i2}\right)\boldsymbol{x}_i + 2\boldsymbol{x}_i^\top \boldsymbol{P}_i^c(\boldsymbol{B}_{i2}\boldsymbol{u}_{i2})\right] \\ &\leq \sum_{i=1}^N \boldsymbol{x}_i^\top\left(2\boldsymbol{P}_i^c\left(\boldsymbol{A}_{c_i} + \boldsymbol{B}_{i1}\boldsymbol{K}_{i2}\right) + 2\boldsymbol{P}_i^c\boldsymbol{P}_i^c\right)\boldsymbol{x}_i + \sum_{i=1}^N\left(\beta_i \sum_{j=1}^N \boldsymbol{x}_j^\top \boldsymbol{x}_j\right) \\ &= \sum_{i=1}^N \boldsymbol{x}_i^\top\left(2\boldsymbol{P}_i^c\left(\boldsymbol{A}_{c_i} + \boldsymbol{B}_{i1}\boldsymbol{K}_{i2}\right) + 2\boldsymbol{P}_i^c\boldsymbol{P}_i^c\right)\boldsymbol{x}_i + \sum_{i=1}^N\left(\sum_{j=1}^N \beta_j\right)\boldsymbol{x}_i^\top \boldsymbol{x}_i \\ &= \sum_{i=1}^N \boldsymbol{x}_i^\top\left(2\boldsymbol{P}_i^c\left(\boldsymbol{A}_{c_i} + \boldsymbol{B}_{i1}\boldsymbol{K}_{i2}\right) + 2\boldsymbol{P}_i^c\boldsymbol{P}_i^c + \beta\boldsymbol{I}_{n_i}\right)\boldsymbol{x}_i, \end{aligned} \quad (24)$$

where β is defined in (13). Substituting $\boldsymbol{K}_{i2}$ given in (18) into (24), we obtain

$$\begin{aligned}\dot{V} &\leq \sum_{i=1}^N \boldsymbol{x}_i^\top\left(2\boldsymbol{P}_i^c\left(\boldsymbol{A}_{c_i} + \boldsymbol{B}_{i1}\boldsymbol{K}_{i2}\right) + 2\boldsymbol{P}_i^c\boldsymbol{P}_i^c + \beta\boldsymbol{I}_{n_i}\right)\boldsymbol{x}_i \\ &= \sum_{i=1}^N \boldsymbol{x}_i^\top\Big(\boldsymbol{A}_{c_i}^\top \boldsymbol{P}_i^c + \boldsymbol{P}_i^c \boldsymbol{A}_{c_i} + 2\boldsymbol{P}_i^c\left(\boldsymbol{I} - \boldsymbol{B}_{i1}\left(\boldsymbol{B}_{i1}^\top \boldsymbol{B}_{i1}\right)^{-1}\boldsymbol{B}_{i1}^\top\right)\boldsymbol{P}_i^c + \beta\boldsymbol{I}_{n_i}\Big)\boldsymbol{x}_i. \\ &= -\sum_{i=1}^N \gamma_i \boldsymbol{x}_i^\top \boldsymbol{x}_i \\ &< 0.\end{aligned}$$

Hence, the closed-loop system is asymptotically stable, which concludes the proof of the theorem. □

4. Local sliding mode observer design

We now assume that only the system outputs $\boldsymbol{y}_i$ are available to us. In such a case, we are not able to implement the decentralised state feedback controller

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presented in the previous section. Thus, we need local state observers to provide estimates of the subsystems' state vectors for the feedback implementation.

If Assumptions 1, 3 and 4 hold for each subsystem, we can design the following sliding mode state observer, described in detail in Hui and Żak (2005), for the ith subsystem,

$$\dot{\hat{\boldsymbol{x}}}_i = (\boldsymbol{A}_i - \boldsymbol{L}_i\boldsymbol{C}_i)\hat{\boldsymbol{x}}_i + \boldsymbol{L}_i\boldsymbol{y}_i + \boldsymbol{B}_{i1}\boldsymbol{u}_{i1} - \boldsymbol{B}_{i2}\boldsymbol{E}_i(\boldsymbol{y}_i, \hat{\boldsymbol{y}}_i, \eta_i) \tag{25}$$

with $\hat{\boldsymbol{y}}_i = \boldsymbol{C}_i\hat{\boldsymbol{x}}_i$ and

$$\boldsymbol{E}_i(\boldsymbol{y}_i, \hat{\boldsymbol{y}}_i, \eta_i) = \begin{cases} \eta_i \dfrac{\boldsymbol{F}_i(\hat{\boldsymbol{y}}_i - \boldsymbol{y}_i)}{\|\boldsymbol{F}_i(\hat{\boldsymbol{y}}_i - \boldsymbol{y}_i)\|_2} & \text{if } \boldsymbol{F}_i(\hat{\boldsymbol{y}}_i - \boldsymbol{y}_i) \neq \boldsymbol{0} \\ \boldsymbol{0} & \text{if } \boldsymbol{F}_i(\hat{\boldsymbol{y}}_i - \boldsymbol{y}_i) = \boldsymbol{0}, \end{cases}$$

where η_i is a design parameter such that $\eta_i \geq \rho_i$, and $\boldsymbol{L}_i \in \mathbb{R}^{n_i \times p_i}$ and $\boldsymbol{F}_i \in \mathbb{R}^{m_{i2} \times p_i}$ satisfy

$$(\boldsymbol{A}_i - \boldsymbol{L}_i\boldsymbol{C}_i)^\top \boldsymbol{P}_i^o + \boldsymbol{P}_i^o(\boldsymbol{A}_i - \boldsymbol{L}_i\boldsymbol{C}_i) < 0 \tag{26}$$

and

$$\boldsymbol{F}_i\boldsymbol{C}_i = \boldsymbol{B}_{i2}^\top \boldsymbol{P}_i^o, \tag{27}$$

for some symmetric positive definite $\boldsymbol{P}_i^o \in \mathbb{R}^{n_i \times n_i}$. It follows from Hui and Żak (2005) that the $\hat{\boldsymbol{x}}_i$ is an asymptotic estimate of the state $\boldsymbol{x}_i$ of the ith subsystem for $\eta_i \geq \rho_i$.

In the following, we present a design procedure for the triple of matrices $(\boldsymbol{L}_i, \boldsymbol{F}_i, \boldsymbol{P}_i^o)$ such that (26) and (27) are satisfied.

It follows from Lemma 2 in Hui and Żak (2005) that if (7) is satisfied, i.e. $\text{rank}(\boldsymbol{B}_{i2}) = \text{rank}(\boldsymbol{C}_i\boldsymbol{B}_{i2}) = r_i$, then there exist non-singular matrices $\boldsymbol{T}_i$ and $\boldsymbol{S}_i$ such that

$$\hat{\boldsymbol{A}}_i = \boldsymbol{T}_i\boldsymbol{A}_i\boldsymbol{T}_i^{-1} = \begin{bmatrix} \boldsymbol{A}_{i_{11}} & \boldsymbol{A}_{i_{12}} \\ \boldsymbol{A}_{i_{21}} & \boldsymbol{A}_{i_{22}} \end{bmatrix}, \tag{28}$$

$$\hat{\boldsymbol{B}}_{i2} = \boldsymbol{T}_i\boldsymbol{B}_{i2} = \begin{bmatrix} \boldsymbol{B}_{i2_1} \\ \boldsymbol{O} \end{bmatrix}, \tag{29}$$

$$\hat{\boldsymbol{C}}_i = \boldsymbol{S}_i\boldsymbol{C}_i\boldsymbol{T}_i^{-1} = \begin{bmatrix} \boldsymbol{I}_{r_i} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{C}_{i_{22}} \end{bmatrix}, \tag{30}$$

where $\boldsymbol{A}_{i_{11}} \in \mathbb{R}^{r_i \times r_i}, \boldsymbol{A}_{i_{22}} \in \mathbb{R}^{(n_i - r_i)\times(n_i - r_i)}, \boldsymbol{B}_{i_{21}} \in \mathbb{R}^{r_i \times m_{i2}}$ and $\boldsymbol{C}_{i_{22}} \in \mathbb{R}^{(p_i - r_i)\times(n_i - r_i)}$. On the other hand, if (8) is satisfied, there exists a matrix $\boldsymbol{L}_{i_{22}}$ such that the eigenvalues of $(\boldsymbol{A}_{i_{22}} - \boldsymbol{L}_{i_{22}}\boldsymbol{C}_{i_{22}})$ are in the open left-hand complex plane. A proof is given in Hui and Żak (2005). Then there exists a symmetric positive definite solution $\boldsymbol{P}_{i_{22}}^o$ to the following Lyapunov equation,

$$\left(\boldsymbol{A}_{i_{22}} - \boldsymbol{L}_{i_{22}}\boldsymbol{C}_{i_{22}}\right)^\top \boldsymbol{P}_{i_{22}}^o + \boldsymbol{P}_{i_{22}}^o\left(\boldsymbol{A}_{i_{22}} - \boldsymbol{L}_{i_{22}}\boldsymbol{C}_{i_{22}}\right) = -\boldsymbol{Q}_{i_{22}}^o$$

for $\boldsymbol{Q}_{i_{22}}^o = \hat{\boldsymbol{Q}}_{i_{22}}^o + 2\delta_i\boldsymbol{I}_{n_i - r_i}$ with $\hat{\boldsymbol{Q}}_{i_{22}}^o$ symmetric positive definite and $\delta_i = \sigma_{\max}^2(\boldsymbol{B}_{i1}\boldsymbol{K}_i)$, where $\boldsymbol{K}_i$ is defined in (10) and $\sigma_{\max}(\bullet)$ denotes the largest singular value of $\bullet$. Next, define

$$\hat{\boldsymbol{P}}_i^o = \begin{bmatrix} \boldsymbol{I}_{r_i} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{P}_{i_{22}}^o \end{bmatrix}, \tag{31}$$

$$\hat{\boldsymbol{F}}_i = \begin{bmatrix} \boldsymbol{B}_{i2_1}^\top & \boldsymbol{O} \end{bmatrix}, \tag{32}$$

$$\hat{\boldsymbol{L}}_i = \begin{bmatrix} (\kappa_i + \delta_i)\boldsymbol{I}_{r_i} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{L}_{i_{22}} \end{bmatrix}, \tag{33}$$

where $\kappa_i > 0$ is a design parameter to be determined.

