PURDUE
UNIVERSITY

Variable Structure Sliding Mode Control

Stanisław H. Żak

School of Electrical and Computer Engineering

ECE 675

Spring 2012

Outline

- What is a variable structure system?
- Basic vocabulary; switching surface, sliding mode
- Switching surface design
- Variable structure sliding mode control law design

What Is a Variable Structure System (VSS)?

- General definition: A dynamical system whose structure changes in accordance with the value of the state
- Our Implementation of VSS: A system composed of independent structures together with a switching logic to switch between each of the structures

Useful Property of a Variable Structure System (VSS)

☞ A VSS may have a property, for example, asymptotic stability, that is not a property of any of its sub-structures

Example 6.1

Double integrator model in state-space format

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -a x_1$$

where *a* is a parameter

Two Structures

- $a = 1/4$

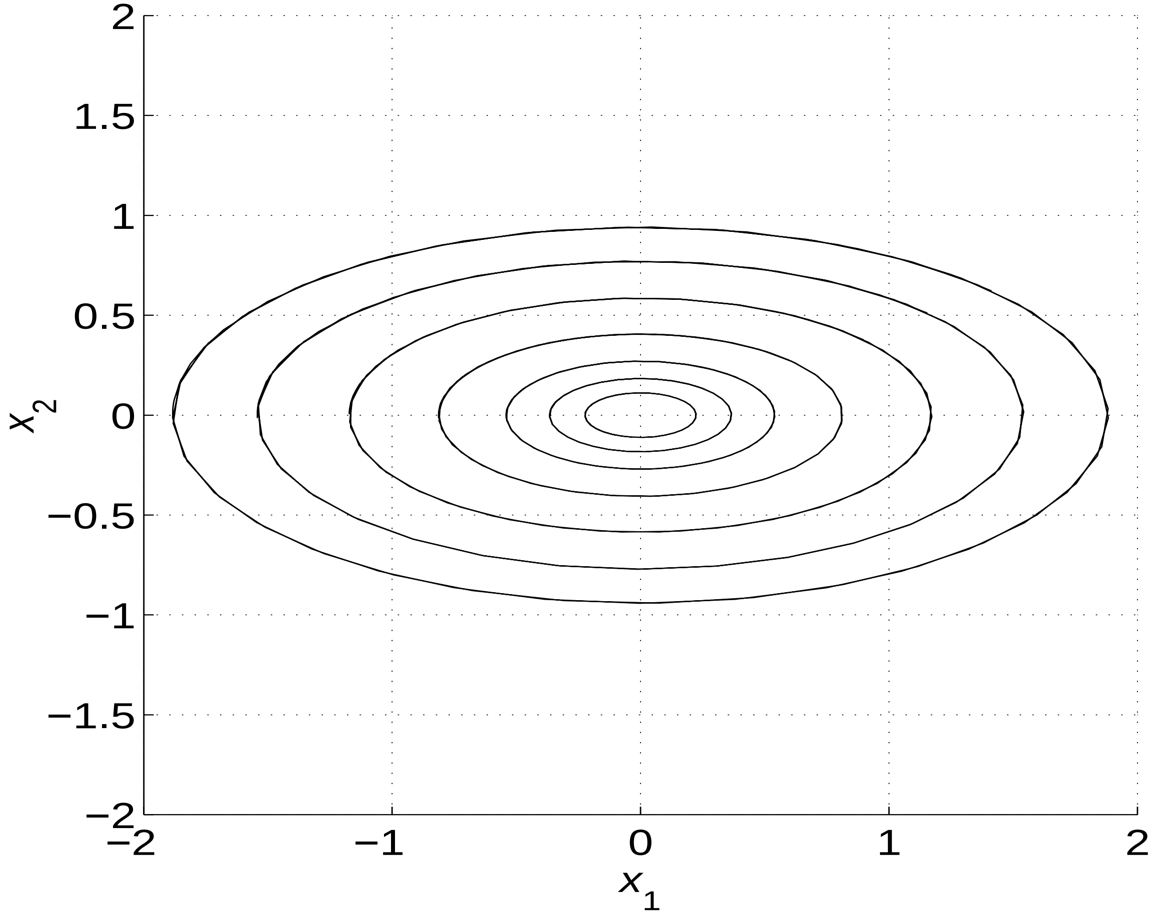
$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 \\ -1/4 & 0 \end{bmatrix} x$$

