9. Control Theory for Automation: Fundamentals

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In this chapter autonomous dynamical systems, stability, asymptotic behavior, dynamical systems with inputs, feedback stabilization of linear systems, feedback stabilization of nonlinear systems, and tracking and regulation are discussed to provide the foundation for control theory for automation.

Modern engineering systems are very complex and comprise a high number of interconnected subcomponents which, thanks to the remarkable development of communications and electronics, can be spread over broad areas and linked through data networks. Each component of this wide interconnected system is a complex system on its own and the good functioning of the overall system relies upon the possibility to efficiently control, estimate or monitor each one of these components. Each component is usually high dimensional, highly nonlinear, and hybrid in nature, and comprises electrical, mechanical or chemical components which interact with computers, decision logics, etc. The behavior of each subsystem is affected by the behavior of part or all of the other components of the system. The control of those complex systems can only be achieved in a decentralized mode, by appropriately designing local controllers for each individual component or small group of components. In this setup, the interactions between components are mostly treated as *commands*, dictated from one particular unit to another one, or as *disturbances*, generated by the operation of other interconnected units. The tasks of the various local controllers are then coordinated by some supervisory unit. Control and computational capabilities being distributed over the system, a steady exchange of data among the components is required, in order for the system to behave properly.

In this setup, each individual component (or small set of components) is viewed as a system whose behavior, in time, is determined or influenced by the behavior of other subsystems. Typically, the physical variables by

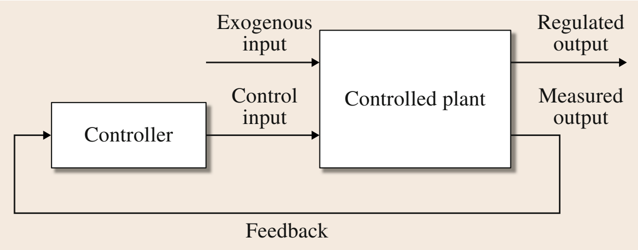

Fig. 9.1 Basic feedback loop

means of which this influence is exerted can be classified into two disjoint sets: one set consisting of all commands and/or disturbances generated by other components (which in this context are usually referred to as *exogenous inputs*) and another set consisting of all variables by means of which the accomplishment of the required tasks is actually imposed (which in this context are usually referred to as *control inputs*). The tasks in question typically comprise the case in which certain variables, called *regulated outputs*, are required to track the behavior of a set of exogenous commands. This leads to the definition, for the variables in question, of a *tracking error*, which should be kept as small as possible, in spite of the possible variation – in time – of the commands and in spite of all exogenous disturbances. The control input, in turn, is provided by a separate subsystem, the *controller*, which processes the information provided by a set of appropriate measurements (the *measured outputs*). The whole control configuration assumes – in this case – the form of a *feedback loop*, as shown in Fig. 9.1.

In any realistic scenario, the control goal has to be achieved in spite of a good number of phenomena which would cause the system to behave differently than expected. As a matter of fact, in addition to the exogenous phenomena already included in the scheme of Fig. 9.1, i.e., the exogenous commands and disturbances, a system may fail to behave as expected also because of endogenous causes, which include the case in which the controlled system responds differently as a consequence of poor knowledge about its behavior due to modeling errors, damages, wear, etc. The ability to handle large uncertainties successfully is one of the main, if not the single most important, reason for choosing the feedback configuration of Fig. 9.1.

To evaluate the overall performances of the system, a number of conventional criteria are chosen. First of all, it must be ensured that the behavior of the variables of the entire system is bounded. In fact, the feedback strategy, which is introduced for the purpose of offsetting exogenous inputs and to attenuate the effect of modeling error, may cause unbounded behaviors, which have to be avoided. Boundedness, and convergence to the desired behavior, are usually analyzed in conventional terms via the concepts of asymptotic *stability* and *steady-state behavior*, discussed in Sects. 9.2–9.3. Since the systems under considerations are systems with inputs (control inputs and exogenous inputs), the influence of such inputs on the behavior of a system also has to be assessed, as discussed in Sect. 9.4. The analytical tools developed in this way are then taken as a basis for the design of a controller, in which – usually – the control structure and free parameters are chosen in such a way as to guarantee that the overall configuration exhibits the desired properties in response to exogenous commands and disturbances and is sufficiently tolerant of any major source of uncertainty. This is discussed in Sects. 9.5–9.8.

9.1 Autonomous Dynamical Systems

In loose terms, a dynamical system is a way to describe how certain physical entities of interest, associated with a natural or artificial process, evolve in time and how their behavior is, or can be, influenced by the evolution of other variables. The most usual point of departure in the analysis of the behavior of a natural or artificial process is the construction of a mathematical model consisting of a set of equations expressing basic physical laws and/or constraints. In the most frequent case, when the study of evolution in time is the issue, the equations in question take the form of an ordinary differential equation, defined on a finite-dimensional Euclidean space. In this chapter, we shall review some fundamental facts underlying the analysis of the solutions of certain ordinary differential equations arising in the study of physical processes.

In this analysis, a convenient point of departure is the case of a mathematical model expressed by means of a first-order differential equation

$$\dot{x} = f(x) , \tag{9.1}$$

in which $\boldsymbol{x} \in \mathbb{R}^n$ is a vector of variables associated with the physical entities of interest, usually referred to as the *state* of the system. A solution of the differential equation (9.1) is a differentiable function $\bar{\boldsymbol{x}} : J \to \mathbb{R}^n$ defined on some interval $J \subset \mathbb{R}$ such that, for all $t \in J$,

$$\frac{\mathrm{d}\bar{\boldsymbol{x}}(t)}{\mathrm{d}t} = f(\bar{\boldsymbol{x}}(t)) .$$

If the map $f : \mathbb{R}^n \to \mathbb{R}^n$ is *locally Lipschitz*, i. e., if for every $\boldsymbol{x} \in \mathbb{R}^n$ there exists a neighborhood U of $\boldsymbol{x}$ and a number $L > 0$ such that, for all $\boldsymbol{x}_1, \boldsymbol{x}_2$ in U,

$$|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| \leq L|\boldsymbol{x}_1 - \boldsymbol{x}_2| ,$$

then, for each $\boldsymbol{x}_0 \in \mathbb{R}^n$ there exists two times $t^- < 0$ and $t^+ > 0$ and a solution $\bar{\boldsymbol{x}}$ of (9.1), defined on the interval $(t^-, t^+) \subset \mathbb{R}$, that satisfies $\bar{\boldsymbol{x}}(0) = \boldsymbol{x}_0$. Moreover, if $\tilde{\boldsymbol{x}}$: $(t^-, t^+) \to \mathbb{R}^n$ is any other solution of (9.1) satisfying $\tilde{\boldsymbol{x}}(0) = \boldsymbol{x}_0$, then necessarily $\tilde{\boldsymbol{x}}(t) = \bar{\boldsymbol{x}}(t)$ for all $t \in (t^-, t^+)$, that is, the solution $\bar{\boldsymbol{x}}$ is unique. In general, the times $t^- < 0$ and $t^+ > 0$ may depend on the point $\boldsymbol{x}_0$. For each $\boldsymbol{x}_0$, there is a maximal open interval $(t_\mathrm{m}^-(\boldsymbol{x}_0), t_\mathrm{m}^+(\boldsymbol{x}_0))$ containing 0 on which is defined a solution $\bar{\boldsymbol{x}}$ with $\bar{\boldsymbol{x}}(0) = \boldsymbol{x}_0$: this is the union of all open intervals on which there is a solution with $\bar{\boldsymbol{x}}(0) = \boldsymbol{x}_0$ (possibly, but not always, $t_\mathrm{m}^-(\boldsymbol{x}_0) = -\infty$ and/or $t_\mathrm{m}^+(\boldsymbol{x}_0) = +\infty$).

Given a differential equation of the form (9.1), associated with a locally Lipschitz map f, define a subset W of $\mathbb{R} \times \mathbb{R}^n$ as follows

$$W = \left\{(t, \boldsymbol{x}) : t \in \left(t_\mathrm{m}^-(\boldsymbol{x}), t_\mathrm{m}^+(\boldsymbol{x})\right), \boldsymbol{x} \in \mathbb{R}^n\right\} .$$

Then define on W a map $\phi : W \to \mathbb{R}^n$ as follows: $\phi(0, \boldsymbol{x}) = \boldsymbol{x}$ and, for each $\boldsymbol{x} \in \mathbb{R}^n$, the function

$$\begin{aligned} \varphi_x : \left(t_\mathrm{m}^-(\boldsymbol{x}), t_\mathrm{m}^+(\boldsymbol{x})\right) &\to \mathbb{R}^n , \\ t &\mapsto \phi(t, \boldsymbol{x}) \end{aligned}$$

is a solution of (9.1). This map is called the *flow* of (9.1). In other words, for each fixed $\boldsymbol{x}$, the restriction of $\phi(t, \boldsymbol{x})$ to the subset of W consisting of all pairs $(t, \boldsymbol{x})$ for which $t \in (t_\mathrm{m}^-(\boldsymbol{x}), t_\mathrm{m}^+(\boldsymbol{x}))$ is the unique (and maximally extended in time) solution of (9.1) passing through $\boldsymbol{x}$ at time $t = 0$.

A dynamical system is said to be *complete* if the set W coincides with the whole of $\mathbb{R} \times \mathbb{R}^n$.

Sometimes, a slightly different notation is used for the flow. This is motivated by the need tom express, within the same context, the flow of a system like (9.1) and the flow of another system, say $\dot{\boldsymbol{y}} = g(\boldsymbol{y})$. In this case, the symbol ϕ, which represents the *map*, must be replaced by two different symbols, one denoting the flow of (9.1) and the other denoting the flow of the other system. The easiest way to achieve this is to use the symbol $\boldsymbol{x}$ to represent the map that characterizes the flow of (9.1) and to use the symbol $\boldsymbol{y}$ to represent the map that characterizes the flow of the other system. In this way, the map characterizing the flow of (9.1) is written $\boldsymbol{x}(t, \boldsymbol{x})$. This notation at first may seem confusing, because the same symbol $\boldsymbol{x}$ is used to represent the map and to represent the second argument of the map itself (the argument representing the initial condition of (9.1)), but this is somewhat inevitable. Once the notation has been understood, though, no further confusion should arise.

In the special case of a *linear* differential equation

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} \tag{9.2}$$

in which $\mathbf{A}$ is an $n \times n$ matrix of real numbers, the flow is given by

$$\phi(t, \boldsymbol{x}) = \mathrm{e}^{\mathbf{A}t}\boldsymbol{x} ,$$

where the matrix exponential $\mathrm{e}^{\mathbf{A}t}$ is defined as the sum of the series

$$\mathrm{e}^{\mathbf{A}t} = \sum_{i=0}^{\infty} \frac{t^i}{i!}\mathbf{A}^i .$$

Let S be a subset of $\mathbb{R}^n$. The set S is said to be *invariant* for (9.1) if, for all $\boldsymbol{x} \in S$, $\phi(t, \boldsymbol{x})$ is defined for all $t \in (-\infty, +\infty)$ and

$$\phi(t, \boldsymbol{x}) \in S , \quad \text{for all } t \in \mathbb{R} .$$

A set S is *positively* (resp. negatively) *invariant* if for all $\boldsymbol{x} \in S$, $\phi(t, \boldsymbol{x})$ is defined for all $t \geq 0$ (resp. for all $t \leq 0$) and $\phi(t, \boldsymbol{x}) \in S$ for all such t.

Equation (9.1) defines a *dynamical system*. To reflect the fact that the map f does not depend on other independent entities (such as the time t or physical entities originated from *external* processes) the system in question is referred to an *autonomous* system. Complex autonomous systems arising in analysis and design of physical processes are usually obtained as a composition of simpler subsystems, each one modeled by equations of the form

$$\begin{aligned} \dot{\boldsymbol{x}}_i &= f_i(\boldsymbol{x}_i, \boldsymbol{u}_i) , \\ \boldsymbol{y}_i &= h_i(\boldsymbol{x}_i, \boldsymbol{u}_i) , \end{aligned} \qquad i = 1, \ldots, N ,$$

in which $\boldsymbol{x}_i \in \mathbb{R}^{n_i}$. Here $\boldsymbol{u}_i \in \mathbb{R}^{m_i}$ and, respectively, $\boldsymbol{y}_i \in \mathbb{R}^{p_i}$ are vectors of variables associated with physical entities by means of which the interconnection of various component parts is achieved.

9.2 Stability and Related Concepts

9.2.1 Stability of Equilibria

Consider an autonomous system as (9.1) and suppose that f is locally Lipschitz. A point $\boldsymbol{x}_e \in \mathbb{R}^n$ is called an *equilibrium* point if $f(\boldsymbol{x}_e) = 0$. Clearly, the constant function $\boldsymbol{x}(t) = \boldsymbol{x}_e$ is a solution of (9.1). Since solutions are unique, no other solution of (9.1) exists passing through $\boldsymbol{x}_e$. The study of equilibria plays a fundamental role in analysis and design of dynamical systems. The most important concept in this respect is that of *stability*, in the sense of *Lyapunov*, specified in the following definition. For $\boldsymbol{x} \in \mathbb{R}^n$, let $|\boldsymbol{x}|$ denote the usual Euclidean norm, that is,

$$|\boldsymbol{x}| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} .$$

Definition 9.1
An equilibrium $\boldsymbol{x}_e$ of (9.1) is *stable* if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\boldsymbol{x}(0) - x_e| \le \delta \Rightarrow |\boldsymbol{x}(t) - \boldsymbol{x}_e| \le \varepsilon , \quad \text{for all } t \ge 0 .$$

An equilibrium x_e of (9.1) is *asymptotically stable* if it is stable and, moreover, there exists a number $d > 0$ such that

$$|\boldsymbol{x}(0) - \boldsymbol{x}_e| \le d \Rightarrow \lim_{t\to\infty} |\boldsymbol{x}(t) - \boldsymbol{x}_e| = 0 .$$

An equilibrium x_e of (9.1) is *globally asymptotically stable* if it is asymptotically stable and, moreover,

$$\lim_{t\to\infty} |\boldsymbol{x}(t) - \boldsymbol{x}_e| = 0 , \quad \text{for every } \boldsymbol{x}(0) \in \mathbb{R}^n .$$

The most elementary, but rather useful in practice, result in stability analysis is described as follows. Assume that $f(\boldsymbol{x})$ is continuously differentiable and suppose, without loss of generality, that $\boldsymbol{x}_e = 0$ (if not, change $\boldsymbol{x}$ into $\bar{\boldsymbol{x}} := \boldsymbol{x} - \boldsymbol{x}_e$ and observe that $\bar{\boldsymbol{x}}$ satisfies the differential equation $\dot{\bar{\boldsymbol{x}}} = f(\bar{\boldsymbol{x}} + \boldsymbol{x}_e)$ in which now $\bar{\boldsymbol{x}} = 0$ is an equilibrium). Expand $f(\boldsymbol{x})$ as follows

$$f(\boldsymbol{x}) = \mathbf{A}\boldsymbol{x} + \tilde{f}(\boldsymbol{x}) , \tag{9.3}$$

in which

$$\mathbf{A} = \frac{\partial f}{\partial \boldsymbol{x}}(0)$$

is the Jacobian matrix of $f(\boldsymbol{x})$, evaluated at $\boldsymbol{x} = 0$, and by construction

$$\lim_{\boldsymbol{x}\to 0} \frac{|\tilde{f}(\boldsymbol{x})|}{|\boldsymbol{x}|} = 0 .$$

The linear system $\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x}$, with the matrix $\mathbf{A}$ defined as indicated, is called the *linear approximation* of the original nonlinear system (9.1) at the equilibrium $\boldsymbol{x} = 0$.