Remark 2: The above design procedure is a modification of that presented in Hui and Żak (2005); where $\boldsymbol{Q}_{i_{22}}^o$ is a symmetric positive definite matrix, and $\hat{\boldsymbol{L}}_i$ is constructed using κ. In our design procedure, a constant $\delta_i > 0$ is included into the construction of $\boldsymbol{Q}_{i_{22}}^o$ and $\hat{\boldsymbol{L}}_i$. This modification is essential for achieving the closed-loop system stability of the combined observer-controller compensators presented in § 5.

It follows from (28), (30) and (33) that

$$\hat{\boldsymbol{A}}_i - \hat{\boldsymbol{L}}_i\hat{\boldsymbol{C}}_i = \begin{bmatrix} \boldsymbol{A}_{i_{11}} - (\kappa_i + \delta_i)\boldsymbol{I}_{r_i} & \boldsymbol{A}_{i_{12}} \\ \boldsymbol{A}_{i_{21}} & \boldsymbol{A}_{i_{22}} - \boldsymbol{L}_{i_{22}}\boldsymbol{C}_{i_{22}} \end{bmatrix}.$$

Let

$$\begin{aligned} \hat{\boldsymbol{Q}}_i^o &= -\left(\hat{\boldsymbol{A}}_i - \hat{\boldsymbol{L}}_i\hat{\boldsymbol{C}}_i\right)^\top \hat{\boldsymbol{P}}_i^o - \hat{\boldsymbol{P}}_i^o\left(\hat{\boldsymbol{A}}_i - \hat{\boldsymbol{L}}_i\hat{\boldsymbol{C}}_i\right) \\ &= \begin{bmatrix} (\kappa_i + \delta_i)\boldsymbol{I}_{r_i} - \boldsymbol{A}_{i_{11}}^\top & -\boldsymbol{A}_{i_{21}}^\top \boldsymbol{P}_{i_{22}}^o \\ -\boldsymbol{A}_{i_{12}}^\top & -\left(\boldsymbol{A}_{i_{22}} - \boldsymbol{L}_{i_{22}}\boldsymbol{C}_{i_{22}}\right)^\top \boldsymbol{P}_{i_{22}}^o \end{bmatrix} \\ &\quad + \begin{bmatrix} (\kappa_i + \delta_i)\boldsymbol{I}_{r_i} - \boldsymbol{A}_{i_{11}} & -\boldsymbol{A}_{i_{12}} \\ -\boldsymbol{P}_{i_{22}}^o\boldsymbol{A}_{i_{21}} & -\boldsymbol{P}_{i_{22}}^o\left(\boldsymbol{A}_{i_{22}} - \boldsymbol{L}_{i_{22}}\boldsymbol{C}_{i_{22}}\right) \end{bmatrix} \\ &= \begin{bmatrix} 2(\kappa_i + \delta_i)\boldsymbol{I}_{r_i} - \boldsymbol{A}_{i_{11}}^\top - \boldsymbol{A}_{i_{11}} & -\boldsymbol{A}_{i_{12}} - \boldsymbol{A}_{i_{21}}^\top \boldsymbol{P}_{i_{22}}^o \\ -\boldsymbol{A}_{i_{12}}^\top - \boldsymbol{P}_{i_{22}}^o\boldsymbol{A}_{i_{21}} & \boldsymbol{Q}_{i_{22}}^o \end{bmatrix} \\ &= \hat{\boldsymbol{Q}}_{i1}^o + \hat{\boldsymbol{Q}}_{i2}^o, \end{aligned} \tag{34}$$

where

$$\hat{\boldsymbol{Q}}_{i1}^o = \begin{bmatrix} 2\kappa_i\boldsymbol{I}_{r_i} - \boldsymbol{A}_{i_{11}}^\top - \boldsymbol{A}_{i_{11}} & -\boldsymbol{A}_{i_{12}} - \boldsymbol{A}_{i_{21}}^\top \boldsymbol{P}_{i_{22}}^o \\ -\boldsymbol{A}_{i_{12}}^\top - \boldsymbol{P}_{i_{22}}^o\boldsymbol{A}_{i_{21}} & \hat{\boldsymbol{Q}}_{i_{22}}^o \end{bmatrix},$$

$$\hat{\boldsymbol{Q}}_{i2}^o = \begin{bmatrix} 2\delta_i\boldsymbol{I}_{r_i} & \boldsymbol{O} \\ \boldsymbol{O} & 2\delta_i\boldsymbol{I}_{n_i - r_i} \end{bmatrix} = 2\delta_i\boldsymbol{I}_{n_i}.$$

Because $\hat{Q}^o_{i_{22}}$ is symmetric positive definite, $\hat{Q}^o_{i1}$ is symmetric positive definite if and only if

$$2\kappa_i I_{r_i} - \left(A^\top_{i_{11}} + A_{i_{11}}\right) - \left(A_{i_{12}} + A^\top_{i_{21}} P^o_{i_{22}}\right) \times \hat{Q}^{o-1}_{i_{22}} \left(A^\top_{i_{12}} + P^o_{i_{22}} A_{i_{21}}\right), \tag{35}$$

which is the Schur complement of $\hat{Q}^o_{i_{22}}$, is symmetric positive definite. Hence, if

$$\kappa_i > \frac{1}{2}\lambda_{\max}\left(A^\top_{i_{11}} + A_{i_{11}} + \left(A_{i_{12}} + A^\top_{i_{21}} P^o_{i_{22}}\right) \times \hat{Q}^{o-1}_{i_{22}} \left(A^\top_{i_{12}} + P^o_{i_{22}} A_{i_{21}}\right)\right),$$

then (35) is symmetric positive definite, which implies that $\hat{Q}^o_{i1}$ is symmetric positive definite. On the other hand, it is clear that $\hat{Q}^o_{i2}$ is also symmetric positive definite. Thus, it follows from (34) that $\hat{Q}^o_i$ is symmetric positive definite. Let

$$P^o_i = T^\top_i \hat{P}^o_i T_i \tag{36}$$

and $F_i = \hat{F}_i S_i$, $L_i = T^{-1}_i \hat{L}_i S_i$ and $Q^o_i = T^\top_i \hat{Q}^o_i T_i$. We have $P^o_i > 0$ and $Q^o_i > 0$ because $\hat{P}^o_i > 0$ and $\hat{Q}^o_i > 0$. It follows from (28)–(33) that

$$\begin{aligned}(A_i - L_i C_i)^\top P^o_i + P^o_i (A_i - L_i C_i) &= T^\top_i \left(\left(\hat{A}_i - \hat{L}_i \hat{C}_i\right)^\top \hat{P}^o_i + \hat{P}^o_i \left(\hat{A}_i - \hat{L}_i \hat{C}_i\right)\right) T_i \\ &= -T^\top_i \hat{Q}^o_i T_i = -Q^o_i\end{aligned} \tag{37}$$

and $F_i C_i = \hat{F}_i \hat{C}_i T_i = \hat{B}^\top_{i2} \hat{P}^o_i T_i = B^\top_{i2} P^o_i$.

5. Decentralised combined observer-controller compensator construction

In this section, we combine the decentralised state feedback controller presented in § 3 and local sliding mode state observers described in § 4 to construct a decentralised combined observer-controller compensator. The local compensator for the ith subsystem is

$$u_{i1} = K_i \hat{x}_i, \tag{38}$$

where K_i is given by (10) and (18), and $\hat{x}_i$ is the estimated state vector of the ith subsystem obtained from the local sliding mode observer (25). The architecture of the decentralised combined controller-observer compensator is shown in Figure 1.

To proceed, we need the following lemma.

Lemma 3: *Let* $Q_i = (B_{i1} K_i)^\top (B_{i1} K_i)$. *We have* $\lambda_{\min}(Q^o_i) > \lambda_{\max}(Q_i)$.

Proof: See Appendix 3. □

Theorem 2: For the interconnected system with the ith subsystem modelled by (6), if Assumptions 1–5 hold, then the closed-loop system driven by the decentralised combined observer-controller compensator (38) is asymptotically stable.