- $a = 4$

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} x$$

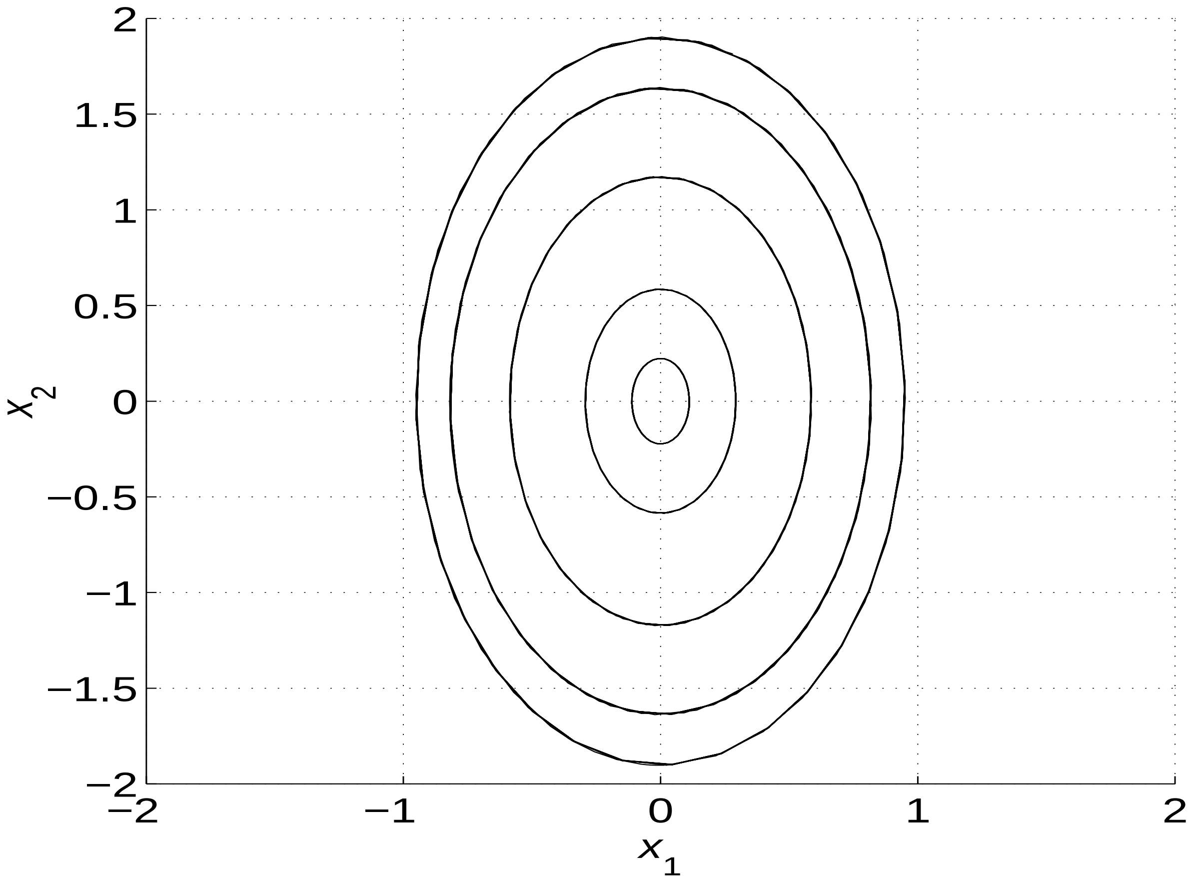
Phase Portrait for a=1/4

The phase portrait is a family of ellipses



Phase Portrait for a=4

The phase portrait is a family of ellipses



Simple Variable Structure System (VSS)

