ECE 675: Dynamic system analysis tools

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Today's Class—State-plane analysis

- Simple System Model

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- Phase portraits

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- The method of isoclines

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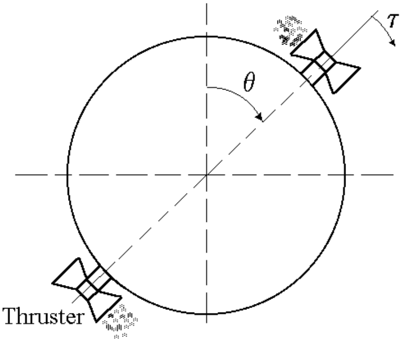
- Phase portraits
- The method of isoclines
- Taylor linearization

Simple System Model

Example 2.2 on page 50—rigid satellite



Mechanical Systems

- Linear rotational systems are analogous to linear translational systems

Translational	Rotational
$F = ma$	$\tau = I\ddot{\theta}$

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- Very simple model of the rigid satellite

$$\tau = I\ddot{\theta}$$

Rigid Satellite Model

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- Hence, $\dot{x}_2 = \ddot{\theta} = \frac{1}{I}\tau$
- State-space model

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{I}\tau = u \end{aligned}$$

Satellite Model in Matrix Format

- $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

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$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},$$
- which is a special case of
$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{u})$$

2D Model Analysis

- In our model

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- We can also plot x_2 vs. x_1 using t as a parameter
- The plane with coordinate axes x_1, x_2 is called the *state plane* or *phase plane*

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- As t varies the RP describes a curve in the state plane, called a *trajectory*
- A family of trajectories is a *phase portrait*

Method of Isoclines—p. 53

- $$\begin{cases} \dot{x}_1 = \frac{dx_1}{dt} = f_1(x_1, x_2) \\ \dot{x}_2 = \frac{dx_2}{dt} = f_2(x_1, x_2) \end{cases}$$

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- Consider the case when

$$\frac{dx_2}{dx_1} = m(x_1, x_2) = m = \text{constant}$$

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- Example: $\ddot{y} + y = 0$

Constructing Isoclines

- Let

$$x_1 = y \quad \text{and} \quad x_2 = \dot{x}_1$$

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$$x_1 = y \quad \text{and} \quad x_2 = \dot{x}_1$$

- We represent $\ddot{y} = -y$ as
$$\begin{cases} \dot{x}_1 &= x_2 = f_1(x_1, x_2) \\ \dot{x}_2 &= -x_1 = f_2(x_1, x_2) \end{cases}$$

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- Construct

$$\frac{dx_2}{dx_1} = -\frac{x_1}{x_2} = m$$

Isoclines' Equation

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- $$x_2 = -\frac{1}{m}x_1$$
- The isoclines for this example are a family of straight lines that pass through the origin
- The line that satisfies the above equation is an isocline corresponding to the trajectories' slope m because a trajectory crossing the isocline will have its slope equal to m

The Isocline Method

- Construct several isoclines in the state plane

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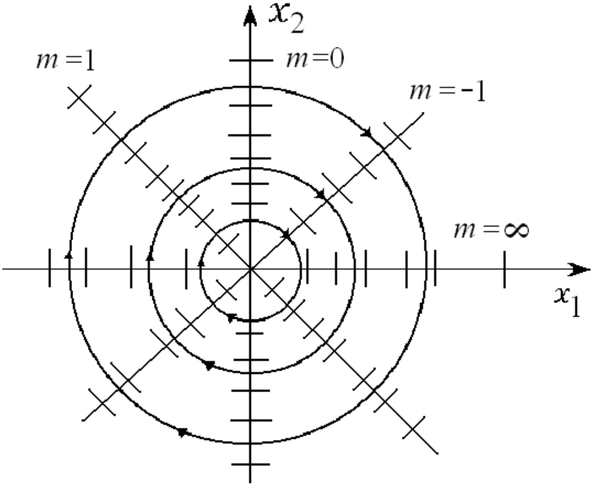
- Construct a field of local tangents m