Theorem 9.1
Let $\boldsymbol{x} = 0$ be an equilibrium of (9.1). Suppose every eigenvalue of $\mathbf{A}$ has real part less than $-c$, with $c > 0$. Then, there are numbers $d > 0$ and $M > 0$ such that

$$|\boldsymbol{x}(0)| \le d \Rightarrow |\boldsymbol{x}(t)| \le M\,\mathrm{e}^{-ct}|\boldsymbol{x}(0)| , \quad \text{for all } t \ge 0 . \tag{9.4}$$

In particular, $\boldsymbol{x} = 0$ is asymptotically stable. If at least one eigenvalue of $\mathbf{A}$ has positive real part, the equilibrium $\boldsymbol{x} = 0$ is not stable.

This property is usually referred to as the *principle of stability in the first approximation*. The equilibrium $\boldsymbol{x} = 0$ is said to be *hyperbolic* if the matrix $\mathbf{A}$ has no eigenvalue with zero real part. Thus, it is seen from the previous Theorem that a hyperbolic equilibrium is either unstable or asymptotically stable.

The inequality on the right-hand side of (9.4) provides a useful bound on the norm of $\boldsymbol{x}(t)$, expressed as a function of the norm of $\boldsymbol{x}(0)$ and of the time t. This bound, though, is very special and restricted to the case of a hyperbolic equilibrium. In general, bounds of this kind can be obtained by means of the so-called *comparison functions*, which are defined as follows.

Definition 9.2
A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. If $a = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$, the function is said to belong to class $\mathcal{K}_\infty$. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the function

$$\alpha : [0, a) \to [0, \infty) , \quad r \mapsto \beta(r, s)$$

belongs to class $\mathcal{K}$ and, for each fixed r, the function

$$\varphi : [0, \infty) \to [0, \infty) , \quad s \mapsto \beta(r, s)$$

is decreasing and $\lim_{s\to\infty} \varphi(s) = 0$.

The composition of two class $\mathcal{K}$ (respectively, class $\mathcal{K}_\infty$) functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$, denoted $\alpha_1(\alpha_2(\cdot))$ or $\alpha_1 \circ \alpha_2(\cdot)$, is a class $\mathcal{K}$ (respectively, class $\mathcal{K}_\infty$) function. If $\alpha(\cdot)$ is a class $\mathcal{K}$ function, defined on $[0, a)$ and $b = \lim_{r\to a} \alpha(r)$, there exists a unique *inverse* function, $\alpha^{-1} : [0, b) \to [0, a)$, namely a function satisfying

$$\alpha^{-1}(\alpha(r)) = r, \quad \text{for all } r \in [0, a)$$

and

$$\alpha(\alpha^{-1}(r)) = r, \quad \text{for all } r \in [0, b)\,.$$

Moreover, $\alpha^{-1}(\cdot)$ is a class $\mathcal{K}$ function. If $\alpha(\cdot)$ is a class $\mathcal{K}_\infty$ function, so is also $\alpha^{-1}(\cdot)$.

The properties of stability, asymptotic stability, and global asymptotic stability can be easily expressed in terms of inequalities involving comparison functions. In fact, it turns out that the equilibrium $\boldsymbol{x} = 0$ is *stable* if and only if there exist a class $\mathcal{K}$ function $\alpha(\cdot)$ and a number $d > 0$ such that

$$|\boldsymbol{x}(t)| \le \alpha(|\boldsymbol{x}(0)|)\,,$$
$$\text{for all } \boldsymbol{x}(0) \text{ such that } |\boldsymbol{x}(0)| \le d \text{ and all } t \ge 0\,,$$

the equilibrium $\boldsymbol{x} = 0$ is *asymptotically stable* if and only if there exist a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a number $d > 0$ such that

$$|\boldsymbol{x}(t)| \le \beta(|\boldsymbol{x}(0)|, t)\,,$$
$$\text{for all } \boldsymbol{x}(0) \text{ such that } |\boldsymbol{x}(0)| \le d \text{ and all } t \ge 0\,,$$

and the equilibrium $\boldsymbol{x} = 0$ is *globally asymptotically stable* if and only if there exist a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that

$$|\boldsymbol{x}(t)| \le \beta(|\boldsymbol{x}(0)|, t)\,, \quad \text{for all } \boldsymbol{x}(0) \text{ and all } t \ge 0\,.$$

9.2.2 Lyapunov Functions

The most important criterion for the analysis of the stability properties of an equilibrium is the criterion of Lyapunov. We introduce first the special form that this criterion takes in the case of a linear system.

Consider the autonomous linear system

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x}$$

in which $\boldsymbol{x} \in \mathbb{R}^n$. Any symmetric $n \times n$ matrix $\mathbf{P}$ defines a *quadratic form*

$$V(\boldsymbol{x}) = \boldsymbol{x}^\top \mathbf{P} \boldsymbol{x}\,.$$

The matrix $\mathbf{P}$ is said to be *positive definite* (respectively, positive semidefinite) if so is the associated quadratic form $V(\boldsymbol{x})$, i. e., if, for all $\boldsymbol{x} \neq 0$,

$$V(\boldsymbol{x}) > 0\,, \quad \text{respectively } V(\boldsymbol{x}) \ge 0\,.$$

The matrix is said to be *negative definite* (respectively, negative semidefinite) if $-\mathbf{P}$ is positive definite (respectively, positive semidefinite). It is easy to show that a matrix $\mathbf{P}$ is positive definite if (and only if) there exist positive numbers $\underline{a}$ and $\overline{a}$ satisfying

$$\underline{a}|\boldsymbol{x}|^2 \le \boldsymbol{x}^\top \mathbf{P} \boldsymbol{x} \le \overline{a}|\boldsymbol{x}|^2\,, \tag{9.5}$$

for all $\boldsymbol{x} \in \mathbb{R}^n$. The property of a matrix $\mathbf{P}$ to be positive definite is usually expressed with the shortened notation $\mathbf{P} > 0$ (which actually means $\boldsymbol{x}^\top \mathbf{P}\boldsymbol{x} > 0$ for all $\boldsymbol{x} \neq 0$).

In the case of linear systems, the criterion of Lyapunov is expressed as follows.

Theorem 9.2
The linear system $\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x}$ is asymptotically stable (or, what is the same, the eigenvalues of $\mathbf{A}$ have negative real part) if there exists a positive-definite matrix $\mathbf{P}$ such that the matrix

$$\mathbf{Q} := \mathbf{PA} + \mathbf{A}^\top \mathbf{P}$$

is negative definite. Conversely, if the eigenvalues of $\mathbf{A}$ have negative real part, then, for any choice of a negative-definite matrix $\mathbf{Q}$, the linear equation

$$\mathbf{PA} + \mathbf{A}^\top \mathbf{P} = \mathbf{Q}$$

has a unique solution $\mathbf{P}$, which is positive definite.

Note that, if $V(\boldsymbol{x}) = \boldsymbol{x}^\top \mathbf{P}\boldsymbol{x}$,

$$\frac{\partial V}{\partial \boldsymbol{x}} = 2\boldsymbol{x}^\top \mathbf{P}$$

and hence

$$\frac{\partial V}{\partial \boldsymbol{x}} \mathbf{A}\boldsymbol{x} = \boldsymbol{x}^\top (\mathbf{PA} + \mathbf{A}^\top \mathbf{P})\boldsymbol{x}\,.$$

Thus, to say that the matrix $\mathbf{PA} + \mathbf{A}^\top \mathbf{P}$ is negative definite is equivalent to say that the form

$$\frac{\partial V}{\partial \boldsymbol{x}} \mathbf{A}\boldsymbol{x}$$

is negative definite.

The general, nonlinear, version of the criterion of Lyapunov appeals to the existence of a positive definite, but not necessarily quadratic, function of $\boldsymbol{x}$. The quadratic lower and upper bounds of (9.5) are therefore replaced by bounds of the form

$$\underline{\alpha}(|\boldsymbol{x}|) \le V(\boldsymbol{x}) \le \overline{\alpha}(|\boldsymbol{x}|)\,, \tag{9.6}$$

in which $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$ are simply class $\mathcal{K}$ functions. The criterion in question is summarized as follows.

Theorem 9.3
Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function satisfying (9.6) for some pair of class $\mathcal{K}$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$. If, for some $d > 0$,

$$\frac{\partial V}{\partial \boldsymbol{x}} f(\boldsymbol{x}) \leq 0\,, \qquad \text{for all } |\boldsymbol{x}| < d\,, \tag{9.7}$$

the equilibrium $\boldsymbol{x} = 0$ of (9.1) is stable. If, for some class $\mathcal{K}$ function $\alpha(\cdot)$ and some $d > 0$,

$$\frac{\partial V}{\partial \boldsymbol{x}} f(\boldsymbol{x}) \leq -\alpha(|\boldsymbol{x}|)\,, \qquad \text{for all } |\boldsymbol{x}| < d\,, \tag{9.8}$$

the equilibrium $\boldsymbol{x} = 0$ of (9.1) is locally asymptotically stable. If $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$ are class $\mathcal{K}_\infty$ functions and the inequality in (9.8) holds for all $\boldsymbol{x}$, the equilibrium $\boldsymbol{x} = 0$ of (9.1) is globally asymptotically stable.

A function $V(\boldsymbol{x})$ satisfying (9.6) and either of the subsequent inequalities is called a *Lyapunov function*. The inequality on the left-hand side of (9.6) is instrumental, together with (9.7), in establishing existence and boundedness of $\boldsymbol{x}(t)$. A simple explanation of the arguments behind the criterion of Lyapunov can be obtained in this way. Suppose (9.7) holds. Then, if $\boldsymbol{x}(0)$ is small, the differentiable function of time $V(\boldsymbol{x}(t))$ is defined for all $t \geq 0$ and nonincreasing along the trajectory $\boldsymbol{x}(t)$. Using the inequalities in (9.6) one obtains

$$\underline{\alpha}(|\boldsymbol{x}(t)|) \leq V(\boldsymbol{x}(t)) \leq V(\boldsymbol{x}(0)) \leq \overline{\alpha}(|\boldsymbol{x}(0)|)$$

and hence $|\boldsymbol{x}(t)| \leq \underline{\alpha}^{-1} \circ \overline{\alpha}(|\boldsymbol{x}(0)|)$, which establishes the stability of the equilibrium $\boldsymbol{x} = 0$.

Similar arguments are very useful in order to establish the *invariance*, in positive time, of certain bounded subsets of $\mathbb{R}^n$. Specifically, suppose the various inequalities considered in Theorem 9.3 hold for $d = \infty$ and let Ω_c denote the set of all $\boldsymbol{x} \in \mathbb{R}^n$ for which $V(\boldsymbol{x}) \leq c$, namely

$$\Omega_c = \{\boldsymbol{x} \in \mathbb{R}^n : V(\boldsymbol{x}) \leq c\}\,.$$

A set of this kind is called a *sublevel set* of the function $V(\boldsymbol{x})$. Note that, if $\underline{\alpha}(\cdot)$ is a class $\mathcal{K}_\infty$ function, then Ω_c is a compact set for all $c > 0$. Now, if

$$\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} f(\boldsymbol{x}) < 0$$

at each point $\boldsymbol{x}$ of the boundary of Ω_c, it can be concluded that, for any initial condition in the interior of Ω_c, the solution $\boldsymbol{x}(t)$ of (9.1) is defined for all $t \geq 0$ and is such that $\boldsymbol{x}(t) \in \Omega_c$ for all $t \geq 0$, that is, the set Ω_c is invariant in positive time. Indeed, existence and uniqueness are guaranteed by the local Lipschitz property so long as $\boldsymbol{x}(t) \in \Omega_c$, because Ω_c is a compact set. The fact that $\boldsymbol{x}(t)$ remains in Ω_c for all $t \geq 0$ is proved by contradiction. For, suppose that, for some trajectory $\boldsymbol{x}(t)$, there is a time t_1 such that $\boldsymbol{x}(t)$ is in the interior of Ω_c at all $t < t_1$ and $\boldsymbol{x}(t_1)$ is on the boundary of Ω_c. Then,

$$V(\boldsymbol{x}(t)) < c\,, \qquad \text{for all } t < t_1 \quad \text{and} \quad V(\boldsymbol{x}(t_1)) = c\,,$$

and this contradicts the previous inequality, which shows that the derivative of $V(\boldsymbol{x}(t))$ is strictly negative at $t = t_1$.

The criterion for asymptotic stability provided by the previous Theorem has a *converse*, namely, the existence of a function $V(\boldsymbol{x})$ having the properties indicated in Theorem 9.3 is *implied* by the property of asymptotic stability of the equilibrium $\boldsymbol{x} = 0$ of (9.1). In particular, the following result holds.

Theorem 9.4
Suppose the equilibrium $\boldsymbol{x} = 0$ of (9.1) is locally asymptotically stable. Then, there exist $d > 0$, a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, and class $\mathcal{K}$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, such that (9.6) and (9.8) hold. If the equilibrium $\boldsymbol{x} = 0$ of (9.1) is globally asymptotically stable, there exist a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, and class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, such that (9.6) and (9.8) hold with $d = \infty$.