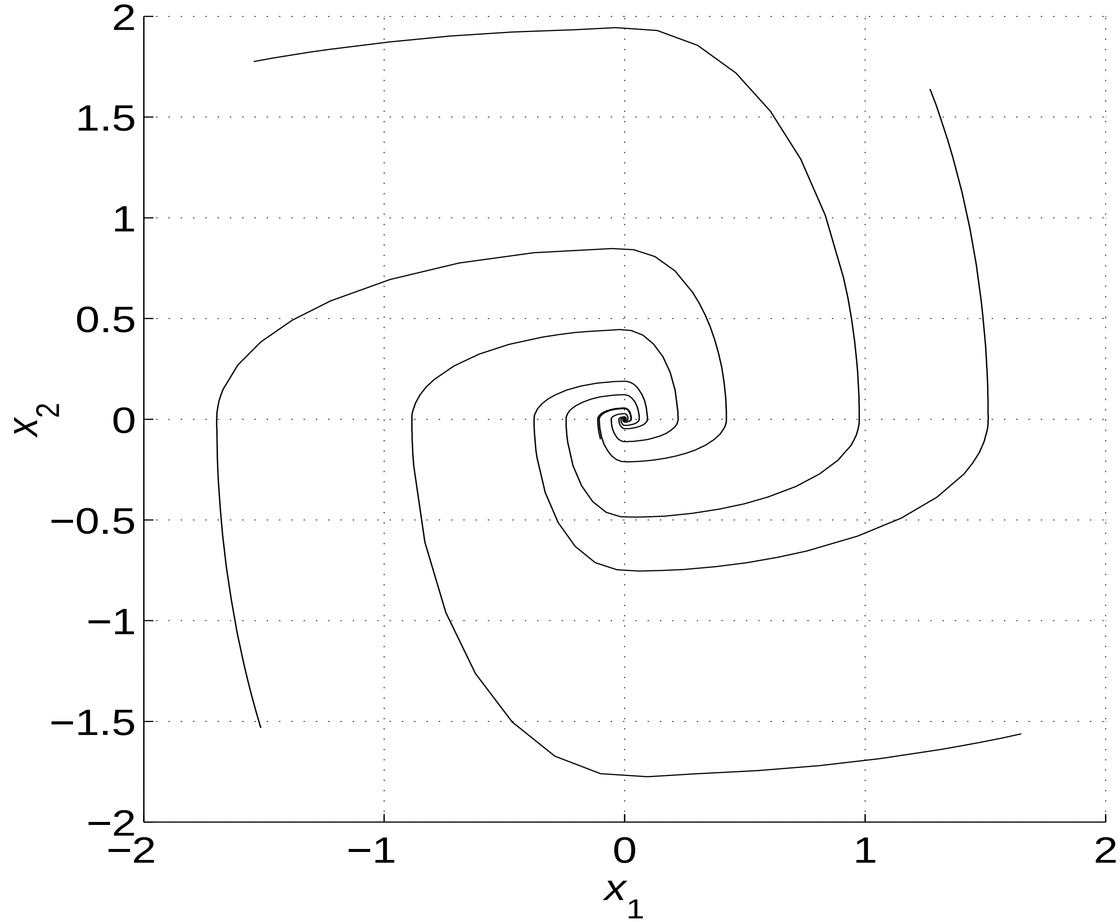
Change the system structure when the trajectory crosses either coordinate, i.e.,

$$\begin{aligned}\frac{d}{dt}x_1 &= x_2 \\ \frac{d}{dt}x_2 &= -ax_1,\end{aligned}$$

$$a = \begin{cases} 4 & \text{if} \quad x_1x_2 > 0 \\ 1/4 & \text{if} \quad x_1x_2 < 0 \end{cases}$$

Phase Portrait of the Simple VSS

Note: The VSS is asymptotically stable!



Example 6.2

Double integrator

$$\begin{aligned} \frac{d}{dt}x_1 &= x_2 \\ \frac{d}{dt}x_2 &= u \\ u &= \pm 2 \end{aligned}$$

Switch about a line

$$x_1 + x_2 = 0$$

Switching Line

Let

$$\sigma(x) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The switching line equation

$$\sigma(x) = 0$$

Switching Control Law

Switching logic

$$u = \begin{cases} -2 & \text{if } \sigma(x) > 0 \\ 2 & \text{if } \sigma(x) < 0 \end{cases}$$

Equivalently

$$u = -2\,\text{sign}(\sigma(x))$$

Discontinuous Control

- Note that

$$\mathrm{sign}(\sigma) = \frac{\sigma}{|\sigma|} = \frac{|\sigma|}{\sigma}$$

- Control u is discontinuous at 0
- The controlled system is governed by differential equations with the discontinuous right hand-side
- Need new methods of analysis for such systems