Proof: Substituting (38) into (6), we obtain

$$\begin{aligned}\dot{x}_i &= A_i x_i + B_{i1} K_i \hat{x}_i + B_{i2} u_{i2}(x) \\ &= A_i x_i + B_{i1} K_i x_i + B_{i1} K_i e_i + B_{i2} u_{i2}(x) \\ &= (A_{ci} + B_{i1} K_{i2}) x_i + B_{i1} K_i e_i + B_{i2} u_{i2}(x).\end{aligned} \tag{39}$$

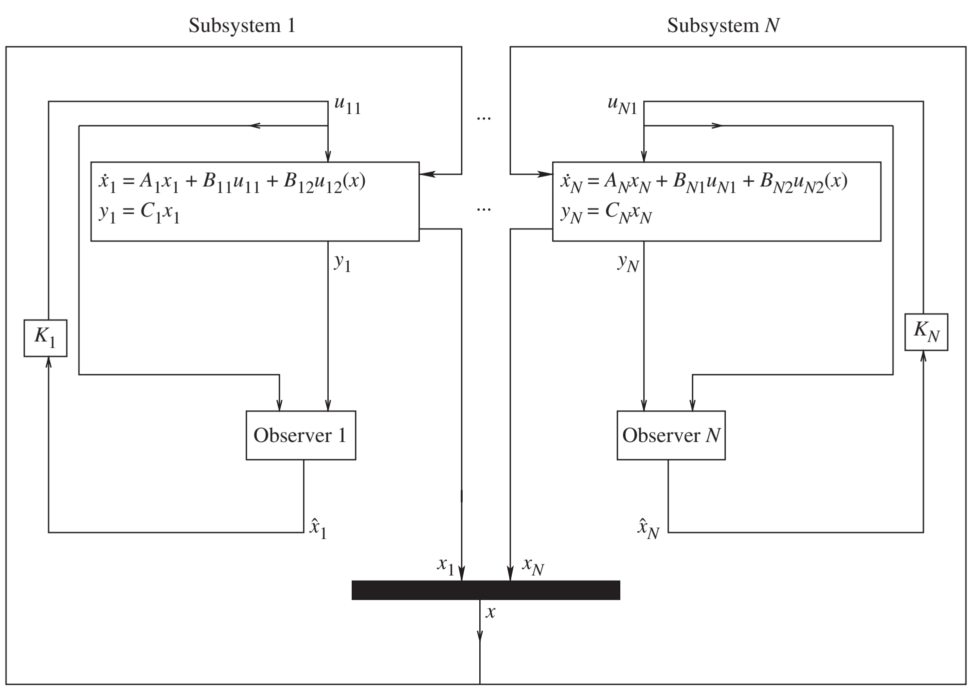

Figure 1. Diagram of the decentralised combined observer-controller compensator.

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Let $e_i = \hat{x}_i - x_i$ be the estimation error for the ith subsystem state vector. Then it follows from (6) and (25) that

$$\dot{e}_i = (A_i - L_iC_i)e_i - B_{i2}u_{i2}(x) - B_{i2}E_i(y_i, \hat{y}_i, \eta_i). \quad (40)$$

Let $P^c = \text{diag}[P_1^c \cdots P_N^c]$ and $P^o = \text{diag}[P_1^o \cdots P_N^o]$, where P_i^c is a symmetric positive definite solution to the quadratic matrix equation (15) for $\gamma_i \in [0\ \gamma_i^*)$ and P_i^o is defined in (36). Now we consider the following Lyapunov function candidate:

$$V = x^\top P^c x + e^\top P^o e = \sum_{i=1}^{N}\left(x_i^\top P_i^c x_i + e_i^\top P_i^o e_i\right).$$

Evaluating the time derivative of V on the solutions of (39) and (40), we obtain:

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{N}\left(2x_i^\top P_i^c \dot{x}_i + 2e_i^\top P_i^o \dot{e}_i\right)\\ &= \sum_{i=1}^{N}\big[2x_i^\top P_i^c(A_{ci} + B_{i1}K_{i2})x_i + 2x_i^\top P_i^c(B_{i1}K_i)e_i\\ &\quad + 2x_i^\top P_i^c(B_{i2}u_{i2})\big] + \sum_{i=1}^{N}\big[2e_i^\top P_i^o(A_i - L_iC_i)e_i\\ &\quad - 2e_i^\top P_i^o B_{i2}E_i(y_i, \hat{y}_i, \eta_i) - 2e_i^\top P_i^o(B_{i2}u_{i2})\big]. \end{aligned} \quad (41)$$

We have

$$2x_i^\top P_i^c(B_{i1}K_i)e_i \le x_i^\top P_i^c P_i^c x_i + e_i^\top (B_{i1}K_i)^\top (B_{i1}K_i)e_i. \quad (42)$$

If follows from (24), (41) and (42) that

$$\begin{aligned}\dot{V} &\le \sum_{i=1}^{N}\big[2x_i^\top P_i^c(A_{ci} + B_{i1}K_{i2})x_i + 2x_i^\top P_i^c P_i^c x_i\\ &\quad + (B_{i2}u_{i2})^\top(B_{i2}u_{i2})\big] + \sum_{i=1}^{N}\big[2e_i^\top P_i^o(A_i - L_iC_i)e_i\\ &\quad - 2e_i^\top P_i^o B_{i2}E_i(y_i, \hat{y}_i, \eta_i) - 2e_i^\top P_i^o(B_{i2}u_{i2})\\ &\quad + e_i^\top(B_{i1}K_i)^\top(B_{i1}K_i)e_i\big]\\ &\le \sum_{i=1}^{N}\big[x_i^\top\left(2P_i^c(A_{ci} + B_{i1}K_{i2}) + 2P_i^cP_i^c + \beta I_{n_i}\right)x_i\big]\\ &\quad + \sum_{i=1}^{N}\big[2e_i^\top P_i^o(A_i - L_iC_i)e_i - 2e_i^\top P_i^o B_{i2}E_i(y_i, \hat{y}_i, \eta_i)\\ &\quad - 2e_i^\top P_i^o(B_{i2}u_{i2}) + e_i^\top(B_{i1}K_i)^\top(B_{i1}K_i)e_i\big]\\ &= \sum_{i=1}^{N}\left(\dot{V}_{ci} + \dot{V}_{oi}\right),\end{aligned}$$

where

$$\dot{V}_{ci} = x_i^\top\left(2P_i^c(A_{ci} + B_{i1}K_{i2}) + 2P_i^cP_i^c + \beta I_{n_i}\right)x_i \quad (43)$$

and

$$\begin{aligned}\dot{V}_{oi} &= 2e_i^\top P_i^o(A_i - L_iC_i)e_i - 2e_i^\top P_i^o B_{i2}E_i(y_i, \hat{y}_i, \eta_i)\\ &\quad - 2e_i^\top P_i^o(B_{i2}u_{i2}) + e_i^\top(B_{i1}K_i)^\top(B_{i1}K_i)e_i. \end{aligned} \quad (44)$$

Substituting K_{i2} given by (18) into (43), we obtain

$$\begin{aligned}\dot{V}_{ci} &= x_i^\top\Big(A_{c_i}^\top P_i^c + P_i^c A_{c_i}\\ &\quad + 2P_i^c\Big(I - B_{i1}\left(B_{i1}^\top B_{i1}\right)^{-1}B_{i1}^\top\Big)P_i^c + \beta I_{n_i}\Big)x_i.\end{aligned}$$

Because P_i^c defined in Theorem 1 satisfies (15), we have $\dot{V}_{ci} = -\gamma_i x_i^\top x_i < 0$. If $F_i(\hat{y}_i - y_i) = 0$, it follows from (27) that $F_iC_ie_i = B_{i2}^\top P_i^o e_i = 0$, and then

$$-2e_i^\top P_i^o B_{i2}E_i(y_i, \hat{y}_i, \eta_i) - 2e_i^\top P_i^o(B_{i2}u_{i2}) = 0. \quad (45)$$

On the other hand, if $F_i(\hat{y}_i - y_i) \neq 0$, it follows from (5), (27) and $\eta_i \ge \rho_i$ that

$$\begin{aligned}&-2e_i^\top P_i^o B_{i2}E_i(y_i, \hat{y}_i, \eta_i) - 2e_i^\top P_i^o(B_{i2}u_{i2})\\ &\quad = -\frac{2\eta_i}{\|F_iC_ie_i\|_2}\left(e_i^\top P_i^o B_{i2}\right)(F_iC_ie_i) - 2e_i^\top P_i^o(B_{i2}u_{i2})\\ &\quad \le -2\eta_i\left\|e_i^\top P_i^o B_{i2}\right\|_2 + 2\rho_i\left\|e_i^\top P_i^o B_{i2}\right\|_2\\ &\quad = -2(\eta_i - \rho_i)\left\|e_i^\top P_i^o B_{i2}\right\|_2\\ &\quad \le 0. \end{aligned} \quad (46)$$

Thus, it follows from (37), (44), (45) and (46) that

$$\begin{aligned}\dot{V}_{oi} &= 2e_i^\top P_i^o(A_i - L_iC_i)e_i - 2e_i^\top P_i^o B_{i2}E_i(y_i, \hat{y}_i, \eta_i)\\ &\quad - 2e_i^\top P_i^o(B_{i2}u_{i2}) + e_i^\top(B_{i1}K_i)^\top(B_{i1}K_i)e_i\\ &\le -e_i^\top Q_i^o e_i + e_i^\top(B_{i1}K_i)^\top(B_{i1}K_i)e_i\\ &= -e_i^\top Q_i^o e_i + e_i^\top Q_i e_i. \end{aligned} \quad (47)$$

Note that $Q_i = Q_i^\top \ge 0$. It follows from Lemma 3 and (47) that

$$\begin{aligned}\dot{V}_{oi} &\le -\lambda_{\min}\left(Q_i^o\right)e_i^\top e_i + \lambda_{\max}(Q_i)e_i^\top e_i\\ &= -\left(\lambda_{\min}\left(Q_i^o\right) - \lambda_{\max}(Q_i)\right)\|e_i\|_2^2\\ &< 0.\end{aligned}$$

Thus, we have

$$\dot{V} \le \sum_{i=1}^{N}\left(\dot{V}_{ci} + \dot{V}_{oi}\right) < 0.$$

Hence, the closed-loop system is asymptotically stable, which concludes the proof of the theorem.