To conclude, observe that, if $\boldsymbol{x} = 0$ is a hyperbolic equilibrium and all eigenvalues of $\mathbf{A}$ have negative real part, $|\boldsymbol{x}(t)|$ is bounded, for small $|\boldsymbol{x}(0)|$, by a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ of the form

$$\beta(r, t) = M\,\mathrm{e}^{-\lambda t} r\,.$$

If the equilibrium $\boldsymbol{x} = 0$ of system (9.1) is globally asymptotically stable and, moreover, there exist numbers $d > 0$, $M > 0$, and $\boldsymbol{\lambda} > 0$ such that

$$|\boldsymbol{x}(t)| \leq M\,\mathrm{e}^{-\lambda t}|\boldsymbol{x}(0)|\,, \qquad \text{for all } |\boldsymbol{x}(0)| \leq d \text{ and all } t \geq 0\,,$$

it is said that this equilibrium is *globally asymptotically and locally exponentially* stable. It can be shown that the equilibrium $\boldsymbol{x} = 0$ of the nonlinear system (9.1) is globally asymptotically and locally exponentially stable if and only if there exists a continuously differentiable function $V(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$, and class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and real numbers $\delta > 0$, $\underline{a} > 0$, $\overline{a} > 0$, $a > 0$, such that

$$\begin{aligned} \underline{\alpha}(|\boldsymbol{x}|) &\le V(\boldsymbol{x}) \le \overline{\alpha}(|\boldsymbol{x}|)\,, \\ \frac{\partial V}{\partial \boldsymbol{x}} f(\boldsymbol{x}) &\le -\alpha(|\boldsymbol{x}|)\,, \end{aligned} \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^n$$

and

$$\underline{\alpha}(s) = \underline{a}\, s^2\,, \quad \alpha(s) = \overline{a}\, s^2\,, \quad \alpha(s) = a\, s^2\,, \quad \text{for all } s \in [0, \delta]\,.$$

9.3 Asymptotic Behavior

9.3.1 Limit Sets

In the analysis of dynamical systems, it is often important to determine whether or not, as time increases, the variables characterizing the motion asymptotically converge to special motions exhibiting some form of recurrence. This is the case, for instance, when a system possesses an asymptotically stable equilibrium: all motions issued from initial conditions in a neighborhood of this point converge to a special motion in which all variables remain constant. A constant motion, or more generally a periodic motion, is characterized by a property of recurrence that is usually referred to as *steady-state* motion or behavior.

The steady-state behavior of a dynamical system can be viewed as a kind of *limit* behavior, approached either as the *actual* time t tends to $+\infty$ or, alternatively, as the *initial* time t_0 tends to $-\infty$. Relevant in this regard are certain concepts introduced by *Birkhoff* in [9.1]. In particular, a fundamental role is played by the concept of ω-limit set of a given point, defined as follows. Consider an *autonomous* dynamical system such as (9.1) and let $\boldsymbol{x}(t, \boldsymbol{x}_0)$ denote its flow. Assume, in particular, that $\boldsymbol{x}(t, \boldsymbol{x}_0)$ is defined for all $t \ge 0$. A point $\boldsymbol{x}$ is said to be an ω-limit *point* of the motion $\boldsymbol{x}(t, \boldsymbol{x}_0)$ if there exists a sequence of times $\{t_k\}$, with $\lim_{k\to\infty} t_k = \infty$, such that

$$\lim_{k\to\infty} \boldsymbol{x}(t_k, \boldsymbol{x}_0) = \boldsymbol{x}\,.$$

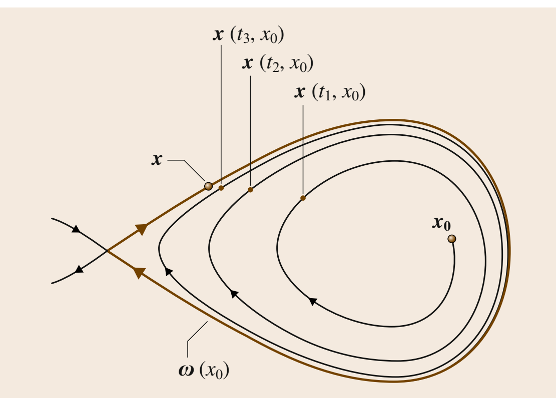

Fig. 9.2 The ω-limit set of a point x_0

The ω-limit *set* of a point $\boldsymbol{x}_0$, denoted $\omega(\boldsymbol{x}_0)$, is *the union* of all ω-limit points of the motion $\boldsymbol{x}(t, \boldsymbol{x}_0)$ (Fig. 9.2).

If $\boldsymbol{x}_e$ is an asymptotically stable equilibrium, then $\boldsymbol{x}_e = \omega(\boldsymbol{x}_0)$ for all $\boldsymbol{x}_0$ in a neighborhood of $\boldsymbol{x}_e$. However, in general, an ω-limit point *is not* necessarily a limit of $\boldsymbol{x}(t, \boldsymbol{x}_0)$ as $t \to \infty$, because the function in question may not admit any limit as $t \to \infty$. It happens though, that if the motion $\boldsymbol{x}(t, \boldsymbol{x}_0)$ is *bounded*, then $\boldsymbol{x}(t, \boldsymbol{x}_0)$ asymptotically approaches *the set* $\omega(\boldsymbol{x}_0)$.

Lemma 9.1

Suppose there is a number M such that $|\boldsymbol{x}(t, \boldsymbol{x}_0)| \le M$ for all $t \ge 0$. Then, $\omega(\boldsymbol{x}_0)$ is a nonempty compact connected set, invariant under (9.1). Moreover, the distance of $\boldsymbol{x}(t, \boldsymbol{x}_0)$ from $\omega(\boldsymbol{x}_0)$ tends to 0 as $t \to \infty$.

It is seen from this that the set $\omega(\boldsymbol{x}_0)$ is filled by motions of (9.1) which are *defined, and bounded, for all backward and forward times*. The other remarkable feature is that $\boldsymbol{x}(t, \boldsymbol{x}_0)$ *approaches* $\omega(\boldsymbol{x}_0)$ as $t \to \infty$, in the sense that the distance *of the point* $\boldsymbol{x}(t, \boldsymbol{x}_0)$ (the value at time t of the solution of (9.1) starting in $\boldsymbol{x}_0$ at time $t = 0$) *to the set* $\omega(\boldsymbol{x}_0)$ tends to 0 as $t \to \infty$. A consequence of this property is that, in a system of the form (9.1), if *all* motions issued from a set B are bounded, all such motions asymptotically approach the set

$$\Omega = \bigcup_{x_0 \in B} \omega(\boldsymbol{x}_0)\,.$$

However, the convergence of $\boldsymbol{x}(t, \boldsymbol{x}_0)$ to Ω is not guaranteed to be *uniform* in $\boldsymbol{x}_0$, even if the set B is compact. There is a larger set, though, which does have this property of uniform convergence. This larger set, known as the ω-limit set *of the set* B, is precisely defined as follows.

Consider again system (9.1), let B be a subset of $\mathbb{R}^n$, and suppose $\boldsymbol{x}(t, \boldsymbol{x}_0)$ is defined for all $t \ge 0$ and all

$x_0 \in B$. The ω-limit set of B, denoted $\omega(B)$, is the set of all points x for which there exists a sequence of pairs $\{x_k, t_k\}$, with $x_k \in B$ and $\lim_{k\to\infty} t_k = \infty$ such that

$$\lim_{k\to\infty} x(t_k, x_k) = x \, .$$

It follows from the definition that, if B consists of only one single point x_0, all x_k in the definition above are necessarily equal to x_0 and the definition in question reduces to the definition of ω-limit set of a point, given earlier. It also follows that, if for some $x_0 \in B$ the set $\omega(x_0)$ is nonempty, all points of $\omega(x_0)$ are points of $\omega(B)$. Thus, in particular, if all motions with $x_0 \in B$ are bounded in positive time,

$$\bigcup_{x_0 \in B} \omega(x_0) \subset \omega(B) \, .$$

However, the converse inclusion is not true in general.

The relevant properties of the ω-limit set of a set, which extend those presented earlier in Lemma 9.1, can be summarized as follows [9.2].

Lemma 9.2
Let B be a nonempty bounded subset of $\mathbb{R}^n$ and suppose there is a number M such that $|x(t, x_0)| \le M$ for all $t \ge 0$ and all $x_0 \in B$. Then $\omega(B)$ is a nonempty compact set, invariant under (9.1). Moreover, the distance of $x(t, x_0)$ from $\omega(B)$ tends to 0 as $t \to \infty$, uniformly in $x_0 \in B$. If B is connected, so is $\omega(B)$.

Thus, as is the case for the ω-limit set of a point, the ω-limit set of a bounded set B, being compact and invariant, is filled with motions which exist for all $t \in (-\infty, +\infty)$ and are bounded backward and forward in time. But, above all, the set in question is *uniformly* approached by motions with initial state $x_0 \in B$. An important corollary of the property of uniform convergence is that, if $\omega(B)$ is contained in the interior of B, then $\omega(B)$ is also asymptotically stable.

Lemma 9.3
Let B be a nonempty bounded subset of $\mathbb{R}^n$ and suppose there is a number M such that $|x(t, x_0)| \le M$ for all $t \ge 0$ and all $x_0 \in B$. Then $\omega(B)$ is a nonempty compact set, invariant under (9.1). Suppose also that $\omega(B)$ is contained in the interior of B. Then, $\omega(B)$ is asymptotically stable, with a domain of attraction that contains B.

9.3.2 Steady-State Behavior

Consider now again system (9.1), with initial conditions in a closed subset $X \subset \mathbb{R}^n$. Suppose the set X is *positively invariant*, which means that, for any initial condition $x_0 \in X$, the solution $x(t, x_0)$ exists for all $t \ge 0$ and $x(t, x_0) \in X$ for all $t \ge 0$. The motions of this system are said to be *ultimately bounded* if there is a bounded subset B with the property that, for every compact subset X_0 of X, there is a time $T > 0$ such that $x(t, x_0) \in B$ for all $t \ge T$ and all $x_0 \in X_0$. In other words, if the motions of the system are ultimately bounded, every motion eventually enters and remains in the bounded set B.

Suppose the motions of (9.1) are ultimately bounded and let $B' \ne B$ be any other bounded subset with the property that, for every compact subset X_0 of X, there is a time $T > 0$ such that $x(t, x_0) \in B'$ for all $t \ge T$ and all $x_0 \in X_0$. Then, it is easy to check that $\omega(B') = \omega(B)$. Thus, in view of the properties described in Lemma 9.2 above, the following definition can be adopted [9.3].

Definition 9.3
Suppose the motions of system (9.1), with initial conditions in a closed and positively invariant set X, are ultimately bounded. A *steady-state* motion is any motion with initial condition $x(0) \in \omega(B)$. The set $\omega(B)$ is the *steady-state locus* of (9.1) and the *restriction* of (9.1) to $\omega(B)$ is the *steady-state behavior* of (9.1).

9.4 Dynamical Systems with Inputs

9.4.1 Input-to-State Stability (ISS)

In this section we show how to determine the stability properties of an interconnected system, on the basis of the properties of each individual component. The easiest interconnection to be analyzed is a cascade connection of two subsystems, namely a system of the form

$$\begin{aligned} \dot{x} &= f(x, z) \, , \\ \dot{z} &= g(z) \, , \end{aligned} \tag{9.9}$$

with $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$ in which we assume $f(0, 0) = 0$, $g(0) = 0$.

If the equilibrium $x = 0$ of $\dot{x} = f(x, 0)$ is locally asymptotically stable and the equilibrium $z = 0$ of the lower subsystem is locally asymptotically stable then the equilibrium $(x, z) = (0, 0)$ of the cascade is locally asymptotically stable. However, in general, *global* asymptotic stability of the equilibrium $x = 0$ of $\dot{x} = f(x, 0)$ and *global* asymptotic stability of the equilibrium $z = 0$ of the lower subsystem *do not* imply *global* asymptotic stability of the equilibrium $(x, z) = (0, 0)$ of the cascade. To infer global asymptotic stability of the cascade, a stronger condition is needed, which expresses a property describing how – in the upper subsystem – the response $x(\cdot)$ is influenced by its *input* $z(\cdot)$.

The property in question requires that, when $z(t)$ is bounded over the semi-infinite time interval $[0, +\infty)$, then also $x(t)$ be bounded, and in particular that, if $z(t)$ asymptotically decays to 0, then also $x(t)$ decays to 0. These requirements altogether lead to the notion of *input-to-state stability*, introduced and studied in [9.4, 5]. The notion in question is defined as follows (see also [9.6, Chap. 10] for additional details). Consider a nonlinear system

$$\dot{x} = f(x, u) , \tag{9.10}$$

with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$, in which $f(0, 0) = 0$ and $f(x, u)$ is locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$. The input function $u : [0, \infty) \to \mathbb{R}^m$ of (9.10) can be any piecewise-continuous bounded function. The set of all such functions, endowed with the supremum norm

$$\|u(\cdot)\|_\infty = \sup_{t \geq 0} |u(t)|$$

is denoted by L^m_∞.

Definition 9.4

System (9.10) is said to be input-to-state stable if there exist a class $\mathcal{KL}$ function $\beta(\cdot, \cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$, called a gain function, such that, for any input $u(\cdot) \in L^m_\infty$ and any $x_0 \in \mathbb{R}^n$, the response $x(t)$ of (9.10) in the initial state $x(0) = x_0$ satisfies

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(\|u(\cdot)\|_\infty) , \quad \text{for all } t \geq 0 . \tag{9.11}$$

It is common practice to replace the wording *input-to-state stable* with the acronym ISS. In this way, a system possessing the property expressed by (9.11) is said to be an ISS system. Since, for any pair $\beta > 0$, $\gamma > 0$, $\max\{\beta, \gamma\} \leq \beta + \gamma \leq \max\{2\beta, 2\gamma\}$, an alternative way to say that a system is input-to-state stable is to say that there exists a class $\mathcal{KL}$ function $\beta(\cdot, \cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that, for any input $u(\cdot) \in L^m_\infty$ and any $x_0 \in \mathbb{R}^n$, the response $x(t)$ of (9.10) in the initial state $x(0) = x_0$ satisfies

$$|x(t)| \leq \max\{\beta(|x_0|, t), \gamma(\|u(\cdot)\|_\infty)\} , \quad \text{for all } t \geq 0 . \tag{9.12}$$

The property, for a given system, of being input-to-state stable, can be given a characterization which extends the criterion of Lyapunov for asymptotic stability. The key tool for this analysis is the notion of *ISS-Lyapunov function*, defined as follows.

Definition 9.5

A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is an ISS-Lyapunov function for system (9.10) if there exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and a class $\mathcal{K}$ function $\chi(\cdot)$ such that

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) , \quad \text{for all } x \in \mathbb{R}^n \tag{9.13}$$

and

$$|x| \geq \chi(|u|) \Rightarrow \frac{\partial V}{\partial x} f(x, u) \leq -\alpha(|x|) , \quad \text{for all } x \in \mathbb{R}^n \text{ and } u \in \mathbb{R}^m . \tag{9.14}$$

An alternative, equivalent, definition is the following one.

Definition 9.6

A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is an ISS-Lyapunov function for system (9.10) if there exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$, $\alpha(\cdot)$, and a class $\mathcal{K}$ function $\sigma(\cdot)$ such that (9.13) holds and

$$\frac{\partial V}{\partial x} f(x, u) \leq -\alpha(|x|) + \sigma(|u|) , \quad \text{for all } x \in \mathbb{R}^n \text{ and all } u \in \mathbb{R}^m . \tag{9.15}$$

The importance of the notion of ISS-Lyapunov function resides in the following criterion, which extends the criterion of Lyapunov for global asymptotic stability to systems with inputs.