Smoothing the Sign Function

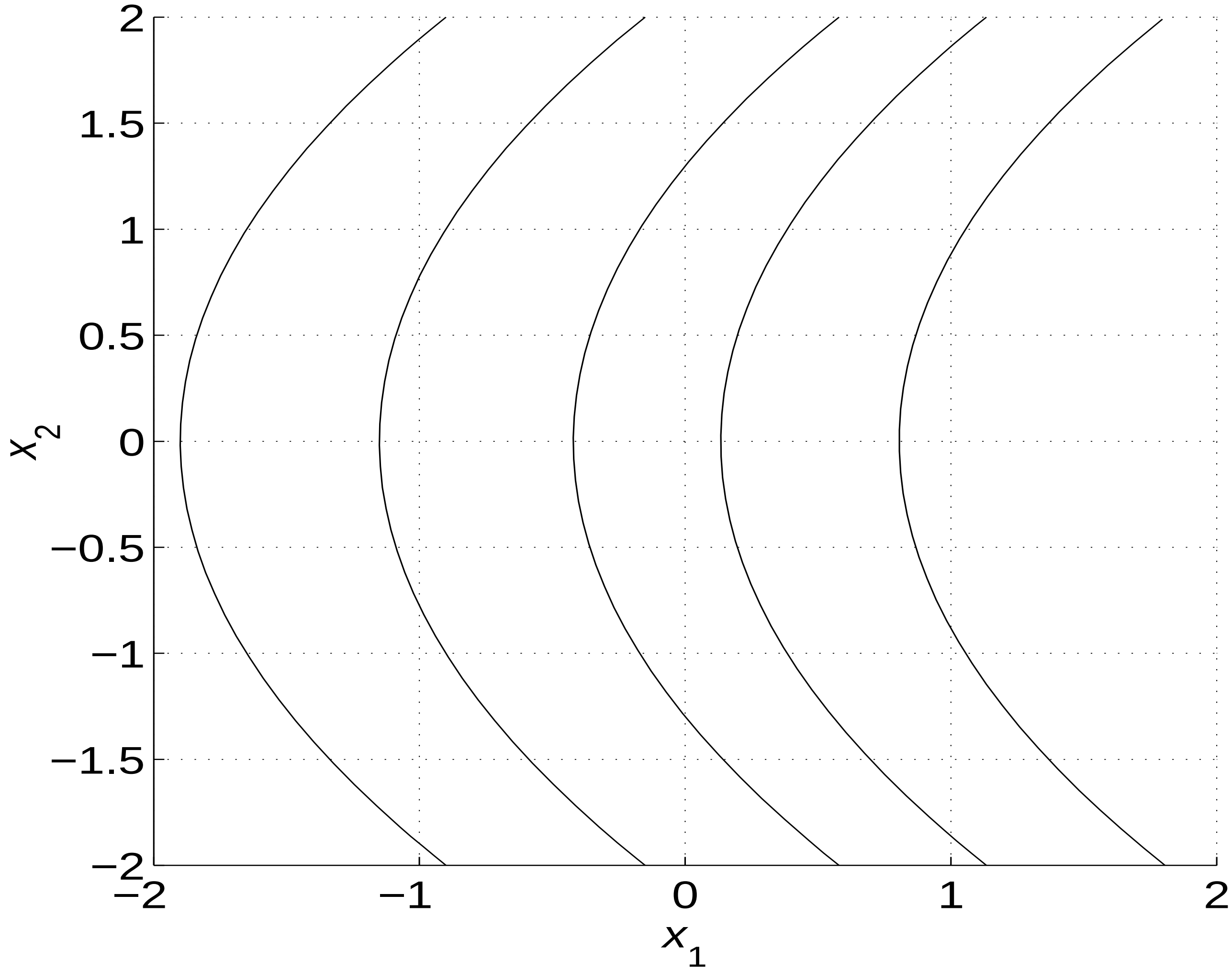
- In many situations the relay control action is not acceptable because it results in chattering behavior
- Approximate the sign function