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We now summarise the main results of this article in the following:

Design Algorithm for the Decentralised Combined Observer-Controller Compensator

(1) Select a pre-feedback gain matrix $\boldsymbol{K}_{i1}$ such that $\boldsymbol{A}_{c_i} = \boldsymbol{A}_i + \boldsymbol{B}_{i1}\boldsymbol{K}_{i1}$ is Hurwitz;

(2) Check if Assumptions 4 and 5 are satisfied. If not, the proposed combined observer-controller compensator cannot be designed; if yes, go to (3);

(3) Choose γ_i small enough to find a symmetric positive definite solution $\boldsymbol{P}_i^c$ to the quadratic matrix equation (15), and then choose $\boldsymbol{K}_{i2} = -(\boldsymbol{B}_{i1}^\top \boldsymbol{B}_{i1})^{-1}\boldsymbol{B}_{i1}^\top \boldsymbol{P}_i^c$;

(4) Find the non-singular matrices $\boldsymbol{T}_i$ and $\boldsymbol{S}_i$ such that the triple $(\boldsymbol{A}_i, \boldsymbol{B}_{i2}, \boldsymbol{C}_i)$ can be transformed into the following form,

$$\hat{\boldsymbol{A}}_i = \boldsymbol{T}_i\boldsymbol{A}_i\boldsymbol{T}_i^{-1} = \begin{bmatrix} \boldsymbol{A}_{i_{11}} & \boldsymbol{A}_{i_{12}} \\ \boldsymbol{A}_{i_{21}} & \boldsymbol{A}_{i_{22}} \end{bmatrix},$$

$$\hat{\boldsymbol{B}}_{i2} = \boldsymbol{T}_i\boldsymbol{B}_{i2} = \begin{bmatrix} \boldsymbol{B}_{i2_1} \\ \boldsymbol{O} \end{bmatrix},$$

$$\hat{\boldsymbol{C}}_i = \boldsymbol{S}_i\boldsymbol{C}_i\boldsymbol{T}_i^{-1} = \begin{bmatrix} \boldsymbol{I}_{r_i} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{C}_{i_{22}} \end{bmatrix},$$

where $\boldsymbol{A}_{i_{11}} \in \mathbb{R}^{r_i \times r_i}$, $\boldsymbol{A}_{i_{22}} \in \mathbb{R}^{(n_i - r_i)\times(n_i - r_i)}$, $\boldsymbol{B}_{i_{21}} \in \mathbb{R}^{r_i \times m_{i2}}$ and $\boldsymbol{C}_{i_{22}} \in \mathbb{R}^{(p_i - r_i)\times(p_i - r_i)}$;

(5) Choose $\boldsymbol{L}_{i_{22}}$ such that $\boldsymbol{A}_{i_{22}} - \boldsymbol{L}_{i_{22}}\boldsymbol{C}_{i_{22}}$ is Hurwitz. Solve $\boldsymbol{P}_{i_{22}}^o$ for

$$\left(\boldsymbol{A}_{i_{22}} - \boldsymbol{L}_{i_{22}}\boldsymbol{C}_{i_{22}}\right)^\top \boldsymbol{P}_{i_{22}}^o + \boldsymbol{P}_{i_{22}}^o\left(\boldsymbol{A}_{i_{22}} - \boldsymbol{L}_{i_{22}}\boldsymbol{C}_{i_{22}}\right) = -\boldsymbol{Q}_{i_{22}}^o,$$

where $\boldsymbol{Q}_{i_{22}}^o = \hat{\boldsymbol{Q}}_{i_{22}}^o + 2\delta_i \boldsymbol{I}_{n_i - r_i}$, $\hat{\boldsymbol{Q}}_{i_{22}}^o$ is a symmetric positive definite matrix and $\delta_i = \sigma_{\max}^2(\boldsymbol{B}_{i1}\boldsymbol{K}_i)$ with $\boldsymbol{K}_i = \boldsymbol{K}_{i1} + \boldsymbol{K}_{i2}$;

(6) Choose κ_i such that

$$\kappa_i > \frac{1}{2}\lambda_{\max}\Big(\boldsymbol{A}_{i_{11}}^\top + \boldsymbol{A}_{i_{11}} + \Big(\boldsymbol{A}_{i_{12}} + \boldsymbol{A}_{i_{21}}^\top \boldsymbol{P}_{i_{22}}^o\Big) \times \hat{\boldsymbol{Q}}_{i_{22}}^{o-1}\Big(\boldsymbol{A}_{i_{12}}^\top + \boldsymbol{P}_{i_{22}}^o \boldsymbol{A}_{i_{21}}\Big)\Big);$$

(7) Construct $\hat{\boldsymbol{P}}_i^o$, $\hat{\boldsymbol{F}}_i$ and $\hat{\boldsymbol{L}}_i$ as

$$\hat{\boldsymbol{P}}_i^o = \begin{bmatrix} \boldsymbol{I}_{r_i} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{P}_{i_{22}}^o \end{bmatrix},$$

$$\hat{\boldsymbol{F}}_i = \begin{bmatrix} \boldsymbol{B}_{i2_1}^\top & \boldsymbol{O} \end{bmatrix},$$

$$\hat{\boldsymbol{L}}_i = \begin{bmatrix} (\kappa_i + \delta_i)\boldsymbol{I}_{r_i} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{L}_{i_{22}} \end{bmatrix}.$$

and compute $\boldsymbol{P}_i^o = \boldsymbol{T}_i^\top \hat{\boldsymbol{P}}_i^o \boldsymbol{T}_i$, $\boldsymbol{F}_i = \hat{\boldsymbol{F}}_i\boldsymbol{S}_i$, $\boldsymbol{L}_i = \boldsymbol{T}_i^{-1}\hat{\boldsymbol{L}}_i\boldsymbol{S}_i$;

(8) Construct local sliding mode observers,

$$\dot{\hat{\boldsymbol{x}}}_i = (\boldsymbol{A}_i - \boldsymbol{L}_i\boldsymbol{C}_i)\hat{\boldsymbol{x}}_i + \boldsymbol{L}_i\boldsymbol{y}_i + \boldsymbol{B}_{i1}\boldsymbol{u}_{i1} - \boldsymbol{B}_{i2}\boldsymbol{E}_i(\boldsymbol{y}_i, \hat{\boldsymbol{y}}_i, \eta_i),$$

with $\hat{\boldsymbol{y}}_i = \boldsymbol{C}_i\hat{\boldsymbol{x}}_i$ and

$$\boldsymbol{E}_i(\boldsymbol{y}_i, \hat{\boldsymbol{y}}_i, \eta_i) = \begin{cases} \eta_i \dfrac{\boldsymbol{F}_i(\hat{\boldsymbol{y}}_i - \boldsymbol{y}_i)}{\left\|\boldsymbol{F}_i(\hat{\boldsymbol{y}}_i - \boldsymbol{y}_i)\right\|_2} & \text{if } \boldsymbol{F}_i(\hat{\boldsymbol{y}}_i - \boldsymbol{y}_i) \neq \boldsymbol{0} \\ \boldsymbol{0} & \text{if } \boldsymbol{F}_i(\hat{\boldsymbol{y}}_i - \boldsymbol{y}_i) = \boldsymbol{0}; \end{cases}$$

(9) Design the decentralised combined observer-controller compensator $\boldsymbol{u}_{i1} = \boldsymbol{K}_i\hat{\boldsymbol{x}}_i$.

6. Examples

In this section, we illustrate the performance of our proposed decentralised combined observer-controller compensator with two examples. In the first example, the non-linear interconnected system used in Pagilla and Zhu (2005) to test their proposed controller is employed. However, even the uncontrolled system is already stable. So, in the second example, we modify the system in Pagilla and Zhu (2005) to make it more challenging to control.

Example 1: The non-linear interconnected system model in Pagilla and Zhu (2005) consists of two subsystems. The first subsystem is a second-order system, and the second subsystem is a third-order system. Let $\boldsymbol{x}_1 = [x_1\ x_2]^\top$ be the state vector of the first subsystem, $\boldsymbol{x}_2 = [x_3\ x_4\ x_5]^\top$ be the state vector of the second subsystem and let $\boldsymbol{x} = [\boldsymbol{x}_1^\top\ \boldsymbol{x}_2^\top]^\top$ be the state vector of the whole system.