Theorem 9.5

System (9.10) is input-to-state stable if and only if there exists an ISS-Lyapunov function.

The comparison functions appearing in the estimates (9.13) and (9.14) are useful to obtain an estimate of the gain function $\gamma(\cdot)$ which characterizes the bound (9.12). In fact, it can be shown that, if system (9.10) possesses an ISS-Lyapunov function $V(\boldsymbol{x})$, the sublevel set

$$\Omega_{\|u(\cdot)\|_\infty} = \{\boldsymbol{x} \in \mathbb{R}^n : V(\boldsymbol{x}) \le \overline{\alpha}(\chi(\|\boldsymbol{u}(\cdot)\|_\infty))\}$$

is invariant in positive time for (9.10). Thus, in view of the estimates (9.13), if the initial state of the system is initially inside this sublevel set, the following estimate holds

$$|\boldsymbol{x}(t)| \le \underline{\alpha}^{-1}\left(\overline{\alpha}(\chi(\|\boldsymbol{u}(\cdot)\|_\infty))\right), \quad \text{for all } t \ge 0 ,$$

and one can obtain an estimate of $\gamma(\cdot)$ as

$$\gamma(r) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(r) .$$

In other words, establishing the existence of an ISS-Lyapunov function $V(\boldsymbol{x})$ is useful not only to check whether or not the system in question is input-to-state stable, but also to determine an estimate of the gain function $\gamma(\cdot)$. Knowing such estimate is important, as will be shown later, in using the concept of input-to-state stability to determine the stability of interconnected systems.

The following simple examples may help understanding the concept of input-to-state stability and the associated Lyapunov-like theorem.

Example 9.1: Consider a linear system

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u} ,$$

with $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{u} \in \mathbb{R}^m$ and suppose that all the eigenvalues of the matrix $\mathbf{A}$ have negative real part. Let $\mathbf{P} > 0$ denote the unique solution of the Lyapunov equation $\mathbf{PA} + \mathbf{A}^\top\mathbf{P} = -\mathbf{I}$. Observe that the function $V(\boldsymbol{x}) = \boldsymbol{x}^\top\mathbf{P}\boldsymbol{x}$ satisfies

$$\underline{a}|\boldsymbol{x}|^2 \le V(\boldsymbol{x}) \le \overline{a}|\boldsymbol{x}|^2 ,$$

for suitable $\underline{a} > 0$ and $\overline{a} > 0$, and that

$$\frac{\partial V}{\partial \boldsymbol{x}}(\mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u}) \le -|\boldsymbol{x}|^2 + 2|\boldsymbol{x}||\mathbf{P}||\mathbf{B}||\boldsymbol{u}| .$$

Pick any $0 < \varepsilon < 1$ and set

$$c = \frac{2}{1-\varepsilon}|\mathbf{P}||\mathbf{B}| , \quad \chi(r) = cr .$$

Then

$$|\boldsymbol{x}| \ge \chi(|\boldsymbol{u}|) \Rightarrow \frac{\partial V}{\partial \boldsymbol{x}}(\mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u}) \le -\varepsilon|\boldsymbol{x}|^2 .$$

Thus, the system is input-to-state stable, with a gain function

$$\gamma(r) = (c\,\overline{a}/\underline{a})\,r$$

which is a *linear* function.

Consider now the simple nonlinear one-dimensional system

$$\dot{\boldsymbol{x}} = -a\boldsymbol{x}^k + \boldsymbol{x}^p\boldsymbol{u} ,$$

in which $k \in \mathbb{N}$ is odd, $p \in \mathbb{N}$ satisfies $p < k$, and $a > 0$. Choose a candidate ISS-Lyapunov function as $V(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^2$, which yields

$$\frac{\partial V}{\partial \boldsymbol{x}} f(\boldsymbol{x}, \boldsymbol{u}) = \\ -a\boldsymbol{x}^{k+1} + \boldsymbol{x}^{p+1}\boldsymbol{u} \le -a|\boldsymbol{x}|^{k+1} + |\boldsymbol{x}|^{p+1}|\boldsymbol{u}| .$$

Set $\nu = k - p$ to obtain

$$\frac{\partial V}{\partial \boldsymbol{x}} f(\boldsymbol{x}, \boldsymbol{u}) \le |\boldsymbol{x}|^{p+1}\left(-a|\boldsymbol{x}|^\nu + |\boldsymbol{u}|\right) .$$

Thus, using the class $\mathcal{K}_\infty$ function $\alpha(r) = \varepsilon r^{k+1}$, with $\varepsilon > 0$, it is deduced that

$$\frac{\partial V}{\partial \boldsymbol{x}} f(\boldsymbol{x}, \boldsymbol{u}) \le -\alpha(|\boldsymbol{x}|)$$

provided that

$$(a - \varepsilon)|\boldsymbol{x}|^\nu \ge |\boldsymbol{u}| .$$

Taking, without loss of generality, $\varepsilon < a$, it is concluded that condition (9.14) holds for the class $\mathcal{K}$ function

$$\chi(r) = \left(\frac{r}{a-\varepsilon}\right)^{\frac{1}{\nu}} .$$

Thus, the system is input-to-state stable.

An important feature of the previous example, which made it possible to prove the system is input-to-state stable, is the inequality $p < k$. In fact, if this inequality does not hold, the system may fail to be input-to-state stable. This can be seen, for instance, in the simple example

$$\dot{\boldsymbol{x}} = -\boldsymbol{x} + \boldsymbol{x}\boldsymbol{u} .$$

To this end, suppose $\boldsymbol{u}(t) = 2$ for all $t \ge 0$. The state response of the system, to this input, from the initial state $\boldsymbol{x}(0) = \boldsymbol{x}_0$ coincides with that of the autonomous system $\dot{\boldsymbol{x}} = \boldsymbol{x}$, i. e., $\boldsymbol{x}(t) = \mathrm{e}^t\boldsymbol{x}_0$, which shows that the bound (9.11) cannot hold.

We conclude with an alternative characterization of the property of input-to-state stability, which is useful in many instances [9.7].

Theorem 9.6
System (9.10) is input-to-state stable if and only if there exist class $\mathcal{K}$ functions $\gamma_0(\cdot)$ and $\gamma(\cdot)$ such that, for any input $\boldsymbol{u}(\cdot) \in L_\infty^m$ and any $\boldsymbol{x}_0 \in \mathbb{R}^n$, the response $\boldsymbol{x}(t)$ in the initial state $\boldsymbol{x}(0) = \boldsymbol{x}_0$ satisfies

$$\|\boldsymbol{x}(\cdot)\|_\infty \leq \max\{\gamma_0(|\boldsymbol{x}_0|), \gamma(\|\boldsymbol{u}(\cdot)\|_\infty)\} ,$$
$$\limsup_{t\to\infty} |\boldsymbol{x}(t)| \leq \gamma(\limsup_{t\to\infty} |\boldsymbol{u}(t)|) .$$

9.4.2 Cascade Connections

The property of input-to-state stability is of paramount importance in the analysis of interconnected systems. The first application consists of the analysis of the *cascade connection*. In fact, the cascade connection of two input-to-state stable systems turns out to be input-to-state stable. More precisely, consider a system of the form (Fig. 9.3)

$$\begin{aligned}\dot{\boldsymbol{x}} &= f(\boldsymbol{x}, z) ,\\ \dot{z} &= g(z, \boldsymbol{u}) ,\end{aligned} \tag{9.16}$$

in which $\boldsymbol{x} \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $f(0,0) = 0$, $g(0,0) = 0$, and $f(\boldsymbol{x}, z)$, $g(z, \boldsymbol{u})$ are locally Lipschitz.

Theorem 9.7
Suppose that system

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, z) , \tag{9.17}$$

viewed as a system with input z and state $\boldsymbol{x}$, is input-to-state stable and that system

$$\dot{z} = g(z, \boldsymbol{u}) , \tag{9.18}$$

viewed as a system with input $\boldsymbol{u}$ and state z, is input-to-state stable as well. Then, system (9.16) is input-to-state stable.

As an immediate corollary of this theorem, it is possible to answer the question of when the cascade connection (9.9) is globally asymptotically stable. In fact, if system

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, z) ,$$

viewed as a system with input z and state $\boldsymbol{x}$, is input-to-state stable and the equilibrium $z = 0$ of the lower subsystem is globally asymptotically stable, the equilibrium $(\boldsymbol{x}, z) = (0, 0)$ of system (9.9) is globally asymptotically stable. This is in particular the case if system (9.9) has the special form

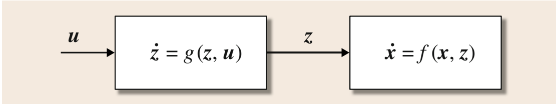

Fig. 9.3 Cascade connection

$$\begin{aligned}\dot{\boldsymbol{x}} &= \mathbf{A}\boldsymbol{x} + p(z) ,\\ \dot{z} &= g(z) ,\end{aligned} \tag{9.19}$$

with $p(0) = 0$ and the matrix $\mathbf{A}$ has all eigenvalues with negative real part. The upper subsystem of the cascade is input-to-state stable and hence, if the equilibrium $z = 0$ of the lower subsystem is globally asymptotically stable, so is the equilibrium $(\boldsymbol{x}, z) = (0, 0)$ of the entire system.

9.4.3 Feedback Connections

In this section we investigate the stability property of nonlinear systems, and we will see that the property of input-to-state stability lends itself to a simple characterization of an important *sufficient condition* under which the feedback interconnection of two globally asymptotically stable systems remains globally asymptotically stable.

Consider the following interconnected system (Fig. 9.3)

$$\begin{aligned}\dot{\boldsymbol{x}}_1 &= f_1(\boldsymbol{x}_1, \boldsymbol{x}_2) ,\\ \dot{\boldsymbol{x}}_2 &= f_2(\boldsymbol{x}_1, \boldsymbol{x}_2, u) ,\end{aligned} \tag{9.20}$$

in which $\boldsymbol{x}_1 \in \mathbb{R}^{n_1}$, $\boldsymbol{x}_2 \in \mathbb{R}^{n_2}$, $\boldsymbol{u} \in \mathbb{R}^m$, and $f_1(0,0) = 0$, $f_2(0,0,0) = 0$. Suppose that the first subsystem, viewed as a system with internal state $\boldsymbol{x}_1$ and input $\boldsymbol{x}_2$, is input-to-state stable. Likewise, suppose that the second subsystem, viewed as a system with internal state $\boldsymbol{x}_2$ and inputs $\boldsymbol{x}_1$ and $\boldsymbol{u}$, is input-to-state stable. In view of the results presented earlier, the hypothesis of input-to-state stability of the first subsystem is equivalent to the existence of functions $\beta_1(\cdot,\cdot)$, $\gamma_1(\cdot)$, the first of class $\mathcal{KL}$

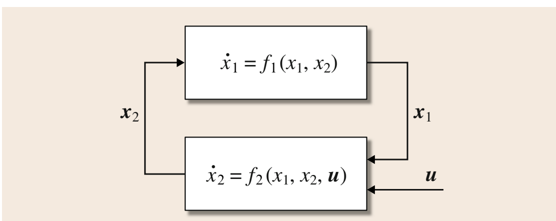

Fig. 9.4 Feedback connection

and the second of class $\mathcal{K}$, such that the response $\boldsymbol{x}_1(\cdot)$ to any input $\boldsymbol{x}_2(\cdot) \in L_\infty^{n_2}$ satisfies

$$|\boldsymbol{x}_1(t)| \leq \max\{\beta_1(\|\boldsymbol{x}_1(0)\|, t), \gamma_1(\|\boldsymbol{x}_2(\cdot)\|_\infty)\} , \quad \text{for all } t \geq 0 . \tag{9.21}$$

Likewise the hypothesis of input-to-state stability of the second subsystem is equivalent to the existence of three class functions $\beta_2(\cdot)$, $\gamma_2(\cdot)$, $\gamma_u(\cdot)$ such that the response $\boldsymbol{x}_2(\cdot)$ to any input $\boldsymbol{x}_1(\cdot) \in L_\infty^{n_1}$, $\boldsymbol{u}(\cdot) \in L_\infty^m$ satisfies

$$|\boldsymbol{x}_2(t)| \leq \max\{\beta_2(\|\boldsymbol{x}_2(0)\|, t), \gamma_2(\|\boldsymbol{x}_1(\cdot)\|_\infty), \gamma_u(\|\boldsymbol{u}(\cdot)\|_\infty)\} , \quad \text{for all } t \geq 0 . \tag{9.22}$$

The important result for the analysis of the stability of the interconnected system (9.20) is that, if the composite function $\gamma_1 \circ \gamma_2(\cdot)$ is a *simple contraction*, i. e., if

$$\gamma_1(\gamma_2(r)) < r , \quad \text{for all } r > 0 , \tag{9.23}$$

the system in question is input-to-state stable. This result is usually referred to as the *small-gain* theorem.

Theorem 9.8
If the condition (9.23) holds, system (9.20), viewed as a system with state $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2)$ and input $\boldsymbol{u}$, is input-to-state stable.

The condition (9.23), i. e., the condition that the composed function $\gamma_1 \circ \gamma_2(\cdot)$ is a contraction, is usually referred to as the *small-gain condition*. It can be written in different alternative ways depending on how the functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ are estimated. For instance, if it is known that $V_1(\boldsymbol{x}_1)$ is an ISS-Lyapunov function for the upper subsystem of (9.20), i. e., a function such

$$\underline{\alpha}_1(|\boldsymbol{x}_1|) \leq V_1(\boldsymbol{x}_1) \leq \overline{\alpha}_1(|\boldsymbol{x}_1|) ,$$
$$|\boldsymbol{x}_1| \geq \chi_1(|\boldsymbol{x}_2|) \Rightarrow \frac{\partial V_1}{\partial \boldsymbol{x}_1} f_1(\boldsymbol{x}_1, \boldsymbol{x}_2) \leq -\alpha(|\boldsymbol{x}_1|) ,$$

then $\gamma_1(\cdot)$ can be estimated by

$$\gamma_1(r) = \underline{\alpha}_1^{-1} \circ \overline{\alpha}_1 \circ \chi_1(r) .$$

Likewise, if $V_2(\boldsymbol{x}_2)$ is a function such that

$$\underline{\alpha}_2(|\boldsymbol{x}_2|) \leq V_2(\boldsymbol{x}_2) \leq \overline{\alpha}_2(|\boldsymbol{x}_2|) ,$$
$$|\boldsymbol{x}_2| \geq \max\{\chi_2(|\boldsymbol{x}_1|), \chi_u(|\boldsymbol{u}|)\} \Rightarrow \frac{\partial V_2}{\partial \boldsymbol{x}_2} f_2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}) \leq -\alpha(|\boldsymbol{x}_2|) ,$$

then $\gamma_2(\cdot)$ can be estimated by

$$\gamma_2(r) = \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \chi_2(r) .$$

If this is the case, the small-gain condition of the theorem can be written in the form

$$\underline{\alpha}_1^{-1} \circ \overline{\alpha}_1 \circ \chi_1 \circ \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \chi_2(r) < r .$$

9.4.4 The Steady-State Response

In this subsection we show how the concept of steady state, introduced earlier, and the property of input-to-state stability are useful in the analysis of the *steady-state response* of a system to inputs generated by a separate autonomous dynamical system [9.8].