$$\text{sign}(\sigma) \approx \frac{\sigma}{|\sigma| + \nu}$$

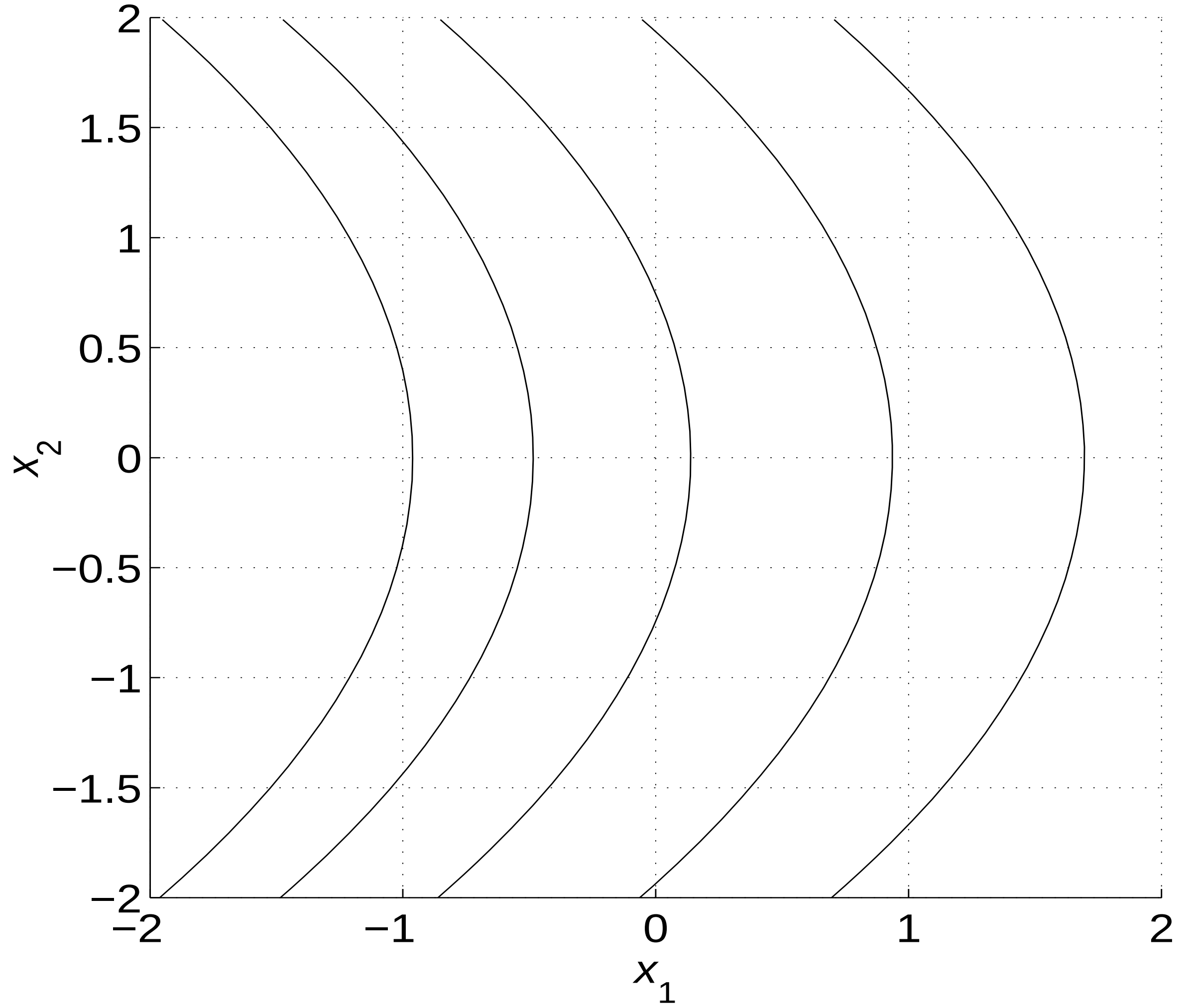
where *v* (nu) is a small parameter

- As *v* tends to 0, the approximation tends to the sign function

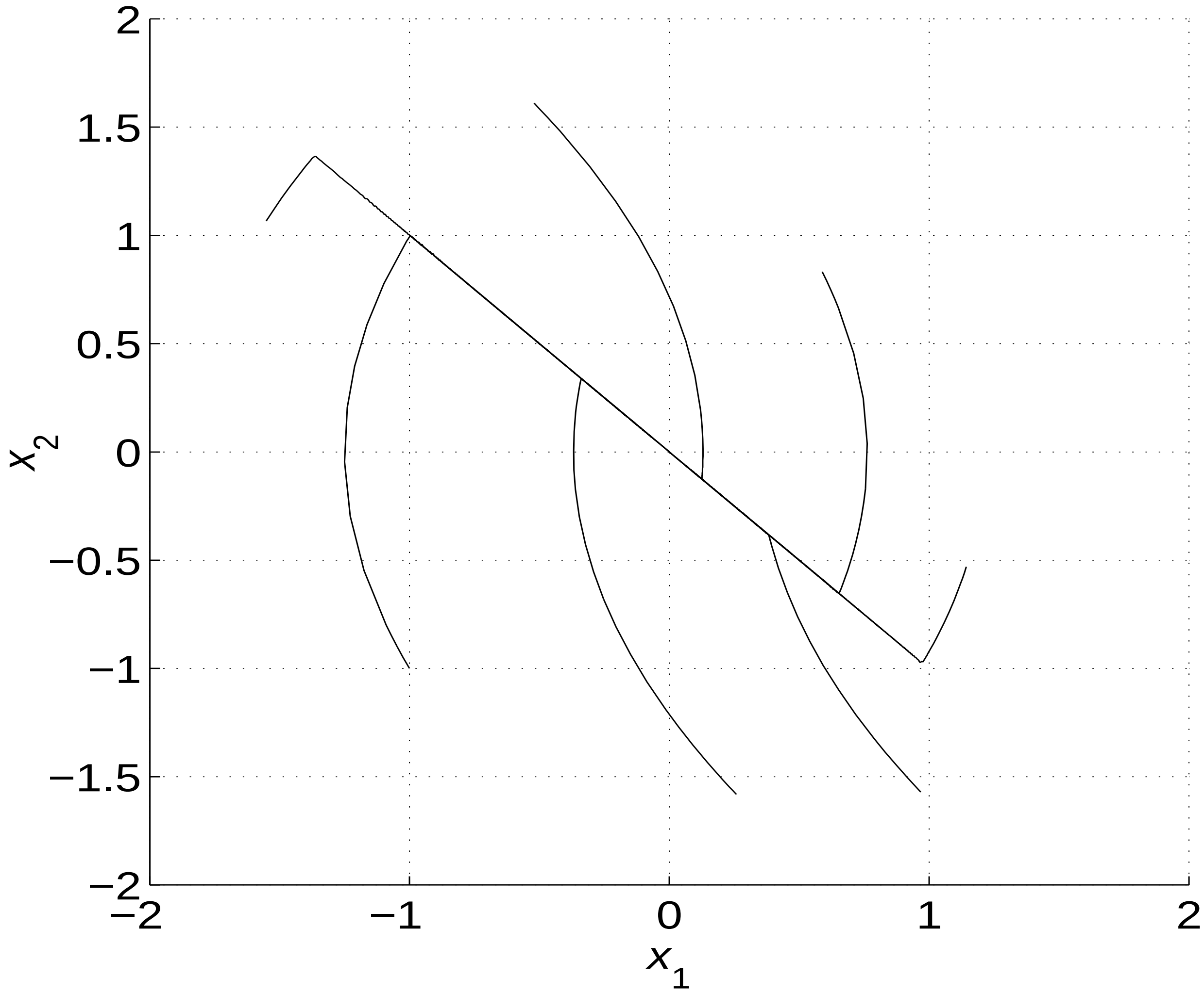
Open-Loop System Behavior for $u=2$



Open-Loop System Behavior for u=-2



Closed-Loop System Phase Portrait



Sliding Mode

- The motion of the system while confined to the switching line is referred to as sliding
- A sliding mode will exist if in the vicinity of the switching surface the state velocity vectors are directed toward the surface, that is, if the switching line is attractive

Attractive Switching Line (Surface)

A switching line (surface) is attractive if

- Any trajectory starting on the surface remains there
- Any trajectory starting outside the surface tends to it at least asymptotically

Sliding Mode Controller Design---A Two Phase Process

- Switching surface design to achieve the desired system behavior, like the stability to the origin, when restricted to the surface