The dynamics of the first subsystem are given by

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= \boldsymbol{A}_1\boldsymbol{x}_1 + \boldsymbol{B}_{11}u_{11} + z_1(\boldsymbol{x}) \\ &= \begin{bmatrix} 0 & 1 \\ -125 & -22.5 \end{bmatrix}\boldsymbol{x}_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u_{11} + z_1(\boldsymbol{x}), \\ y_1 &= \boldsymbol{c}_1\boldsymbol{x}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}\boldsymbol{x}_1, \end{aligned}$$

where $z_1(\boldsymbol{x}) = \alpha_1 \cos(x_4)\boldsymbol{Z}_1\boldsymbol{x}$ with $\alpha_1 = 0.2$ and

$$\boldsymbol{Z}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

We can represent the dynamics of the first subsystem in the form

$$\dot{\boldsymbol{x}}_1 = \begin{bmatrix} 0 & 1 \\ -125 & -22.5 \end{bmatrix}\boldsymbol{x}_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u_{11} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u_{12}(\boldsymbol{x}).$$

where $u_{12}(\boldsymbol{x}) = 0.2\cos(x_4)\sum_{i=1}^5 x_i/\sqrt{10}$.

The second subsystem's dynamics are given by

$$\begin{aligned} \dot{\boldsymbol{x}}_2 &= \boldsymbol{A}_2\boldsymbol{x}_2 + \boldsymbol{B}_{21}u_{21} + z_2(\boldsymbol{x}) \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -37.5 & -50 & -13.5 \end{bmatrix}\boldsymbol{x}_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u_{21} + z_2(\boldsymbol{x}), \\ y_2 &= \boldsymbol{c}_2\boldsymbol{x}_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\boldsymbol{x}_2, \end{aligned}$$

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where $z_2(\boldsymbol{x}) = \alpha_2 \cos(x_1)\boldsymbol{Z}_2\boldsymbol{x}$ with $\alpha_2 = 0.2$ and

$$\boldsymbol{Z}_2 = \frac{1}{\sqrt{15}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

We represent the dynamics of the second subsystem as

$$\dot{\boldsymbol{x}}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -37.5 & -50 & -13.5 \end{bmatrix}\boldsymbol{x}_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u_{21} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}u_{22}(\boldsymbol{x}).$$

where $u_{22}(\boldsymbol{x}) = 0.2\cos(x_1)\sum_{i=1}^{5} x_i/\sqrt{15}$.

It is easy to check that for each subsystem, Assumptions 1, 2 and 4 hold. Because $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are stable, we take the pre-feedback gain matrices $\boldsymbol{K}_{11}$ and $\boldsymbol{K}_{21}$ to be zero, that is, $\boldsymbol{A}_{c1} = \boldsymbol{A}_1$ and $\boldsymbol{A}_{c2} = \boldsymbol{A}_1$. It follows from (13) that $\beta = 0.08$. Then we get

$$\delta\left(\boldsymbol{A}_{c1}, \sqrt{2\beta}\boldsymbol{B}_{11}\left(\boldsymbol{B}_{11}^{\top}\boldsymbol{B}_{11}\right)^{-\frac{1}{2}}\right) = 0.9841 > \sqrt{2\beta},$$

and

$$\delta\left(\boldsymbol{A}_{c2}, \sqrt{2\beta}\boldsymbol{B}_{21}\left(\boldsymbol{B}_{21}^{\top}\boldsymbol{B}_{21}\right)^{-\frac{1}{2}}\right) = 0.5865 > \sqrt{2\beta},$$

which implies that Assumption 5 holds for each subsystem. As far as Assumption 3 is concerned, we select $\rho_1 = \rho_2 = 10.0$ based on the rough knowledge of the system operating range. We select $\gamma_1 = 0.1$ and $\gamma_2 = 0.01$ such that they satisfy the condition of Lemma 1, and then we compute $\boldsymbol{P}_1^c$ and $\boldsymbol{P}_2^c$ to be

$$\boldsymbol{P}_1^c = \begin{bmatrix} 0.5957 & 0.0036 \\ 0.0036 & 0.0042 \end{bmatrix},$$

$$\boldsymbol{P}_2^c = \begin{bmatrix} 0.2552 & 0.1932 & 0.0039 \\ 0.1932 & 0.2791 & 0.0071 \\ 0.0039 & 0.0071 & 0.0039 \end{bmatrix}.$$

Using $\boldsymbol{P}_1^c$ and $\boldsymbol{P}_2^c$, we obtain

$$\boldsymbol{K}_{12} = \begin{bmatrix} -0.0036 & -0.0042 \end{bmatrix},$$

and

$$\boldsymbol{K}_{22} = \begin{bmatrix} -0.0039 & -0.0071 & -0.0039 \end{bmatrix}.$$

The initial conditions for two subsystems are chosen to be $\boldsymbol{x}_1(0) = [5\ \ 5]^{\top}$ and $\boldsymbol{x}_2(0) = [5\ \ 5\ \ 5]^{\top}$, respectively. Then, according to the design procedure presented in § 4, we obtain

$$\boldsymbol{L}_1 = \begin{bmatrix} 2.3362 & 2.3362 \end{bmatrix}^{\top},$$

and

$$\boldsymbol{L}_2 = \begin{bmatrix} 9.0197 & 9.0197 & 9.0197 \end{bmatrix}^{\top},$$

and $\boldsymbol{F}_1 = \boldsymbol{F}_2 = 1$. The total simulation time is 10 s. Simulation results for the first subsystem and the second subsystem are shown, respectively, in Figures 2 and 3. In Figure 4, plots of the control inputs u_{11} and u_{21} are shown.

Example 2: For the modified system, the first subsystem's dynamics are

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= \boldsymbol{A}_1\boldsymbol{x}_1 + \boldsymbol{B}_{11}u_{11} + z_1(\boldsymbol{x}) \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{x}_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u_{11} + z_1(\boldsymbol{x}), \\ \boldsymbol{y}_1 &= \boldsymbol{c}_1\boldsymbol{x}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}\boldsymbol{x}_1, \end{aligned}$$

While the second subsystem's dynamics are

$$\begin{aligned} \dot{\boldsymbol{x}}_2 &= \boldsymbol{A}_2\boldsymbol{x}_2 + \boldsymbol{B}_{21}u_{21} + z_2(\boldsymbol{x}) \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -40.8 & -41.5 & -9.35 \end{bmatrix}\boldsymbol{x}_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u_{21} + z_2(\boldsymbol{x}), \\ \boldsymbol{y}_2 &= \boldsymbol{c}_2\boldsymbol{x}_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\boldsymbol{x}_2, \end{aligned}$$

where $z_1(\boldsymbol{x})$ and $z_2(\boldsymbol{x})$ are the same as in Example 1. The initial conditions for two subsystems are still chosen to be $\boldsymbol{x}_1(0) = [5\ \ 5]^{\top}$ and $\boldsymbol{x}_2(0) = [5\ \ 5\ \ 5]^{\top}$, respectively.

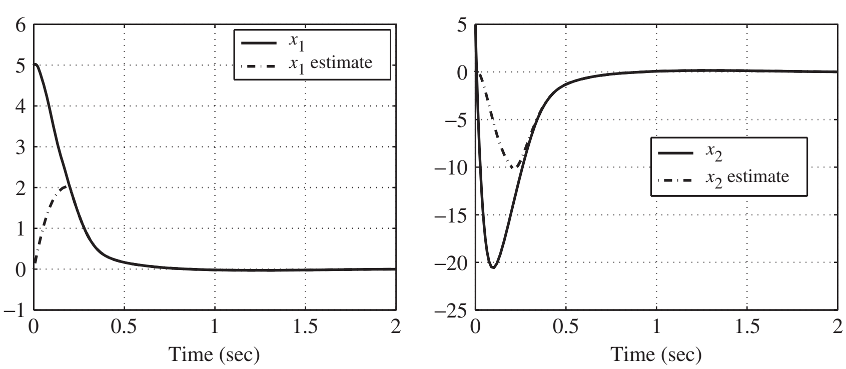

Figure 2. Decentralised combined observer-controller compensator performance for the first subsystem in Example 1.