Example 9.2: Consider an n-dimensional, single-input, *asymptotically stable* linear system

$$\dot{z} = \mathbf{F}z + \mathbf{G}u \tag{9.24}$$

forced by the harmonic input $\boldsymbol{u}(t) = u_0 \sin(\omega t + \phi_0)$. A simple method to analyze the asymptotic behavior of (9.24) consists of viewing the forcing input $\boldsymbol{u}(t)$ as provided by an autonomous *signal generator* of the form

$$\dot{\boldsymbol{w}} = \mathbf{S}\boldsymbol{w} ,$$
$$\boldsymbol{u} = \mathbf{Q}\boldsymbol{w} ,$$

in which

$$\mathbf{S} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} , \quad \mathbf{Q} = \begin{pmatrix} 1 & 0 \end{pmatrix} ,$$

and in analyzing the state-state behavior of the associated *augmented* system

$$\dot{\boldsymbol{w}} = \mathbf{S}\boldsymbol{w} ,$$
$$\dot{z} = \mathbf{F}z + \mathbf{G}\mathbf{Q}\boldsymbol{w} . \tag{9.25}$$

As a matter of fact, let $\boldsymbol{\Pi}$ be the unique solution of the Sylvester equation $\boldsymbol{\Pi}\mathbf{S} = \mathbf{F}\boldsymbol{\Pi} + \mathbf{G}\mathbf{Q}$ and observe that the graph of the linear map $z = \boldsymbol{\Pi}\boldsymbol{w}$ is an invariant subspace for the system (9.25). Since all trajectories of (9.25) approach this subspace as $t \to \infty$, the limit behavior of (9.25) is determined by the restriction of its motion to this invariant subspace.

Revisiting this analysis from the viewpoint of the more general notion of steady-state introduced earlier, let $W \subset \mathbb{R}^2$ be a set of the form

$$W = \{\boldsymbol{w} \in \mathbb{R}^2 : \|\boldsymbol{w}\| \leq c\} , \tag{9.26}$$

in which c is a fixed number, and suppose the set of initial conditions for (9.25) is $W \times \mathbb{R}^n$. This is in fact the case when the problem of evaluating the periodic response of (9.24) to harmonic inputs whose amplitude does not exceed a fixed number c is addressed. The set W is compact and invariant for the upper subsystem of (9.25) and, as is easy to check, the ω-limit set of W under the motion of the upper subsystem of (9.25) is the subset W itself.

The set $W \times \mathbb{R}^n$ is closed and positively invariant for the full system (9.25) and, moreover, since the lower subsystem of (9.25) is input-to-state stable, the motions of system of (9.25), for initial conditions taken in $W \times \mathbb{R}^n$, are ultimately bounded. It is easy to check that

$$\omega(\mathbf{B}) = \{(\boldsymbol{w}, z) \in \mathbb{R}^2 \times \mathbb{R}^n : \boldsymbol{w} \in W, z = \boldsymbol{\Pi} \boldsymbol{w}\} ,$$

i. e., that $\omega(\mathbf{B})$ is the graph of the restriction of the map $z = \boldsymbol{\Pi}\boldsymbol{w}$ to the set W. The restriction of (9.25) to the invariant set $\omega(\mathbf{B})$ characterizes the steady-state behavior of (9.24) under the family of all harmonic inputs of fixed angular frequency ω and amplitude not exceeding c.

Example 9.3: A similar result, namely the fact that the *steady-state locus* is the *graph* of a map, can be reached if the *signal generator* is any nonlinear system, with initial conditions chosen in a compact invariant set W. More precisely, consider an augmented system of the form

$$\begin{aligned} \dot{\boldsymbol{w}} &= s(\boldsymbol{w}) , \\ \dot{z} &= \mathbf{F}z + \mathbf{G}q(\boldsymbol{w}) , \end{aligned} \tag{9.27}$$

in which $\boldsymbol{w} \in W \subset \mathbb{R}^r$, $\boldsymbol{x} \in \mathbb{R}^n$, and assume that: (i) all eigenvalues of F have negative real part, and (ii) the set W is a compact set, invariant for the the upper subsystem of (9.27).

As in the previous example, the ω-limit set of W under the motion of the upper subsystem of (9.27) is the subset W itself. Moreover, since the lower subsystem of (9.27) is input-to-state stable, the motions of system (9.27), for initial conditions taken in $W \times \mathbb{R}^n$, are ultimately bounded. It is easy to check that the steady-state locus of (9.27) is the graph of the map

$$\begin{aligned} \pi : W &\to \mathbb{R}^n , \\ \boldsymbol{w} &\mapsto \pi(\boldsymbol{w}) , \end{aligned}$$

defined by

$$\pi(\boldsymbol{w}) = \lim_{T\to\infty} \int_{-T}^{0} \mathrm{e}^{-F\tau} \mathbf{G} q(\boldsymbol{w}(\tau, \boldsymbol{w}))\,\mathrm{d}\tau . \tag{9.28}$$

There are various ways in which the result discussed in the previous example can be generalized; for instance, it can be extended to describe the steady-state response of a nonlinear system

$$\dot{z} = f(z, \boldsymbol{u}) \tag{9.29}$$

in the neighborhood of a locally exponentially stable equilibrium point. To this end, suppose that $f(0,0) = 0$ and that the matrix

$$\mathbf{F} = \left[\frac{\partial f}{\partial z}\right](0,0)$$

has all eigenvalues with negative real part. Then, it is well known (see, e.g., [9.9, p. 275]) that it is always possible to find a compact subset $Z \subset \mathbb{R}^n$, which contains $z = 0$ in its interior and a number $\sigma > 0$ such that, if $|z_0| \in Z$ and $\|\boldsymbol{u}(t)\| \leq \sigma$ for all $t \geq 0$, the solution of (9.29) with initial condition $z(0) = z_0$ satisfies $|z(t)| \in Z$ for all $t \geq 0$. Suppose that the input $\boldsymbol{u}$ to (9.29) is produced, as before, by a signal generator of the form

$$\begin{aligned} \dot{\boldsymbol{w}} &= s(\boldsymbol{w}) , \\ \boldsymbol{u} &= q(\boldsymbol{w}) , \end{aligned} \tag{9.30}$$

with initial conditions chosen in a compact invariant set W and, moreover, suppose that, $\|q(\boldsymbol{w})\| \leq \sigma$ for all $\boldsymbol{w} \in W$. If this is the case, the set $W \times Z$ is positively invariant for

$$\begin{aligned} \dot{\boldsymbol{w}} &= s(\boldsymbol{w}) , \\ \dot{z} &= f(z, q(\boldsymbol{w})) , \end{aligned} \tag{9.31}$$

and the motions of the latter are ultimately bounded, with $\mathbf{B} = W \times Z$. The set $\omega(\mathbf{B})$ may have a complicated structure but it is possible to show, by means of arguments similar to those which are used in the proof of the center manifold theorem, that if Z and $\mathbf{B}$ are small enough, the set in question can still be expressed as the graph of a map $z = \pi(\boldsymbol{w})$. In particular, the graph in question is precisely the center manifold of (9.31) at $(0,0)$ if $s(0) = 0$, and the matrix

$$S = \left[\frac{\partial s}{\partial \boldsymbol{w}}\right](0)$$

has all eigenvalues on the imaginary axis.

A common feature of the examples discussed above is the fact that the steady-state locus of a system of the form (9.31) can be expressed as the graph of a map $z = \pi(w)$. This means that, so long as this is the case, a system of this form has a *unique* well-defined *steady-state response* to the input $u(t) = q(w(t))$. As a matter of fact, the response in question is precisely $z(t) = \pi(w(t))$. Of course, this may not always be the case and *multiple* steady-state responses to a given input may occur. In general, the following property holds.

Lemma 9.4
Let W be a compact set, invariant under the flow of (9.30). Let Z be a closed set and suppose that the motions of (9.31) with initial conditions in $W \times Z$ are ultimately bounded. Then, the steady-state locus of (9.31) is the graph of a set-valued map defined on the whole of W.

9.5 Feedback Stabilization of Linear Systems

9.5.1 Stabilization by Pure State Feedback

Consider a linear system, modeled by equations of the form

$$\dot{x} = \mathbf{A}x + \mathbf{B}u \, ,$$
$$y = \mathbf{C}x \, , \tag{9.32}$$

in which $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$, and in which $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are matrices with real entries.

We begin by analyzing the influence, on the response of the system, of control law of the form

$$u = \mathbf{F}x \, , \tag{9.33}$$

in which $\mathbf{F}$ is an $n \times m$ matrix with real entries. This type of control is usually referred to as *pure state feedback* or *memoryless state feedback*. The imposition of this control law on the first equation of (9.32) yields the autonomous linear system

$$\dot{x} = (\mathbf{A} + \mathbf{BF})x \, .$$

The purpose of the design is to choose $\mathbf{F}$ so as to obtain, if possible, a prescribed asymptotic behavior. In general, two options are sought: (i) the n eigenvalues of $(\mathbf{A} + \mathbf{BF})$ have negative real part, (ii) the n eigenvalues of $(\mathbf{A} + \mathbf{BF})$ coincide with the n roots of an arbitrarily fixed polynomial

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots a_1\lambda + a_0$$

of degree n, with real coefficients. The first option is usually referred to as the *stabilization* problem, while the second is usually referred to as the *eigenvalue assignment* problem.

The conditions for the existence of solutions of these problems can be described as follows. Consider the $n \times (n+m)$ polynomial matrix

$$\mathbf{M}(\lambda) = \big((\mathbf{A} - \lambda\mathbf{I}) \; \mathbf{B} \big) \, . \tag{9.34}$$

Definition 9.7
System (9.32) is said to be *stabilizable* if, for all λ which is an eigenvalue of $\mathbf{A}$ and has nonnegative real part, the matrix $\mathbf{M}(\lambda)$ has rank n. This system is said to be *controllable* if, for all λ which is an eigenvalue of $\mathbf{A}$, the matrix $\mathbf{M}(\lambda)$ has rank n.

The two properties thus identified determine the existence of solutions of the problem of stabilization and, respectively, of the problem of eigenvalue assignment. In fact, the following two results hold.

Theorem 9.9
There exists a matrix $\mathbf{F}$ such that $\mathbf{A} + \mathbf{BF}$ has all eigenvalues with negative real part if and only if system (9.32) is stabilizable.

Theorem 9.10
For any choice of a polynomial $p(\lambda)$ of degree n with real coefficients there exists a matrix $\mathbf{F}$ such that the n eigenvalues of $\mathbf{A} + \mathbf{BF}$ coincide with the n roots of $p(\lambda)$ if and only if system (9.32) is controllable.

The actual construction of the matrix $\mathbf{F}$ usually requires a preliminary transformation of the equations describing the system. As an example, we illustrate how this is achieved in the case of a single-input system, for the problem of eigenvalue assignment. If the input of a system is one dimensional, the system is controllable if and only if the $n \times n$ matrix

$$\mathbf{P} = \big(\mathbf{B} \;\; \mathbf{AB} \;\; \cdots \;\; \mathbf{A}^{n-1}\mathbf{B} \big) \tag{9.35}$$

is nonsingular. Assuming that this is the case, let γ denote the last row of $\mathbf{P}^{-1}$, that is, the unique solution of the set of equations

$$\gamma\mathbf{B} = \gamma\mathbf{AB} = \cdots = \gamma\mathbf{A}^{n-2}\mathbf{B} = 0,$$
$$\gamma\mathbf{A}^{n-1}\mathbf{B} = 1 .$$

Then, simple manipulations show that the change of coordinates

$$\tilde{\boldsymbol{x}} = \begin{pmatrix} \gamma \\ \gamma\mathbf{A} \\ \cdots \\ \gamma\mathbf{A}^{n-1} \end{pmatrix} \boldsymbol{x}$$

transforms system (9.32) into a system of the form

$$\dot{\tilde{\boldsymbol{x}}} = \tilde{\mathbf{A}}\tilde{\boldsymbol{x}} + \tilde{\mathbf{B}}u ,$$
$$\boldsymbol{y} = \tilde{\mathbf{C}}\tilde{\boldsymbol{x}} \tag{9.36}$$

in which

$$\tilde{\mathbf{A}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ d_0 & d_1 & d_2 & \cdots & d_{n-2} & d_{n-1} \end{pmatrix} , \quad \tilde{\mathbf{B}} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix} .$$

This form is known as *controllability canonical form* of the equations describing the system. If a system is written in this form, the solution of the problem of eigenvalue assignment is straightforward. If suffices, in fact, to pick a control law of the form

$$\boldsymbol{u} = -(d_0 + a_0)\tilde{x}_1 - (d_1 + a_1)\tilde{x}_2 - \cdots$$
$$- (d_{n-1} + a_{n-1})\tilde{x}_n := \tilde{\mathbf{F}}\tilde{\boldsymbol{x}} \tag{9.37}$$

to obtain a system

$$\dot{\tilde{\boldsymbol{x}}} = (\tilde{\mathbf{A}} + \tilde{\mathbf{B}}\tilde{\mathbf{F}})\tilde{\boldsymbol{x}}$$

in which

$$\tilde{\mathbf{A}} + \tilde{\mathbf{B}}\tilde{\mathbf{F}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix} .$$

The characteristic polynomial of this matrix coincides with the prescribed polynomial $p(\lambda)$ and hence the problem is solved. Rewriting the law (9.37) in the original coordinates, one obtains a formula that directly expresses the matrix $\mathbf{F}$ in terms of the parameters of the system (the $n \times n$ matrix $\mathbf{A}$ and the $1 \times n$ row vector γ) and of the coefficients of the prescribed polynomial $p(\lambda)$

$$\boldsymbol{u} = -\gamma\Big[(d_0 + a_0)\mathbf{I} + (d_1 + a_1)\mathbf{A} + \cdots$$
$$+ (d_{n-1} + a_{n-1})\mathbf{A}^{n-1}\Big]\boldsymbol{x}$$
$$= -\gamma\Big[a_0\mathbf{I} + a_1\mathbf{A} + \cdots$$
$$+ a_{n-1}\mathbf{A}^{n-1} + \mathbf{A}^n\Big]\boldsymbol{x} := \mathbf{F}\boldsymbol{x} .$$

The latter is known as Ackermann's formula.