- Controller design---select feedback controller so that the system under control is stable to the sliding surface, that is, the controller forces the system trajectories towards the switching surface

Plant Model---Single Input

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

Switching Surface Design---Single Input

$$
\begin{aligned}
\sigma(x) &= sx \\
&= \begin{bmatrix} s_1 & s_2 & \dots & s_{n-1} & 1 \end{bmatrix} x \\
&= s_1 x_1 + s_2 x_2 + \cdots + s_{n-1} x_{n-1} + x_n \\
&= 0
\end{aligned}
$$

Dynamics of the Plant in Sliding Mode

- From the switching surface equation

$$x_n = -s_1 x_1 - s_2 x_2 - \cdots - s_{n-1} x_{n-1}$$

- From the plant model

$$\frac{d}{dt} x_{n-1} = x_n$$

Plant Dynamics in Sliding Mode

Combine the two equations from the previous slide with the plant model

$$
\begin{aligned}
\frac{d}{dt}x_1 &= x_2 \\
\frac{d}{dt}x_2 &= x_3 \\
&\vdots \\
\frac{d}{dt}x_{n-1} &= -s_1x_1 - s_2x_2 - \cdots - s_{n-1}x_{n-1}
\end{aligned}
$$

Dynamics in Sliding

Equivalent representation of the plant dynamics in sliding mode---note the order reduction

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -s_1 & -s_2 & -s_3 & \dots & -s_{n-1} \end{bmatrix} x$$