The Isocline Method

- Construct several isoclines in the state plane
- Construct a field of local tangents m
- The trajectory passing through any given point in the state plane is obtained by drawing a continuous curve following the directions of the field

The Isocline Method—Example



Interactive Phase Portrait—Prep

```
t0=0;tf=20;tspan=tf-t0;
x0=[-4 -4]';
button=1;
p=4*[-1 0;1 0];
clf;plot(p(:,1),p(:,2))
hold on
plot(p(:,2),p(:,1))
axis(4*[-1 1 -1 1])
```

Interactive Phase Portrait

```
while(button==1)
[t,x]=ode45(@my_xdot,tspan,x0);
plot(x(:,1),x(:,2))
[x1,x2,button]=ginput(1);
x0=[x1 x2]';
end
```

Interactive Phase Portrait—ODE

```
function xdot=Diff_eq(t,x)
xdot=[x(2);-2*x(2)-x(1)];
```

Taylor Linearization—Sec 2.3

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- When we use a linear model of a physical system, we are employing some form of linearization
- A linear model accurately models a physical system in some range about an operating point about which the system is linearized

Taylor Linearization—Sec 2.3

- Non-linear time-invariant system

$$\left.\begin{array}{rcl} \frac{dx_1}{dt} & = & f_1(x_1, x_2, \ldots, x_n) \\ \frac{dx_2}{dt} & = & f_2(x_1, x_2, \ldots, x_n) \\ & \vdots & \\ \frac{dx_n}{dt} & = & f_n(x_1, x_2, \ldots, x_n) \end{array}\right\}$$

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- Compact notation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

Equilibrium state

Definition

A point $\mathbf{x}_e$ is an equilibrium state, or an equilibrium point, of the system modeled by $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ if it has the property that when the system starts at $\mathbf{x}_e$, it will stay at $\mathbf{x}_e$ for all future time. The equilibrium points are obtained by solving the algebraic equation

$$\mathbf{f}(\mathbf{x}) = \mathbf{0}.$$

r-neighborhood

Definition

The r-neighborhood is a set of points

$$\{\mathbf{x} : \|\mathbf{x} - \mathbf{x}_e\| < r\},$$

where $\|\cdot\|$ can be any p-norm on $\mathbb{R}^n$

Isolated Equilibrium State

Definition

An equilibrium state $\mathbf{x}_e$ is isolated if there is an $r > 0$ such that the r-neighborhood of $\mathbf{x}_e$ contains no equilibrium state other than $\mathbf{x}_e$.

Taylor Linearization About Equilibrium

$$\frac{d}{dt}\mathbf{x} = \mathbf{f}(\mathbf{x}_e) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_e)\,\Delta\mathbf{x} + \text{higher-order terms}$$

where

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_e)$$

is the Jacobian matrix of $\mathbf{f}$ with respect $\mathbf{x}$, evaluated at the equilibrium state $\mathbf{x}_e$, and $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}_e$

Taylor Linearization About Equilibrium—Manipulations

- Observe that

$$\frac{d}{dt}\mathbf{x} = \frac{d}{dt}\mathbf{x}_e + \frac{d}{dt}\Delta\mathbf{x} = \frac{d}{dt}\Delta\mathbf{x}$$

because $\mathbf{x}_e$ is constant

Taylor Linearization About Equilibrium—Manipulations

- Observe that

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  because $\mathbf{x}_e$ is constant

- Use $\mathbf{f}(\mathbf{x}_e) = \mathbf{0}$ to obtain

$$\frac{d}{dt}\Delta\mathbf{x} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_e)\,\Delta\mathbf{x}$$

Taylor Linearized Model

- Let
$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_e)$$

Taylor Linearized Model

- Let
$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}_e)$$
- Linear model in $\Delta \mathbf{x}$,
$$\frac{d}{dt}\Delta \mathbf{x} = \mathbf{A}\,\Delta \mathbf{x}$$