9.5.2 Observers and State Estimation

The imposition of a control law of the form (9.33) requires the availability of all n components of the state $\boldsymbol{x}$ of system (9.32) for measurement, which is seldom the case. Thus, the issue arises of when and how the components in question could be, at least asymptotically, estimated by means of an appropriate auxiliary dynamical system driven by the only variables that are actually accessible for measurement, namely the input $\boldsymbol{u}$ and the output $\boldsymbol{y}$.

To this end, consider a n-dimensional system thus defined

$$\dot{\hat{\boldsymbol{x}}} = \mathbf{A}\hat{\boldsymbol{x}} + \mathbf{B}\boldsymbol{u} + \mathbf{G}(\boldsymbol{y} - \mathbf{C}\hat{\boldsymbol{x}}) , \tag{9.38}$$

viewed as a system with state $\hat{\boldsymbol{x}} \in \mathbb{R}^n$, driven by the *inputs* $\boldsymbol{u}$ and $\boldsymbol{y}$. This system can be interpreted as a *copy* of the original dynamics of (9.32), namely

$$\dot{\hat{\boldsymbol{x}}} = \mathbf{A}\hat{\boldsymbol{x}} + \mathbf{B}\boldsymbol{u}$$

corrected by a term proportional, through the $n \times p$ weighting matrix $\mathbf{G}$, to the *effect* that a possible difference between $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ has on the only available measurement. The idea is to determine $\mathbf{G}$ in such a way that $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ asymptotically converge. Define the difference

$$\boldsymbol{E} = \boldsymbol{x} - \hat{\boldsymbol{x}} ,$$

which is called *observation error*. Simple algebra shows that

$$\dot{\boldsymbol{e}} = (\mathbf{A} - \mathbf{GC})\boldsymbol{e} .$$

Thus, the observation error obeys an autonomous linear differential equation, and its asymptotic behavior is completely determined by the eigenvalues of $(\mathbf{A} - \mathbf{GC})$. In general, two options are sought: (i) the n eigenvalues of $(\mathbf{A} - \mathbf{GC})$ have negative real part, (ii) the n eigenvalues of $(\mathbf{A} - \mathbf{GC})$ coincide with the n roots of an arbitrarily fixed polynomial of degree n having real coefficients. The first option is usually referred to as the *asymptotic state estimation* problem, while the second does not carry a special name.

Note that, if the eigenvalues of $(\mathbf{A}-\mathbf{GC})$ have negative real part, the state $\hat{\boldsymbol{x}}$ of the auxiliary system (9.38) satisfies

$$\lim_{t\to\infty}[\boldsymbol{x}(t)-\hat{\boldsymbol{x}}(t)]=0\,,$$

i.e., it asymptotically tracks the state $\boldsymbol{x}(t)$ of (9.32) regardless of what the initial states $\boldsymbol{x}(0)$, $\hat{\boldsymbol{x}}(0)$ and the input $u(t)$ are. System (9.38) is called an asymptotic state estimator or a Luenberger *observer*.

The conditions for the existence of solutions of these problems can be described as follows. Consider the $(n+p)\times n$ polynomial matrix

$$\mathbf{N}(\boldsymbol{\lambda})=\begin{pmatrix}(\mathbf{A}-\boldsymbol{\lambda}\mathbf{I})\\ \mathbf{C}\end{pmatrix}\,. \tag{9.39}$$

Definition 9.8
System (9.32) is said to be *detectable* if, for all $\boldsymbol{\lambda}$ which is an eigenvalue of $\mathbf{A}$ and has nonnegative real part, the matrix $\mathbf{N}(\boldsymbol{\lambda})$ has rank n. This system is said to be *observable* if, for all $\boldsymbol{\lambda}$ which is an eigenvalue of $\mathbf{A}$, the matrix $\mathbf{N}(\boldsymbol{\lambda})$ has rank n.

Theorem 9.11
There exists a matrix $\mathbf{G}$ such that $\mathbf{A}-\mathbf{GC}$ has all eigenvalues with negative real part if and only if system (9.32) is detectable.

Theorem 9.12
For any choice of a polynomial $p(\boldsymbol{\lambda})$ of degree n with real coefficients there exists a matrix G such that the n eigenvalues of $\mathbf{A}-\mathbf{GC}$ coincide with the n roots of $p(\boldsymbol{\lambda})$ if and only if system (9.32) is observable.

In this case, also, the actual construction of the matrix $\mathbf{G}$ is made simple by transforming the equations describing the system. If the output of a system is one dimensional, the system is observable if and only if the $n\times n$ matrix

$$\mathbf{Q}=\begin{pmatrix}\mathbf{C}\\ \mathbf{CA}\\ \cdots\\ \mathbf{CA}^{n-1}\end{pmatrix} \tag{9.40}$$

is nonsingular. Let this be the case and let β denote the last column of $\mathbf{Q}^{-1}$, that is, the unique solution of the set of equations

$$\mathbf{C}\beta=\mathbf{CA}\beta=\cdots=\mathbf{CA}^{n-2}\beta=0,\quad \mathbf{CA}^{n-1}\beta=1\,.$$

Then, simple manipulations show that the change of coordinates

$$\tilde{\boldsymbol{x}}=\begin{pmatrix}\mathbf{A}^{n-1}\beta & \cdots & \mathbf{A}\beta & \beta\end{pmatrix}^{-1}\boldsymbol{x}$$

transforms system (9.32) into a system of the form

$$\begin{aligned}\dot{\tilde{x}}&=\tilde{\mathbf{A}}\tilde{\boldsymbol{x}}+\tilde{\mathbf{B}}\boldsymbol{u}\,,\\ \boldsymbol{y}&=\tilde{\mathbf{C}}\tilde{\boldsymbol{x}}\end{aligned} \tag{9.41}$$

in which

$$\tilde{\mathbf{A}}=\begin{pmatrix}d_{n-1} & 1 & 0 & \cdots & 0 & 0\\ d_{n-2} & 0 & 1 & \cdots & 0 & 0\\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot\\ d_1 & 0 & 0 & \cdots & 0 & 1\\ d_0 & 0 & 0 & \cdots & 0 & 0\end{pmatrix},$$

$$\tilde{\mathbf{C}}=\begin{pmatrix}1 & 0 & \cdots & 0 & 0\end{pmatrix}.$$

This form is known as *observability canonical form* of the equations describing the system. If a system is written in this form, it is straightforward to write a matrix $\tilde{\mathbf{G}}$ assigning the eigenvalues to $(\tilde{\mathbf{A}}-\tilde{\mathbf{G}}\tilde{\mathbf{C}})$. If suffices, in fact, to pick a

$$\tilde{\mathbf{G}}=\begin{pmatrix}d_{n-1}+a_{n-1}\\ d_{n-2}+a_{n-2}\\ \cdots\\ d_0+a_0\end{pmatrix} \tag{9.42}$$

to obtain a matrix

$$\tilde{\mathbf{A}}-\tilde{\mathbf{G}}\tilde{\mathbf{C}}=\begin{pmatrix}-a_{n-1} & 1 & 0 & \cdots & 0 & 0\\ -a_{n-2} & 0 & 1 & \cdots & 0 & 0\\ \cdot & \cdot & \cdot & \cdots & \cdot & 0\\ -a_1 & 0 & 0 & \cdots & 0 & 1\\ -a_0 & 0 & 0 & \cdots & 0 & 0\end{pmatrix},$$

whose characteristic polynomial coincides with the prescribed polynomial $p(\boldsymbol{\lambda})$.

9.5.3 Stabilization via Dynamic Output Feedback

Replacing, in the control law (9.33), the true state $\boldsymbol{x}$ by the estimate $\hat{\boldsymbol{x}}$ provided by the asymptotic observer

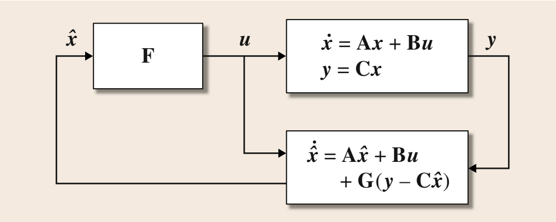

Fig. 9.5 Observer-based control

(9.38) yields a dynamic, output-feedback, control law of the form

$$u = \mathbf{F}\hat{x} ,$$
$$\dot{\hat{x}} = (\mathbf{A}+\mathbf{BF}-\mathbf{GC})\hat{x}+\mathbf{G}y . \quad (9.43)$$

Controlling system (9.32) by means of (9.43) yields the closed-loop system (Fig. 9.5)

$$\begin{pmatrix}\dot{x}\\ \dot{\hat{x}}\end{pmatrix} = \begin{pmatrix}\mathbf{A} & \mathbf{BF}\\ \mathbf{GC} & \mathbf{A}+\mathbf{BF}-\mathbf{GC}\end{pmatrix}\begin{pmatrix}x\\ \hat{x}\end{pmatrix} . \quad (9.44)$$

It is straightforward to check that the eigenvalues of the system thus obtained coincide with those of the two matrices $(\mathbf{A}+\mathbf{BF})$ and $(\mathbf{A}-\mathbf{GC})$. To this end, in fact, it suffices to replace $\hat{x}$ by $e = x - \hat{x}$, which changes system (9.44) into an equivalent system

$$\begin{pmatrix}\dot{x}\\ \dot{e}\end{pmatrix} = \begin{pmatrix}\mathbf{A}+\mathbf{BF} & -\mathbf{BF}\\ 0 & \mathbf{A}-\mathbf{GC}\end{pmatrix}\begin{pmatrix}x\\ e\end{pmatrix} \quad (9.45)$$

in block-triangular form.

From this argument, it can be concluded that the dynamic feedback law (9.43) suffices to yield a closed-loop system whose $2n$ eigenvalues either have negative real part (if system (9.32) is stabilizable and detectable) or even coincide with the roots of a pair of prescribed polynomials of degree n (if (9.32) is controllable and observable). In particular, the result in question can be achieved by means of a *separate* design of $\mathbf{F}$ and $\mathbf{G}$, the former to control the eigenvalues of $(\mathbf{A}+\mathbf{BF})$ and the latter to control the eigenvalues of $(\mathbf{A}-\mathbf{GC})$. This possibility is usually referred to as the *separation principle* for stabilization via (dynamic) output feedback.

It can be concluded from this argument that, if a system is stabilizable and detectable, there exists a dynamic, output feedback, law yielding a closed-loop system with all eigenvalues with negative real part. It is important to observe that also the converse of this property is true, namely the existence of a dynamic, output feedback, law yielding a closed-loop system with all eigenvalues with negative real part requires the controlled system to be *stabilizable and detectable*. The proof of this converse result is achieved by taking any arbitrary dynamic output-feedback law

$$\dot{\xi} = \bar{\mathbf{F}}\xi + \bar{\mathbf{G}}y ,$$
$$u = \bar{\mathbf{H}}\xi + \bar{\mathbf{K}}y ,$$

yielding a closed-loop system

$$\begin{pmatrix}\dot{x}\\ \dot{\xi}\end{pmatrix} = \begin{pmatrix}\mathbf{A}+\mathbf{B}\bar{\mathbf{K}}\mathbf{C} & \mathbf{B}\bar{\mathbf{H}}\\ \bar{\mathbf{G}}\mathbf{C} & \bar{\mathbf{F}}\end{pmatrix}\begin{pmatrix}x\\ \xi\end{pmatrix}$$

and proving, via the converse Lyapunov theorem for linear systems, that, if the eigenvalues of the latter have negative real part, *necessarily* there exist two matrices $\mathbf{F}$ and $\mathbf{G}$ such that the eigenvalues of $(\mathbf{A}+\mathbf{BF})$ and, respectively, $(\mathbf{A}-\mathbf{GC})$ have negative real part.

9.6 Feedback Stabilization of Nonlinear Systems

9.6.1 Recursive Methods for Global Stability

Stabilization of nonlinear systems is a very difficult task and general methods are not available. Only if the equations of the system exhibit a special structure do there exist systematic methods for the design of pure state feedback (or, if necessary, dynamic, output feedback) laws yielding global asymptotic stability of an equilibrium. In this section we review some of these special design procedures.

We begin by a simple *modular* property which can be recursively used to stabilize systems in *triangular form* (see [9.10, Chap. 9] for further details).

Lemma 9.5
Consider a system described by equations of the form

$$\dot{z} = f(z,\xi) ,$$
$$\dot{\xi} = q(z,\xi)+b(z,\xi)u , \quad (9.46)$$

in which $(z,\xi)\in\mathbb{R}^n\times\mathbb{R}$, and the functions $f(z,\xi)$, $q(z,\xi)$, $b(z,\xi)$ are continuously differentiable functions. Suppose that $b(z,\xi)\neq 0$ for all (z,ξ) and that $f(0,0)=0$ and $q(0,0)=0$. If $z=0$ is a globally asymptotically stable equilibrium of $\dot{z}=f(z,0)$, there exists a differentiable function $u = u(z,\xi)$ with

$\boldsymbol{u}(0,0)=0$ such that the equilibrium at $(z,\boldsymbol{\xi})=(0,0)$

$$\dot{z}=f(z,\boldsymbol{\xi})\,,$$
$$\dot{\boldsymbol{\xi}}=q(z,\boldsymbol{\xi})+b(z,\boldsymbol{\xi})\boldsymbol{u}(z,\boldsymbol{\xi})\,,$$

is globally asymptotically stable.

The construction of the stabilizing feedback $\boldsymbol{u}(z,\boldsymbol{\xi})$ is achieved as follows. First of all observe that, using the assumption $b(z,\boldsymbol{\xi})\neq 0$, the imposition of the preliminary feedback law

$$\boldsymbol{u}(z,\boldsymbol{\xi})=\frac{1}{b(z,\boldsymbol{\xi})}\left(-q(z,\boldsymbol{\xi})+v\right)$$

yields the simpler system

$$\dot{z}=f(z,\boldsymbol{\xi})\,,$$
$$\dot{\boldsymbol{\xi}}=v\,.$$

Then, express $f(z,\boldsymbol{\xi})$ in the form

$$f(z,\boldsymbol{\xi})=f(z,0)+p(z,\boldsymbol{\xi})\boldsymbol{\xi}\,,$$

in which $p(z,\boldsymbol{\xi})=[f(z,\boldsymbol{\xi})-f(z,0)]/\boldsymbol{\xi}$ is at least continuous.