Order Reduction In Sliding

- The poles of the reduced-order system are the roots of the equation

$$\lambda^{n-1} + s_{n-1}\lambda^{n-2} + \cdots + s_2\lambda + s_1 = 0$$

- The system response in sliding is completely specified by the coefficients of the switching surface

$$s_1,\ s_2, \cdots, s_{n-1}$$

Example---Double Integrator

- Double integrator state-space model

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

- Phase one---Switching surface design

$$\sigma(x) = \begin{bmatrix} s_1 & 1 \end{bmatrix} x = 0$$

- System in sliding

$$\frac{d}{dt}x_1 = -s_1 x_1$$

Phase Two---Controller Design

- Controller objective is to forces the system trajectory toward the sliding surface
- We can use

$$\|\boldsymbol{\sigma}\|^2 = \boldsymbol{\sigma}^T \boldsymbol{\sigma}$$

as a distance measure from the surface $\{\boldsymbol{x} : \boldsymbol{\sigma} = \mathbf{0}\}$

- Note that for single input plants

$$\|\sigma\|^2 = \sigma^T \sigma = \sigma^2$$

Forcing the System Trajectory Towards the Sliding Surface

- Design the controller to reduce the distance from the surface by forcing the negative rate of distance increase from the switching surface, that is,

$$\frac{d}{dt}\sigma^2 < 0$$

- Note that for single input case

$$\sigma \frac{d}{dt}\sigma = \frac{1}{2}\frac{d}{dt}\sigma^2$$

Selecting Controller's Structure

- Use

$$u = k_1^{\pm} x_1 + \cdots + k_n^{\pm} x_n$$

- Select gains so that the rate of distance increase from the switching surface is negative, that is, use the *reachability condition*,

$$\sigma \left(\frac{d}{dt} \sigma \right) < 0$$

Selecting Controller's Gains

Use the condition

$$\sigma\left(\frac{d}{dt}\sigma\right) = \sigma\begin{bmatrix} s_1 & s_2 & \ldots & 1 \end{bmatrix}\frac{d}{dt}x$$

$$= \sigma\left(s_1\frac{dx_1}{dt} + \cdots + \frac{dx_n}{dt}\right)$$

Perform Simple Manipulations

- First substitute from the plant model

$$\begin{aligned}\sigma\left(\frac{d}{dt}\sigma\right) &= \sigma\left(s_1x_2+\cdots+s_{n-1}x_n+\frac{dx_n}{dt}\right)\\ &= \sigma(s_1x_2+\cdots+s_{n-1}x_n\\ &\qquad -a_0x_1-\cdots-a_{n-1}x_n+u)\end{aligned}$$

- Continue

$$\begin{aligned}\sigma\left(\frac{d}{dt}\sigma\right) &= \sigma(s_1x_2+\cdots+s_{n-1}x_n\\ &\qquad -a_0x_1-\cdots-a_{n-1}x_n\\ &\qquad +k_1^{\pm}x_1+\cdots+k_n^{\pm}x_n)\\ &< 0 \quad \text{for} \quad \sigma\neq 0\end{aligned}$$

More Manipulations

$$
\begin{aligned}
\sigma\left(\frac{d}{dt}\sigma\right) &= \sigma(s_1 x_2 + \cdots + s_{n-1} x_n \\
&\quad - a_0 x_1 - \cdots - a_{n-1} x_n \\
&\quad + k_1^{\pm} x_1 + \cdots + k_n^{\pm} x_n) \\
&= \sigma x_1 \left(a_0 + k_1^{\pm}\right) \\
&\quad + \sigma x_2 \left(a_1 + s_1 + k_2^{\pm}\right) \\
&\quad \vdots \\
&\quad + \sigma x_n \left(a_{n-1} + s_{n-1} + k_n^{\pm}\right) \\
&< 0
\end{aligned}
$$

A Method for Selecting Gains

Force each term to be negative, that is,

$$\begin{aligned}
\sigma x_1 \left(a_0 + k_1^{\pm}\right) &< 0 \\
\sigma x_2 \left(a_1 + s_1 + k_2^{\pm}\right) &< 0 \\
&\vdots \\
\sigma x_n \left(a_{n-1} + s_{n-1} + k_n^{\pm}\right) &< 0
\end{aligned}$$