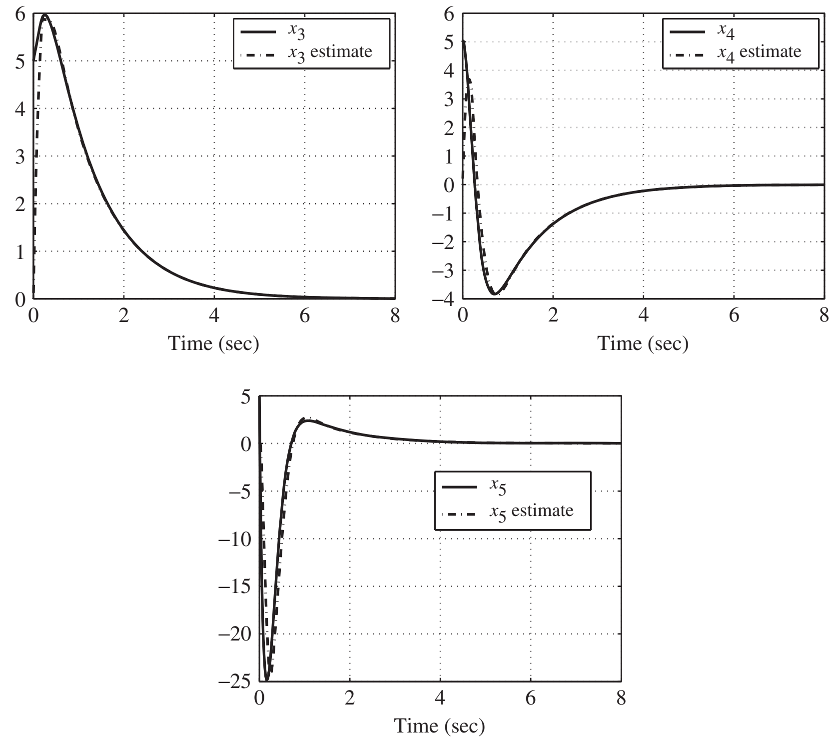

Figure 3. Decentralised combined observer-controller compensator performance for the second subsystem in Example 1.

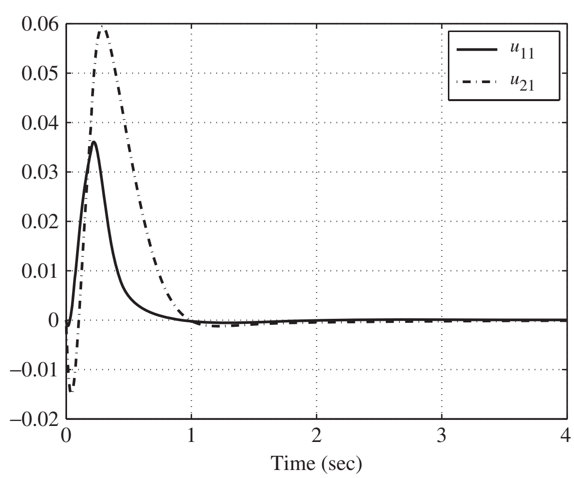

Figure 4. Control inputs u_{11} and u_{12} in Example 1.

Plots of the uncontrolled system states are shown in Figure 5. We can see that the uncontrolled system is unstable.

To stabilise $\boldsymbol{A}_1$, we choose $\boldsymbol{K}_{11}=[-1.5\ -1.25]$ so that $\boldsymbol{A}_{c1}=\boldsymbol{A}_1+\boldsymbol{B}_{11}\boldsymbol{K}_{11}$ is stable. On the other hand, $\boldsymbol{A}_2$ is stable, so we take $\boldsymbol{K}_{21}$ to be zero, that is, $\boldsymbol{A}_{c2}=\boldsymbol{A}_2$. We still have $\beta=0.08$ for the modified system, and then we check

$$\delta\left(\boldsymbol{A}_{c1},\sqrt{2\beta}\boldsymbol{B}_{11}\left(\boldsymbol{B}_{11}^{\top}\boldsymbol{B}_{11}\right)^{-\frac{1}{2}}\right)=0.5660>\sqrt{2\beta},$$

and

$$\delta\left(\boldsymbol{A}_{c2},\sqrt{2\beta}\boldsymbol{B}_{21}\left(\boldsymbol{B}_{21}^{\top}\boldsymbol{B}_{21}\right)^{-\frac{1}{2}}\right)=0.5578>\sqrt{2\beta},$$

We choose $\rho_1=\rho_2=10.0$, $\gamma_1=0.1$ and $\gamma_2=0.01$. Then we compute $\boldsymbol{P}_1^c$ and $\boldsymbol{P}_2^c$ to be

$$\boldsymbol{P}_1^c=\begin{bmatrix}0.4872 & 0.2182\\ 0.2182 & 0.2847\end{bmatrix},$$

$$\boldsymbol{P}_2^c=\begin{bmatrix}4.7139 & 3.9301 & 0.9243\\ 3.9301 & 3.8380 & 0.8229\\ 0.9243 & 0.8229 & 0.2566\end{bmatrix}.$$

Using $\boldsymbol{P}_1^c$ and $\boldsymbol{P}_2^c$, we obtain

$$\boldsymbol{K}_{12}=\begin{bmatrix}-0.2182 & -0.2847\end{bmatrix},$$

and

$$\boldsymbol{K}_{22}=\begin{bmatrix}-0.9243 & -0.8229 & -0.2566\end{bmatrix}.$$

Then we obtain

$$\boldsymbol{L}_1=\begin{bmatrix}7.8700 & 7.8700\end{bmatrix}^{\top},$$

and

$$\boldsymbol{L}_2=\begin{bmatrix}14.2412 & 14.2412 & 14.2412\end{bmatrix}^{\top},$$

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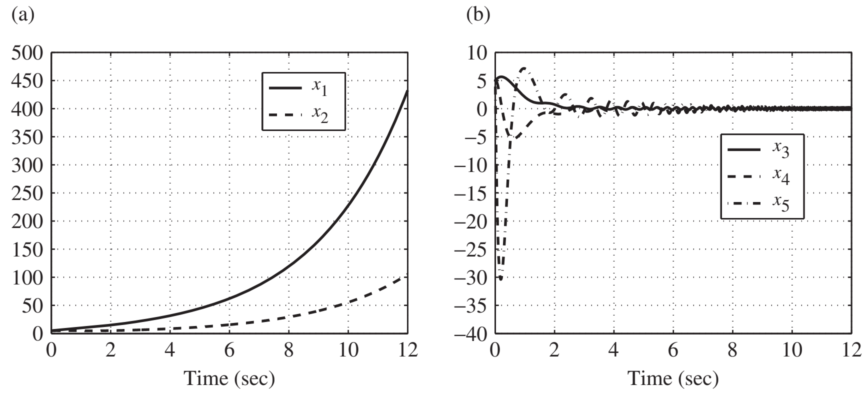

Figure 5. Plots of the uncontrolled system states in Example 2. (a) first subsystem; (b) second subsystem.

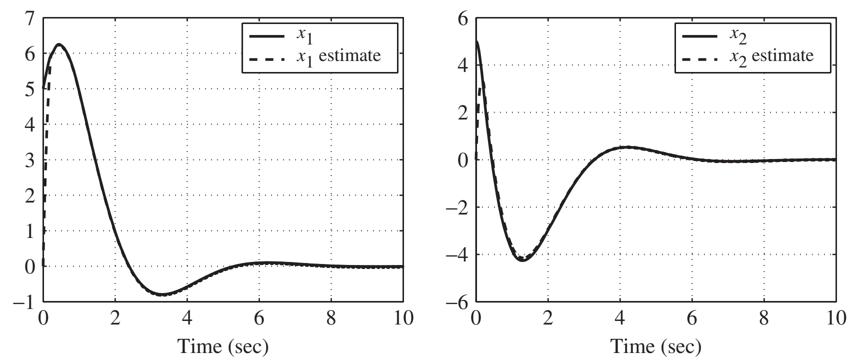

Figure 6. Decentralised combined observer-controller compensator performance for the first subsystem in Example 2.

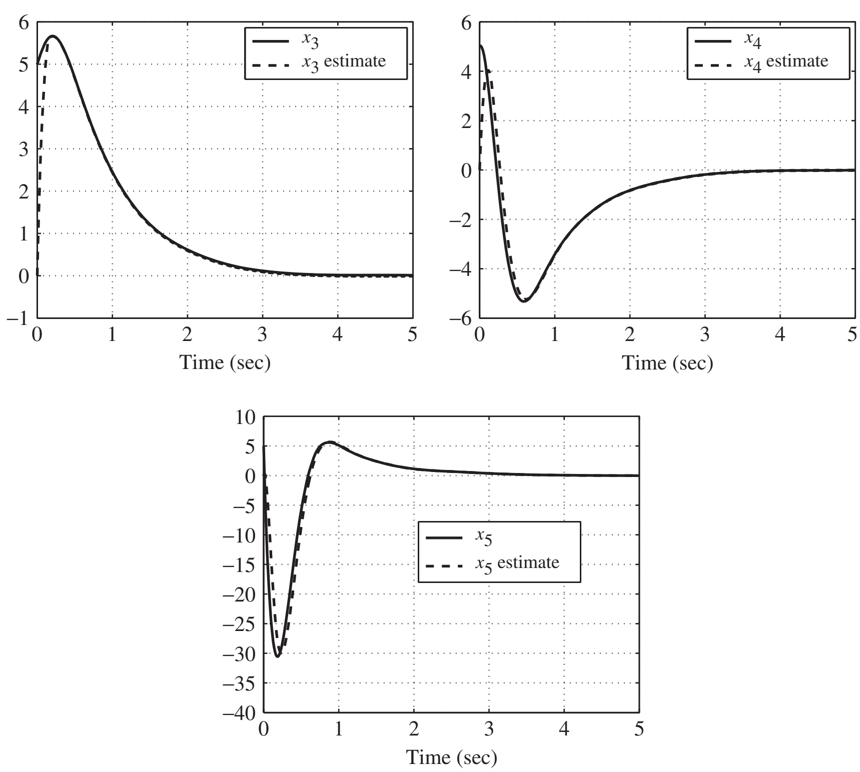

Figure 7. Decentralised combined observer-controller compensator performance for the second subsystem in Example 2.