Since by assumption $z=0$ is a globally asymptotically stable equilibrium of $\dot{z}=f(z,0)$, by the converse Lyapunov theorem there exists a smooth real-valued function $V(z)$, which is positive definite and proper, satisfying

$$\frac{\partial V}{\partial z}f(z,0)<0\,,$$

for all nonzero z. Now, consider the positive-definite and proper function

$$W(z,\boldsymbol{\xi})=V(z)+\frac{1}{2}\boldsymbol{\xi}^2\,,$$

and observe that

$$\frac{\partial W}{\partial z}\dot{z}+\frac{\partial W}{\partial \boldsymbol{\xi}}\dot{\boldsymbol{\xi}}=\frac{\partial V}{\partial z}f(z,0)+\frac{\partial V}{\partial z}p(z,\boldsymbol{\xi})\boldsymbol{\xi}+\boldsymbol{\xi}v\,.$$

Choosing

$$v=-\boldsymbol{\xi}-\frac{\partial V}{\partial z}p(z,\boldsymbol{\xi}) \qquad (9.47)$$

yields

$$\frac{\partial W}{\partial z}\dot{z}+\frac{\partial W}{\partial \boldsymbol{\xi}}\dot{\boldsymbol{\xi}}=\frac{\partial V}{\partial z}f(z,0)-\boldsymbol{\xi}^2<0\,,$$

for all nonzero $(z,\boldsymbol{\xi})$ and this, by the direct Lyapunov criterion, shows that the feedback law

$$\boldsymbol{u}(z,\boldsymbol{\xi})=\frac{1}{b(z,\boldsymbol{\xi})}\left[-q(z,\boldsymbol{\xi})-\boldsymbol{\xi}-\frac{\partial V}{\partial z}p(z,\boldsymbol{\xi})\right]$$

globally asymptotically stabilizes the equilibrium $(z,\boldsymbol{\xi})=(0,0)$ of the associated closed-loop system.

In the next Lemma (which contains the previous one as a particular case) this result is extended by showing that, for the purpose of stabilizing the equilibrium $(z,\boldsymbol{\xi})=(0,0)$ of system (9.46), it suffices to assume that the equilibrium $z=0$ of

$$\dot{z}=f(z,\boldsymbol{\xi})$$

is *stabilizable* by means of a *virtual* control law $\boldsymbol{\xi}=v^\star(z)$.

Lemma 9.6
Consider again the system described by equations of the form (9.46). Suppose there exists a continuously differentiable function

$$\boldsymbol{\xi}=v^\star(z)\,,$$

with $v^\star(0)=0$, which globally asymptotically stabilizes the equilibrium $z=0$ of $\dot{z}=f(z,v^\star(z))$. Then there exists a differentiable function $\boldsymbol{u}=\boldsymbol{u}(z,\boldsymbol{\xi})$ with $\boldsymbol{u}(0,0)=0$ such that the equilibrium at $(z,\boldsymbol{\xi})=(0,0)$

$$\dot{z}=f(z,\boldsymbol{\xi})\,,$$
$$\dot{\boldsymbol{\xi}}=q(z,\boldsymbol{\xi})+b(z,\boldsymbol{\xi})\boldsymbol{u}(z,\boldsymbol{\xi})$$

is globally asymptotically stable.

To prove the result, and to construct the stabilizing feedback, it suffices to consider the (globally defined) change of variables

$$\boldsymbol{y}=\boldsymbol{\xi}-v^\star(z)\,,$$

which transforms (9.46) into a system

$$\dot{z}=f(z,v^\star(z)+\boldsymbol{y})\,,$$
$$\dot{\boldsymbol{y}}=-\frac{\partial v^\star}{\partial z}f(z,v^\star(z)+\boldsymbol{y})+q(v^\star(z)+\boldsymbol{y},\boldsymbol{\xi})$$
$$+b(v^\star(z)+\boldsymbol{y},\boldsymbol{\xi})\boldsymbol{u}\,, \qquad (9.48)$$

which meets the assumptions of Lemma 9.5, and then follow the construction of a stabilizing feedback as described. Using repeatedly the property indicated in Lemma 9.6 it is straightforward to derive the expression of a globally stabilizing feedback for a system in *triangular* form

$$\dot{z}=f(z,\xi_1)\,,$$
$$\dot{\xi}_1=q_1(z,\xi_1)+b_1(z,\xi_1)\xi_2\,,$$
$$\dot{\xi}_2=q_2(z,\xi_1,\xi_2)+b_2(z,\xi_1,\xi_2)\xi_3\,,$$
$$\dots$$
$$\dot{\xi}_r=q_r(z,\xi_1,\xi_2,\dots,\xi_r)+b_r(z,\xi_1,\xi_2,\dots,\xi_r)\boldsymbol{u}\,. \qquad (9.49)$$

To this end, in fact, it suffices to assume that the equilibrium $z = 0$ of $\dot{z} = f(z, \boldsymbol{\xi})$ is stabilizable by means of a virtual law $\boldsymbol{\xi} = v^\star(z)$, and that $b_1(z, \xi_1), b_2(z, \xi_1, \xi_2), \ldots, b_r(z, \xi_1, \xi_2, \ldots, \xi_r)$ are nowhere zero.

9.6.2 Semiglobal Stabilization via Pure State Feedback

The global stabilization results presented in the previous section are indeed conceptually appealing but the actual implementation of the feedback law requires the explicit knowledge of a Lyapunov function $V(z)$ for the system $\dot{z} = f(z, 0)$ (or for the system $\dot{z} = f(z, v^*(z))$ in the case of Lemma 9.6). This function, in fact, explicitly determines the structure of the feedback law which globally asymptotically stabilizes the system. Moreover, in the case of systems of the form (9.49) with $r > 1$, the computation of the feedback law is somewhat cumbersome, in that it requires to iterate a certain number of times the manipulations described in the proof of Lemmas 9.5 and 9.6. In this section we show how these drawbacks can be overcome, in a certain sense, if a less ambitious design goal is pursued, namely if instead of seeking global stabilization one is interested in a feedback law capable of asymptotically steering to the equilibrium point all trajectories which have origin in a *a priori fixed* (and hence possibly large) *bounded set*.

Consider again a system satisfying the assumptions of Lemma 9.5. Observe that $b(z, \boldsymbol{\xi})$, being continuous and nowhere zero, has a well-defined sign. Choose a simple control law of the form

$$\boldsymbol{u} = -k \,\mathrm{sign}(b)\, \boldsymbol{\xi} \tag{9.50}$$

to obtain the system

$$\begin{aligned} \dot{z} &= f(z, \boldsymbol{\xi})\,, \\ \dot{\boldsymbol{\xi}} &= q(z, \boldsymbol{\xi}) - k|b(z, \boldsymbol{\xi})|\boldsymbol{\xi}\,. \end{aligned} \tag{9.51}$$

Assume that the equilibrium $z = 0$ of $\dot{z} = f(z, 0)$ is globally asymptotically but also *locally exponentially* stable. If this is the case, then the linear approximation of the first equation of (9.51) at the point $(z, \boldsymbol{\xi}) = (0, 0)$ is a system of the form

$$\dot{z} = \mathbf{F}z + \mathbf{G}\boldsymbol{\xi}\,,$$

in which $\mathbf{F}$ is a Hurwitz matrix. Moreover, the linear approximation of the second equation of (9.51) at the point $(z, \boldsymbol{\xi}) = (0, 0)$ is a system of the form

$$\dot{\boldsymbol{\xi}} = \mathbf{Q}z + \mathbf{R}\boldsymbol{\xi} - kb_0\boldsymbol{\xi}\,,$$

in which $b_0 = |b(0, 0)|$. It follows that the linear approximation of system (9.51) at the equilibrium $(z, \boldsymbol{\xi}) = (0, 0)$ is a linear system $\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x}$ in which

$$\mathbf{A} = \begin{pmatrix} \mathbf{F} & \mathbf{G} \\ \mathbf{Q} & (\mathbf{R} - kb_0) \end{pmatrix}.$$

Standard arguments show that, if the number k is large enough, the matrix in question has all eigenvalues with negative real part (in particular, as k increases, n eigenvalues approach the n eigenvalues of $\mathbf{F}$ and the remaining one is a real eigenvalue that tends to $-\infty$). It is therefore concluded, from the principle of stability in the first approximation, that if k is sufficiently large the equilibrium $(z, \boldsymbol{\xi}) = (0, 0)$ of the closed-loop system (9.51) is locally asymptotically (actually locally exponentially) stable.

However, a stronger result holds. It can be proven that, for any arbitrary compact subset K of $\mathbb{R}^n \times \mathbb{R}$, there exists a number k^*, such that, for all $k \geq k^*$, the equilibrium $(z, \boldsymbol{\xi}) = (0, 0)$ of the closed-loop system (9.51) is locally asymptotically stable and all initial conditions in K produce a trajectory that asymptotically converges to this equilibrium. In other words, the basin of attraction of the equilibrium $(z, \boldsymbol{\xi}) = (0, 0)$ of the closed-loop system contains the set K. Note that the number k^* depends on the choice of the set K and, in principle, it increases as the size of K increases. The property in question can be summarized as follows (see [9.10, Chap. 9] for further details). A system

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u})$$

is said to be *semiglobally stabilizable* (an equivalent, but longer, terminology is *asymptotically stabilizable with guaranteed basin of attraction*) at a given point $\bar{\boldsymbol{x}}$ if, for each *compact* subset $K \subset \mathbb{R}^n$, there exists a feedback law $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$, *which in general depends on* K, such that in the corresponding closed-loop system

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}))$$

the point $\boldsymbol{x} = \bar{\boldsymbol{x}}$ is a locally asymptotically stable equilibrium, and

$$\boldsymbol{x}(0) \in K \Rightarrow \lim_{t\to\infty} \boldsymbol{x}(t) = \bar{x}$$

(i. e., the compact subset K is contained in the basin of attraction of the equilibrium $\boldsymbol{x} = \bar{\boldsymbol{x}}$). The result described above shows that system (9.46), under the said assumptions, is semiglobally stabilizable at $(z, \boldsymbol{\xi}) = (0, 0)$, by means of a feedback law of the form (9.50).

The arguments just shown can be iterated to deal with a system of the form (9.49). In fact, it is easy to realize that, if the equilibrium $z = 0$ of $\dot{z} = f(z, 0)$ is globally asymptotically and also locally exponentially stable, if $q_i(z, \xi_1, \xi_2, \ldots, \xi_i)$ vanishes at $(z, \xi_1, \xi_2, \ldots, \xi_i) = (0, 0, 0, \ldots, 0)$ and $b_i(z, \xi_1, \xi_2, \ldots, \xi_i)$ is nowhere zero, for all $i = 1, \ldots, r$, system (9.49) is semiglobally stabilizable at the point $(z, \xi_1, \xi_2, \ldots, \xi_r) = (0, 0, 0, \ldots, 0)$, actually by means of a control law that has the following structure

$$\boldsymbol{u} = \alpha_1 \xi_1 + \alpha_2 \xi_2 + \cdots + \alpha_r \xi_r .$$

The coefficients $\alpha_1, \ldots, \alpha_r$ that characterize this control law can be determined by means of recursive iteration of the arguments described above.

9.6.3 Semiglobal Stabilization via Dynamic Output Feedback

System (9.49) can be semiglobally stabilized, at the equilibrium $(z, \xi_1, \ldots, \xi_r) = (0, 0, \ldots, 0)$, by means of a simple feedback law, which is a linear function of the *partial state* $(\xi_1, \ldots, \xi_r)$. If these variables are not directly available for feedback, one may wish to use instead an estimate – as is possible in the case of linear systems – provided by a dynamical system driven by the measured output. This is actually doable if the output y of (9.49) coincides with the state variable ξ_1. For the purpose of stabilizing system (9.49) by means of dynamic output feedback, it is convenient to reexpress the equations describing this system in a simpler form, known as *normal form*. Set $\eta_1 = \xi_1$ and define

$$\eta_2 = q_1(z, \xi_1) + b_1(z, \xi_1)\xi_2 ,$$

by means of which the second equation of (9.49) is changed into $\dot{\eta}_1 = \eta_2$. Set now

$$\eta_3 = \frac{\partial(q_1 + b_1\xi_2)}{\partial z} f(z, \xi_1) + \frac{\partial(q_1 + b_1\xi_2)}{\partial \xi_1}[q_1 + b_1\xi_2] + b_1[q_2 + b_2\xi_3] ,$$

by means of which the third equation of (9.49) is changed into $\dot{\eta}_2 = \eta_3$. Proceeding in this way, it is easy to conclude that the system (9.49) can be changed into a system modeled by

$$\begin{aligned}
\dot{z} &= f(z, \eta_1) , \\
\dot{\eta}_1 &= \eta_2 , \\
\dot{\eta}_2 &= \eta_3 , \\
&\cdots \\
\dot{\eta}_r &= q(z, \eta_1, \eta_2, \ldots, \eta_r) + b(z, \eta_1, \eta_2, \ldots, \eta_r)\boldsymbol{u} , \\
\boldsymbol{y} &= \eta_1 ,
\end{aligned} \tag{9.52}$$

in which $q(0, 0, 0, \ldots, 0) = (0, 0, 0, \ldots, 0)$ and $b(z, \eta_1, \eta_2, \ldots, \eta_r)$ is nowhere zero.

It has been shown earlier that, if the equilibrium $z = 0$ of $\dot{z} = f(z, 0)$ is globally asymptotically and also locally exponentially stable, this system is semiglobally stabilizable, by means of a feedback law

$$\boldsymbol{u} = h_1\eta_1 + h_2\eta_2 + \ldots + h_r\eta_r , \tag{9.53}$$

which is a linear function of the states $\eta_1, \eta_2, \ldots, \eta_r$. The feedback in question, if the coefficients are appropriately chosen, is able to steer at the equilibrium $(z, \eta_1, \ldots, \eta_r) = (0, 0, \ldots, 0)$ all trajectories with initial conditions in a given compact set K (whose size influences, as stressed earlier, the actual choice of the parameters $h_1, \ldots, h_r$). Note that, since all such trajectories will never exit, in positive time, a (possibly larger) compact set, there exists a number L such that

$$|h_1\eta_1(t) + h_2\eta_2(t) + \ldots + h_r\eta_r(t)| \le L ,$$

for all $t \ge 0$ whenever the initial condition of the closed loop is in K. Thus, to the extent of achieving asymptotic stability with a basin of attraction including K, the feedback law (9.53) could be replaced with a (nonlinear) law of the form

$$\boldsymbol{u} = \sigma_L(h_1\eta_1 + h_2\eta_2 + \ldots + h_r\eta_r) , \tag{9.54}$$

in which $\sigma(r)$ is any *bounded* function that coincides with r when $|r| \le L$. The advantage of having a feedback law whose amplitude is guaranteed not to exceed a fixed bound is that, when the partial states η_i will be replaced by approximate estimates, possibly large errors in the estimates will not cause dangerously large control efforts.