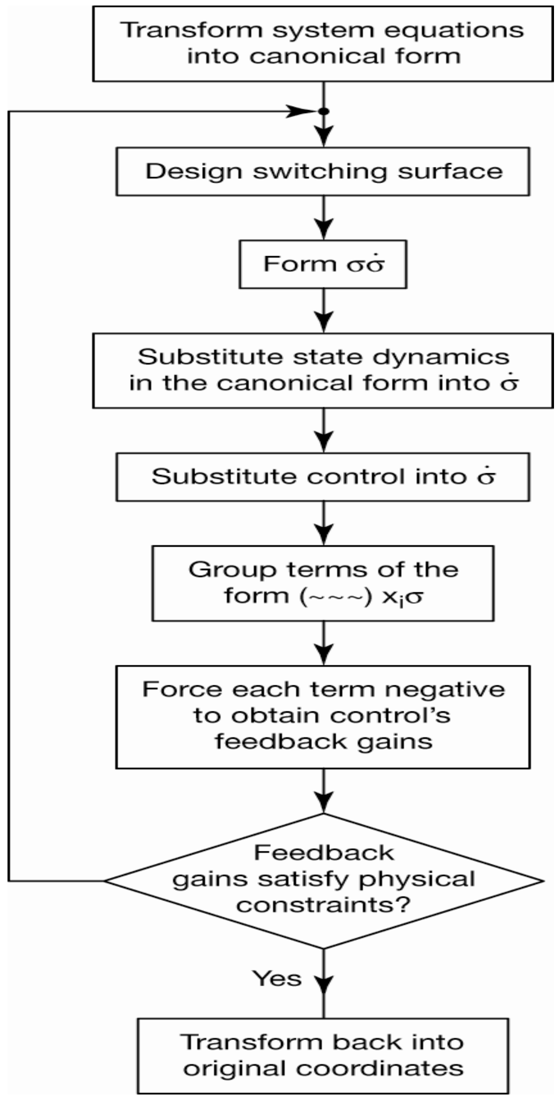
The Resulting Gains

$$
\begin{aligned}
k_1^+ &< -|a_0|, \quad \sigma x_1 > 0 \\
k_1^- &> |a_0|, \quad \sigma x_1 < 0 \\
&\vdots \\
k_n^+ &< -(|a_{n-1}| + |s_{n-1}|), \quad \sigma x_n > 0 \\
k_n^- &> |a_{n-1}| + |s_{n-1}|, \quad \sigma x_n < 0
\end{aligned}
$$

Possible Controller Implementation

$$
\begin{aligned}
k_1^{\pm} &= -(|a_0| + \varepsilon)\,\mathrm{sign}(x_1\sigma) \\
&\vdots \\
k_n^{\pm} &= -(|a_{n-1} + |s_{n-1}| + \varepsilon)\,\mathrm{sign}(x_n\sigma) \\
u &= k_1^{\pm}x_1 + \cdots + k_n^{\pm}x_n
\end{aligned}
$$

Summary of Sliding Mode Controller Design



Uncertain System Model

- Uncertain linear time invariant system

$$\frac{d}{dt}x = Ax + B(u + \xi)$$

- The vector ξ models matched uncertainties or nonlinearities
- Matched uncertainties---affect the system's dynamics through the input channels only

Multi-Input Case

- The switching surface

$$\{x \ : \ Sx = 0\} = \ker S$$

where rank ***S*** = *m*

- System in sliding

$$\begin{aligned} \frac{d}{dt}\sigma(x) &= 0 \\ \sigma(x) &= 0 \end{aligned}$$

Dynamics in Sliding Mode

$$\begin{aligned}\frac{d}{dt}\sigma(x) &= S\frac{d}{dt}x \\ &= S\left(Ax + Bu + B\xi\right) \\ &= SAx + SBu + SB\xi = 0\end{aligned}$$

Equivalent Control

- Time derivative of $\boldsymbol{\sigma}$ is

$$\frac{d\sigma}{dt} = SAx + SBu + SB\xi = 0$$

- Assumption

$$\det(SB) \neq 0$$

- Solve for ***u***, called the equivalent control, to obtain

$$u_{eq} = -(SB)^{-1} SAx - \xi$$

Sliding Dynamics

- Substitute the equivalent control

$$\begin{aligned}\frac{d}{dt}x &= Ax + B(u+\xi) \\ u_{eq} &= -(SB)^{-1}SAx - \xi\end{aligned}$$

- Manipulate

$$\frac{d}{dt}x = Ax - B(SB)^{-1}SAx - B\xi + B\xi$$

System Dynamics in Sliding

- Recall the equations describing the system dynamics in sliding

$$\begin{aligned} \frac{d}{dt}\sigma(x) &= 0 \\ \sigma(x) &= 0 \end{aligned}$$

- Equivalent description

$$\begin{aligned} \frac{d}{dt}x &= \left(I_n - B\left(SB\right)^{-1} S\right) Ax \\ Sx &= 0 \end{aligned}$$

Benefits of Sliding Motion

- The system dynamics are not affected by matched uncertainties
- The system dynamics are governed by a reduced-order model

☞ **Sliding mode controllers** are robust and order reduction facilitates sliding mode controller design