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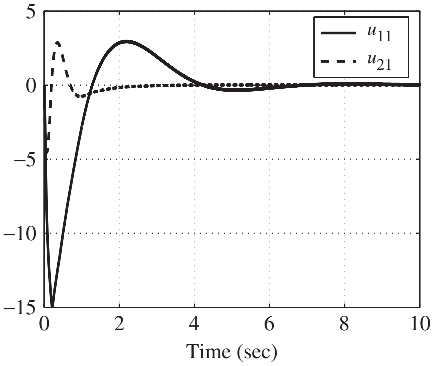

Figure 8. Control inputs u_{11} and u_{12} in Example 2.

and $\boldsymbol{F}_1 = \boldsymbol{F}_2 = 1$. The total simulation time is 10 s. Simulation results for the first subsystem and the second subsystem are shown, respectively, in Figures 6 and 7. In Figure 8, plots of the control inputs u_{11} and u_{21} are shown.

7. Conclusions

In this article, we consider the stabilisation problem of class of non-linear interconnected systems. The interconnection functions are assumed to satisfy quadratic constraints of the state vector of the whole system. A decentralised combined observer-controller compensator is proposed and analysed. The proposed control strategy incorporates local sliding mode observers to estimate the state vector of subsystems for feedback implementation.

It would be interesting to consider the application of the decentralised compensator proposed in this article to the control of power systems given in Wang, Hill and Guo (1998) and Šiljak, Stipanovic and Zecevic (2002). The systems considered in Wang et al. (1998), Šiljak et al. (2002) fall into the class of interconnected systems considered herein. It remains to verify if the above systems satisfy the assumptions stated in our article. Any interconnected system consisting of non-linear subsystems with non-linear bounded interconnection functions as opposed to linear subsystems with non-linear interconnection functions would be a natural extension of the research reported in this article. In addition, the design of the compensator is not restricted to just non-linear interconnected systems, but can also be used in fault detection and isolation of dynamical systems.

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Appendix 1

Proof of Lemma 1

We first show by contradiction that the distance function for a given pair $(\boldsymbol{A},\boldsymbol{B})$, $\delta(\boldsymbol{A},\epsilon\boldsymbol{B})$, is a continuous function of ϵ. Let

$$\sigma(\omega,\epsilon)=\sigma_{\min}\left(\begin{bmatrix} j\omega\boldsymbol{I}-\boldsymbol{A} & \epsilon\boldsymbol{B}\end{bmatrix}\right).$$

We have $\delta(\boldsymbol{A},\epsilon\boldsymbol{B})=\min_{\omega\in\mathbb{R}}\sigma(\omega,\epsilon)$. Because the smallest singular value of a matrix is a continuous function of its elements and $\sigma(\omega,\epsilon)\to\infty$ as $\omega\to\infty$ for a fixed ϵ (Rajamani 1998; Rajamani and Cho 1998; Aboky et al. 2002), there exists a finite ω^* such that

$$\delta(\boldsymbol{A},\epsilon\boldsymbol{B})=\min_{\omega\in\mathbb{R}}\sigma(\omega,\epsilon)=\sigma(\omega^*,\epsilon). \tag{48}$$

To proceed, let $B_r(\epsilon_1)=\{\epsilon:|\epsilon-\epsilon_1|<r\}$ and

$$B_r(\omega_1^*,\epsilon_1)=\left\{(\omega,\epsilon):\left(\omega-\omega_1^*\right)^2+(\epsilon-\epsilon_1)^2<r^2\right\}.$$

Suppose that $\delta(\boldsymbol{A},\epsilon\boldsymbol{B})$ is discontinuous at some ϵ_1, that is, there is an $\epsilon>0$ so that for each $\tau>0$ there exists an $\epsilon_2\in B_\tau(\epsilon_1)$ such that

$$|\delta(\boldsymbol{A},\epsilon_2\boldsymbol{B})-\delta(\boldsymbol{A},\epsilon_1\boldsymbol{B})|>\varepsilon. \tag{49}$$

It follows from (48) and (49) that

$$\left|\sigma\left(\omega_2^*,\epsilon_2\right)-\sigma\left(\omega_1^*,\epsilon_1\right)\right|>\varepsilon$$

for some $\omega_1^*\in\mathbb{R}$ and $\omega_2^*\in\mathbb{R}$. If $\sigma(\omega_2^*,\epsilon_2)>\sigma(\omega_1^*,\epsilon_1)$, we have

$$\sigma\left(\omega_2^*,\epsilon_2\right)>\sigma\left(\omega_1^*,\epsilon_1\right)+\varepsilon. \tag{50}$$

Because $\sigma(\omega,\epsilon)$ is a continuous function, there exists a $\tau_1>0$ for the given ε so that $|\sigma(\omega,\epsilon)-\sigma(\omega_1^*,\epsilon_1)|<\varepsilon$ for any $(\omega,\epsilon)\in B_{\tau_1}(\omega_1^*,\epsilon_1)$. Choose $\tau<\tau_1$. We can find ω_2 such that $(\omega_2,\epsilon_2)\in B_{\tau_1}(\omega_1^*,\epsilon_1)$ and $|\sigma(\omega_2,\epsilon_2)-\sigma(\omega_1^*,\epsilon_1)|<\varepsilon$, which implies that

$$\sigma(\omega_2,\epsilon_2)<\sigma\left(\omega_1^*,\epsilon_1\right)+\varepsilon. \tag{51}$$

It follows from (50) and (51) that

$$\sigma(\omega_2,\epsilon_2)<\sigma\left(\omega_1^*,\epsilon_1\right)+\varepsilon<\sigma\left(\omega_2^*,\epsilon_2\right). \tag{52}$$

However, there is a contradiction between (52) and the fact that

$$\sigma\left(\omega_2^*,\epsilon_2\right)=\min_{\omega\in\mathbb{R}}\sigma(\omega,\epsilon_2)\le\sigma(\omega_2,\epsilon_2).$$

If, on the other hand, $\sigma(\omega_2^*,\epsilon_2)\le\sigma(\omega_1^*,\epsilon_1)$, we have

$$\sigma\left(\omega_1^*,\epsilon_1\right)>\sigma\left(\omega_2^*,\epsilon_2\right)+\varepsilon. \tag{53}$$

Then, for the given ε, there exists a $\tau_2>0$ so that $|\sigma(\omega,\epsilon)-\sigma(\omega_2^*,\epsilon_2)|<\varepsilon$ for any $(\omega,\epsilon)\in B_{\tau_2}(\omega_2^*,\epsilon_2)$. Choose $\tau<\tau_2$. We can find ω_1 such that $(\omega_1,\epsilon_1)\in B_{\tau_2}(\omega_2^*,\epsilon_2)$ and $|\sigma(\omega_1,\epsilon_1)-\sigma(\omega_2^*,\epsilon_2)|<\varepsilon$, which implies that

$$\sigma(\omega_1,\epsilon_1)<\sigma\left(\omega_2^*,\epsilon_2\right)+\varepsilon. \tag{54}$$

It follows from (53) and (54) that

$$\sigma(\omega_1,\epsilon_1)<\sigma\left(\omega_2^*,\epsilon_2\right)+\varepsilon<\sigma\left(\omega_1^*,\epsilon_1\right). \tag{55}$$

We also obtain a contradiction between (55) and the fact that

$$\sigma\left(\omega_1^*,\epsilon_1\right)=\min_{\omega\in\mathbb{R}}\sigma(\omega,\epsilon_1)\le\sigma(\omega_1,\epsilon_1).$$

Hence, we conclude that $\delta(\boldsymbol{A},\epsilon\boldsymbol{B})$ is a continuous function of ϵ.

Let $f(\epsilon)=\delta(\boldsymbol{A},\sqrt{2\epsilon}\boldsymbol{B})-\sqrt{2\epsilon}$. Because $\delta(\boldsymbol{A},\epsilon\boldsymbol{B})$ and $\sqrt{2\epsilon}$ are continuous functions of ϵ, the composite function, $\delta(\boldsymbol{A},\sqrt{2\epsilon}\boldsymbol{B})$, is also a continuous function of ϵ. Hence, $f(\epsilon)$ is a continuous function of ϵ. By the hypothesis of the lemma that $\delta(\boldsymbol{A},\sqrt{2\beta}\boldsymbol{B})>\sqrt{2\beta}$, we have $f(\beta)>0$. It therefore follows from the continuity of $f(\epsilon)$ that there exists a $\gamma_i^*>0$ such that $f(\beta+\gamma_i)>0$ for $\gamma_i\in[0\ \gamma_i^*)$.

Thus, $\delta(\boldsymbol{A}, \sqrt{2(\beta+\gamma_i)}\boldsymbol{B}) > \sqrt{2(\beta+\gamma_i)}$ for $\gamma_i \in [0\ \gamma_i^*)$, which concludes the proof of the lemma.

Appendix 2

The following proof of the fact that the Hamiltonian matrix $\boldsymbol{H}_i$ is hyperbolic is a modification of the proof of Lemma 4 in Pagilla and Zhu (2005). We first compute the determinant of the matrix $(s\boldsymbol{I}_{2n_i} - \boldsymbol{H}_i)$,

$$\begin{aligned}
&\det(s\boldsymbol{I}_{2n_i} - \boldsymbol{H}_i)\\
&= \det\begin{bmatrix} s\boldsymbol{I}_{n_i} - \boldsymbol{A}_{ci} & -\boldsymbol{R}_i \\ (\beta+\gamma_i)\boldsymbol{I}_{n_i} & s\boldsymbol{I}_{n_i} + \boldsymbol{A}_{ci}^\top \end{bmatrix}\\
&= (-1)^{n_i}\det\begin{bmatrix} (\beta+\gamma_i)\boldsymbol{I}_{n_i} & s\boldsymbol{I}_{n_i} + \boldsymbol{A}_{ci}^\top \\ s\boldsymbol{I}_{n_i} - \boldsymbol{A}_{ci} & -\boldsymbol{R}_i \end{bmatrix}\\
&= (-1)^{n_i}\det\left(\begin{bmatrix} (\beta+\gamma_i)\boldsymbol{I}_{n_i} & s\boldsymbol{I}_{n_i} + \boldsymbol{A}_{ci}^\top \\ s\boldsymbol{I}_{n_i} - \boldsymbol{A}_{ci} & -\boldsymbol{R}_i \end{bmatrix}\right.\\
&\quad\left.\times\begin{bmatrix} \boldsymbol{I}_{n_i} & -\dfrac{1}{\beta+\gamma_i}\left(s\boldsymbol{I}_{n_i} + \boldsymbol{A}_{ci}^\top\right) \\ \boldsymbol{O} & \boldsymbol{I}_{n_i} \end{bmatrix}\right)\\
&= (-1)^{n_i}\det\begin{bmatrix} (\beta+\gamma_i)\boldsymbol{I}_{n_i} & \boldsymbol{O} \\ s\boldsymbol{I}_{n_i} - \boldsymbol{A}_{ci} & -\left\{\begin{array}{c} \dfrac{1}{\beta+\gamma_i}\left(s\boldsymbol{I}_{n_i} - \boldsymbol{A}_{ci}\right) \\ \times\left(s\boldsymbol{I}_{n_i} + \boldsymbol{A}_{ci}^\top\right) - \boldsymbol{R}_i \end{array}\right\} \end{bmatrix}\\
&= (-1)^{n_i}\det\left(-(\beta+\gamma_i)\boldsymbol{R}_i - \left(s\boldsymbol{I}_{n_i} - \boldsymbol{A}_{ci}\right)\left(s\boldsymbol{I}_{n_i} + \boldsymbol{A}_{ci}^\top\right)\right)\\
&= (-1)^{n_i}\det\boldsymbol{G}(s),
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{G}(s) = &-2(\beta+\gamma_i)\left(\boldsymbol{I}_{n_i} - \boldsymbol{B}_{i1}\left(\boldsymbol{B}_{i1}^\top\boldsymbol{B}_{i1}\right)^{-1}\boldsymbol{B}_{i1}^\top\right)\\
&- \left(s\boldsymbol{I}_{n_i} - \boldsymbol{A}_{ci}\right)\left(s\boldsymbol{I}_{n_i} + \boldsymbol{A}_{ci}^\top\right).
\end{aligned}$$

If $\boldsymbol{H}_i$ has eigenvalues on the imaginary axis, then $\boldsymbol{G}(-j\omega)$ is singular for some $\omega \in \mathbb{R}$. It follows from the expression of $\boldsymbol{G}(s)$ that

$$\boldsymbol{G}(-j\omega) = -\boldsymbol{G}_1 + \boldsymbol{G}_2(\omega),$$

where $\boldsymbol{G}_1 = 2(\beta+\gamma_i)\boldsymbol{I}_{n_i}$ and

$$\begin{aligned}
\boldsymbol{G}_2(\omega) = &\left[j\omega\boldsymbol{I}_{n_i} - \boldsymbol{A}_{ci} \quad \sqrt{2(\beta+\gamma_i)}\boldsymbol{B}_{i1}\left(\boldsymbol{B}_{i1}^\top\boldsymbol{B}_{i1}\right)^{-\frac{1}{2}}\right]\\
&\times\left[j\omega\boldsymbol{I}_{n_i} - \boldsymbol{A}_{ci} \quad \sqrt{2(\beta+\gamma_i)}\boldsymbol{B}_{i1}\left(\boldsymbol{B}_{i1}^\top\boldsymbol{B}_{i1}\right)^{-\frac{1}{2}}\right]^{\mathrm{H}}
\end{aligned}$$

with the superscript H denotes the Hermitian transpose. Thus, $\boldsymbol{G}(-j\omega) = \boldsymbol{G}(-j\omega)^{\mathrm{H}}$, that is, $\boldsymbol{G}(-j\omega)$ is Hermitian. Hence, the eigenvalues of $\boldsymbol{G}(-j\omega)$ are all real. It follows from (16) that

$$\begin{aligned}
\lambda_{\min}(\boldsymbol{G}_2(\omega)) &= \left(\sigma_{\min}\left(\left[j\omega\boldsymbol{I}_{n_i} - \boldsymbol{A}_{ci} \quad \sqrt{2\beta+\gamma_i}\boldsymbol{B}_{i1}\left(\boldsymbol{B}_{i1}^\top\boldsymbol{B}_{i1}\right)^{-\frac{1}{2}}\right]\right)\right)^2\\
&\geq \left(\delta\left(\boldsymbol{A}_{ci}, \sqrt{2\beta+\gamma_i}\boldsymbol{B}_{i1}\left(\boldsymbol{B}_{i1}^\top\boldsymbol{B}_{i1}\right)^{-\frac{1}{2}}\right)\right)^2\\
&> 2(\beta+\gamma_i).
\end{aligned}$$

Using the above, we obtain

$$\begin{aligned}
\boldsymbol{v}^\top\boldsymbol{G}(-j\omega)\boldsymbol{v} &= -\boldsymbol{v}^\top\boldsymbol{G}_1\boldsymbol{v} + \boldsymbol{v}^\top\boldsymbol{G}_2(\omega)\boldsymbol{v}\\
&\geq (-2(\beta+\gamma_i) + \lambda_{\min}(\boldsymbol{G}_2(\omega)))\|\boldsymbol{v}\|^2\\
&> 0,
\end{aligned}$$

for any non-zero vector $\boldsymbol{v} \in \mathbb{R}^{n_i}$, which implies that $\boldsymbol{G}(-j\omega) > 0$ for all $\omega \in \mathbb{R}$. Therefore, $\boldsymbol{H}_i$ has no eigenvalues on the imaginary axis and the proof is completed.

Appendix 3

Proof of Lemma 3

Recall from §4 that $\boldsymbol{Q}_i^o = \boldsymbol{T}_i^\top\hat{\boldsymbol{Q}}_i^o\boldsymbol{T}_i$, where

$$\hat{\boldsymbol{Q}}_i^o = \hat{\boldsymbol{Q}}_{i1}^o + \hat{\boldsymbol{Q}}_{i2}^o,$$

with symmetric positive definite $\hat{\boldsymbol{Q}}_{i1}^o$ defined in (35) and $\hat{\boldsymbol{Q}}_{i2}^o = 2\delta_i\boldsymbol{I}_{n_i}$, where $\delta_i = \sigma_{\max}^2(\boldsymbol{B}_{i1}\boldsymbol{K}_i)$. Because $\hat{\boldsymbol{Q}}_{i1}^o$ is symmetric positive definite, we can represent $\hat{\boldsymbol{Q}}_{i1}^o$ as $\hat{\boldsymbol{Q}}_{i1}^o = \boldsymbol{U}_i\boldsymbol{D}_i\boldsymbol{U}_i^\top$, where $\boldsymbol{U}_i = \boldsymbol{U}_i^\top$ is an orthogonal matrix consisting of the eigenvectors of $\hat{\boldsymbol{Q}}_{i1}^o$ and $\boldsymbol{D}_i$ is a diagonal matrix consisting of eigenvalues of $\hat{\boldsymbol{Q}}_{i1}^o$. On the other hand, we have $\boldsymbol{U}_i\hat{\boldsymbol{Q}}_{i2}^o = \hat{\boldsymbol{Q}}_{i2}^o\boldsymbol{U}_i$, so $\hat{\boldsymbol{Q}}_{i2}^o = \boldsymbol{U}_i\hat{\boldsymbol{Q}}_{i2}^o\boldsymbol{U}_i^\top$. Thus, we can represent $\hat{\boldsymbol{Q}}_i^o$ as

$$\hat{\boldsymbol{Q}}_i^o = \boldsymbol{U}_i\left(\boldsymbol{D}_i + \hat{\boldsymbol{Q}}_{i2}^o\right)\boldsymbol{U}_i^\top.$$

Then we have

$$\lambda_{\min}\left(\hat{\boldsymbol{Q}}_i^o\right) = \lambda_{\min}(\boldsymbol{D}_i) + 2\delta_i > \delta_i,$$

because $\lambda_{\min}(\boldsymbol{D}_i) > 0$. Therefore,

$$\lambda_{\min}\left(\boldsymbol{Q}_i^o\right) = \lambda_{\min}\left(\hat{\boldsymbol{Q}}_i^o\right) > \delta_i = \lambda_{\max}(\boldsymbol{Q}_i),$$

which concludes the proof of the lemma.

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