Inspection of the equations (9.52) reveals that the state variables used in the control law (9.54) coincide with the measured output $\mathbf{y}$ and its derivatives with respect to time, namely

$$\eta_i = \mathbf{y}^{(i-1)} , \quad i = 1, 2, \ldots, r .$$

It is therefore reasonable to expect that these variables could be asymptotically estimated in some simple way by means of a dynamical system driven by the measured output itself. The system in question is actually of the form

$$\begin{aligned}
\dot{\tilde{\eta}}_1 &= \tilde{\eta}_2 - \kappa c_{r-1}(\boldsymbol{y} - \tilde{\eta}_1) , \\
\dot{\tilde{\eta}}_2 &= \tilde{\eta}_3 - \kappa^2 c_{r-2}(\boldsymbol{y} - \tilde{\eta}_1) , \\
&\cdots \\
\dot{\tilde{\eta}}_r &= -\kappa^r c_0(\boldsymbol{y} - \tilde{\eta}_1) .
\end{aligned} \tag{9.55}$$

It is easy to realize that, if $\tilde{\eta}_1(t) = \boldsymbol{y}(t)$, then all $\tilde{\eta}_i(t)$, for $i = 2, \ldots, r$, coincide with $\eta_i(t)$. However, there is no a priori guarantee that this can be achieved and hence system (9.55) cannot be regarded as a true observer of the partial state $\eta_1, \dots, \eta_r$ of (9.52). It happens, though, that if the reason why this partial state needs to be estimated is only the implementation of the feedback law (9.54), then an *approximate* observer such as (9.55) can be successfully used.

The fact is that, if the coefficients $c_0, \dots, c_{r-1}$ are coefficients of a Hurwitz polynomial

$$p(\lambda) = \lambda^r + c_{r-1}\lambda^{r-1} + \dots + c_1\lambda + c_0 ,$$

and if the parameter κ is sufficiently large, the *rough estimates* $\tilde{\eta}_i$ of η_i provided by (9.55) can be used to replace the true states η_i in the control law (9.54). This results in a controller, which is a dynamical system modeled by equations of the form (Fig. 9.6)

$$\begin{aligned} \dot{\tilde{\eta}} &= \tilde{\mathbf{F}}\tilde{\eta} + \tilde{\mathbf{G}}\boldsymbol{y} , \\ \boldsymbol{u} &= \sigma_L(\mathbf{H}\boldsymbol{\eta}) , \end{aligned} \tag{9.56}$$

able to solve a problem of semiglobal stabilization for (9.52), if its parameters are appropriately chosen (see [9.6, Chap. 12] and [9.11, 12] for further details).

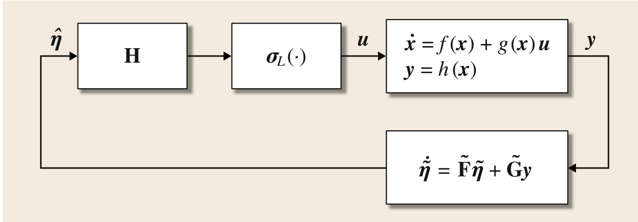

Fig. 9.6 Control via partial-state estimator

9.6.4 Observers and Full State Estimation

The design of observers for nonlinear systems modeled by equations of the form

$$\begin{aligned} \dot{\boldsymbol{x}} &= f(\boldsymbol{x}, u) , \\ \boldsymbol{y} &= h(\boldsymbol{x}, \boldsymbol{u}) , \end{aligned} \tag{9.57}$$

with state $\boldsymbol{x} \in \mathbb{R}^n$, input $\boldsymbol{u} \in \mathbb{R}^m$, and output $\boldsymbol{y} \in \mathbb{R}$ usually requires the preliminary transformation of the equations describing the system, in a form that suitably corresponds to the observability canonical form describe earlier for linear systems. In fact, a key requirement for the existence of observers is the existence of a global changes of coordinates $\tilde{\boldsymbol{x}} = \Phi(\boldsymbol{x})$ carrying system (9.57) into a system of the form

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{f}_1(\tilde{x}_1, \tilde{x}_2, \boldsymbol{u}) , \\ \dot{\tilde{x}}_2 &= \tilde{f}_2(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \boldsymbol{u}) , \\ &\cdots \\ \dot{\tilde{x}}_{n-1} &= \tilde{f}_{n-1}(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n, \boldsymbol{u}) , \\ \dot{\tilde{x}}_n &= \tilde{f}_n(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n, \boldsymbol{u}) , \\ \boldsymbol{y} &= \tilde{h}(\tilde{x}_1, \boldsymbol{u}) , \end{aligned} \tag{9.58}$$

in which the $\tilde{h}(\tilde{x}_1, \boldsymbol{u})$ and $\tilde{f}_i(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{i+1}, \boldsymbol{u})$ satisfy

$$\frac{\partial \tilde{h}}{\partial \tilde{x}_1} \neq 0 , \quad \text{and} \quad \frac{\partial \tilde{f}_i}{\partial \tilde{x}_{i+1}} \neq 0 , \quad \text{for all} \quad i = 1, \dots, n-1 \tag{9.59}$$

for all $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$, and all $\boldsymbol{u} \in \mathbb{R}^m$. This form is usually referred to as the *uniform observability canonical form*.

The existence of canonical forms of this kind can be obtained as follows [9.13, Chap. 2]. Define – recursively – a sequence of real-valued functions $\varphi_i(\boldsymbol{x}, \boldsymbol{u})$ as follows

$$\begin{aligned} \varphi_1(\boldsymbol{x}, \boldsymbol{u}) &:= h(\boldsymbol{x}, \boldsymbol{u}) , \\ &\vdots \\ \varphi_i(\boldsymbol{x}, \boldsymbol{u}) &:= \frac{\partial \varphi_{i-1}}{\partial \boldsymbol{x}} f(\boldsymbol{x}, \boldsymbol{u}) , \end{aligned}$$

for $i = 1, \dots, n$. Using these functions, define a sequence of i-vector-valued functions $\Phi_i(\boldsymbol{x}, \boldsymbol{u})$ as follows

$$\Phi_i(\boldsymbol{x}, \boldsymbol{u}) = \begin{pmatrix} \varphi_1(\boldsymbol{x}, \boldsymbol{u}) \\ \vdots \\ \varphi_i(\boldsymbol{x}, \boldsymbol{u}) \end{pmatrix} ,$$

for $i = 1, \dots, n$. Finally, for each of the $\Phi_i(\boldsymbol{x}, \boldsymbol{u})$, compute the subspace

$$K_i(\boldsymbol{x}, \boldsymbol{u}) = \ker\left[\frac{\partial \Phi_i}{\partial \boldsymbol{x}}\right]_{(\boldsymbol{x}, \boldsymbol{u})} ,$$

in which ker$[M]$ denotes the subspace consisting of all vectors $\boldsymbol{v}$ such that $M\boldsymbol{v} = 0$, that is the so-called null space of the matrix M. Note that, since the entries of the matrix

$$\frac{\partial \Phi_i}{\partial \boldsymbol{x}}$$

are in general dependent on $(\boldsymbol{x}, \boldsymbol{u})$, so is its null space $K_i(\boldsymbol{x}, \boldsymbol{u})$.

The role played by the objects thus defined in the construction of the change of coordinates yielding an observability canonical form is explained in this result.

Lemma 9.7
Consider system (9.57) and the map $\tilde{\boldsymbol{x}} = \Phi(\boldsymbol{x})$ defined by

$$\Phi(\boldsymbol{x}) = \begin{pmatrix} \varphi_1(\boldsymbol{x}, 0) \\ \varphi_2(\boldsymbol{x}, 0) \\ \vdots \\ \varphi_n(\boldsymbol{x}, 0) \end{pmatrix} .$$

Suppose that $\Phi(\boldsymbol{x})$ has a globally defined and continuously differentiable inverse. Suppose also that, for all $i = 1, \ldots, n$,

$$\begin{aligned} &\dim[K_i(\boldsymbol{x}, u)] = n - i , \\ &\qquad \text{for all } \boldsymbol{u} \in \mathbb{R}^m \\ &\qquad \text{and for all } \boldsymbol{x} \in \mathbb{R}^n \\ &K_i(\boldsymbol{x}, \boldsymbol{u}) = \text{independent of } \boldsymbol{u} . \end{aligned}$$

Then, system (9.57) is globally transformed, via $\Phi(\boldsymbol{x})$, into a system in uniform observability canonical form.

Once a system has been changed into its observability canonical form, an asymptotic observer can be built as follows. Take a *copy* of the dynamics of (9.58), corrected by an *innovation* term proportional to the difference between the output of (9.58) and the output of the copy. More precisely, consider a system of the form

$$\begin{aligned} \dot{\hat{x}}_1 &= \tilde{f}_1(\hat{x}_1, \hat{x}_2, \boldsymbol{u}) + \kappa c_{n-1}(\boldsymbol{y} - h(\hat{x}_1, \boldsymbol{u})) , \\ \dot{\hat{x}}_2 &= \tilde{f}_2(\hat{x}_1, \hat{x}_2, \hat{x}_3, \boldsymbol{u}) + \kappa^2 c_{n-2}(\boldsymbol{y} - h(\hat{x}_1, \boldsymbol{u})) , \\ &\ldots \\ \dot{\hat{x}}_{n-1} &= \tilde{f}_{n-1}(\hat{\boldsymbol{x}}, \boldsymbol{u}) + \kappa^{n-1} c_1(\boldsymbol{y} - h(\hat{x}_1, \boldsymbol{u})) , \\ \dot{\hat{x}}_n &= \tilde{f}_n(\hat{\boldsymbol{x}}, \boldsymbol{u}) + \kappa^n c_0(\boldsymbol{y} - h(\hat{x}_1, \boldsymbol{u})) , \end{aligned} \tag{9.60}$$

in which κ and $c_{n-1}, c_{n-2}, \ldots, c_0$ are design parameters.

The state of the system thus defined is able to asymptotically track, no matter what the initial conditions $\boldsymbol{x}(0)$, $\tilde{\boldsymbol{x}}(0)$ and the input $\boldsymbol{u}(t)$ are, the state of system (9.58) provided that the two following technical hypotheses hold:

(i) Each of the maps $\tilde{f}_i(\tilde{x}_1, \ldots, \tilde{x}_i, \tilde{x}_{i+1}, \boldsymbol{u})$, for $i = 1, \ldots, n$, is globally Lipschitz with respect to $(\tilde{x}_1, \ldots, \tilde{x}_i)$, uniformly in $\tilde{x}_{i+1}$ and $\boldsymbol{u}$,
(ii) There exist two real numbers α, β, with $0 < \alpha < \beta$, such that
$$\alpha \le \left| \frac{\partial \tilde{h}}{\partial \tilde{x}_1} \right| \le \beta , \quad \text{and} \quad \alpha \le \left| \frac{\partial \tilde{f}_i}{\partial \tilde{x}_{i+1}} \right| \le \beta , \quad \text{for all } i = 1, \ldots, n-1 ,$$
for all $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$, and all $\boldsymbol{u} \in \mathbb{R}^m$.

Let the *observation error* be defined as

$$e_i = \hat{x}_i - \tilde{x}_i , \quad i = 1, 2, \ldots, n .$$

The fact is that, if the two assumptions above hold, there is a choice of the coefficients $c_0, c_1, \ldots, c_{n-1}$ and there is a number κ^* such that, if $\kappa \ge \kappa^*$, the observation error asymptotically decays to zero as time tends to infinity, regardless of what the initial states $\tilde{\boldsymbol{x}}(0)$, $\hat{\boldsymbol{x}}(0)$ and the input $\boldsymbol{u}(t)$ are. For this reason the observer in question is called a *high-gain* observer (see [9.13, Chap. 6] for further details).

The availability of such an observer makes it possible to design a dynamic, output feedback, stabilizing control law, thus extending to the case of nonlinear systems the separation principle for stabilization of linear systems. In fact, consider a system in canonical form (9.58), rewritten as

$$\begin{aligned} \dot{\tilde{\boldsymbol{x}}} &= \tilde{f}(\tilde{\boldsymbol{x}}, \boldsymbol{u}) , \\ \boldsymbol{y} &= \tilde{h}(\tilde{\boldsymbol{x}}, \boldsymbol{u}) . \end{aligned}$$

Suppose a feedback law is known $\boldsymbol{u} = \alpha(\tilde{\boldsymbol{x}})$ that globally asymptotic stabilizes the equilibrium point $\tilde{\boldsymbol{x}} = 0$ of the closed-loop system

$$\dot{\tilde{\boldsymbol{x}}} = \tilde{f}(\tilde{\boldsymbol{x}}, \alpha(\tilde{\boldsymbol{x}})) .$$

Then, an output feedback controller of the form (Fig. 9.7)

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}} &= \tilde{f}(\hat{\boldsymbol{x}}, \boldsymbol{u}) + \mathbf{G}[\boldsymbol{y} - \tilde{h}(\hat{\boldsymbol{x}}, \boldsymbol{u})] , \\ \boldsymbol{u} &= \sigma_L(\alpha(\hat{\boldsymbol{x}})) , \end{aligned}$$

whose dynamics are those of system (9.60) and $\sigma_L : \mathbb{R} \to \mathbb{R}$ is a bounded function satisfying $\sigma_L(r) = r$ for all $|r| \le L$, is able to stabilize the equilibrium $(\tilde{\boldsymbol{x}}, \hat{\boldsymbol{x}}) = (0, 0)$ of the closed-loop system, with a basin of attraction that includes any a priori fixed compact set $K \times K$, if its parameters (the coefficients $c_0, c_1, \ldots, c_{n-1}$ and the parameter κ of (9.60) and the parameter L of $\sigma_L(\cdot)$) are appropriately chosen (see [9.13, Chap. 7] for details).

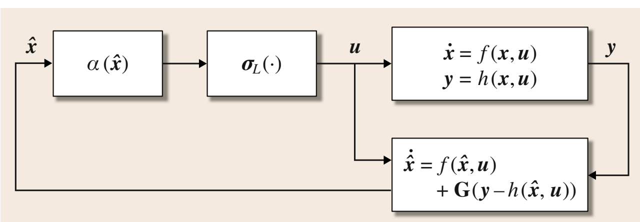

Fig. 9.7 Observer-based control for a nonlinear system

9.7 Tracking and Regulation

9.7.1 The Servomechanism Problem

A central problem in control theory is the design of feedback controllers so as to have certain outputs of a given plant *to track* prescribed reference trajectories. In any realistic scenario, this control goal has to be achieved in spite of a good number of phenomena which would cause the system to behave differently than expected. These phenomena could be endogenous, for instance, parameter variations, or exogenous, such as additional undesired inputs affecting the behavior of the plant. In numerous design problems, the trajectory to be tracked (or the disturbance to be rejected) is not available for measurement, nor is it known ahead of time. Rather, it is only known that this trajectory is simply an (undefined) member in a set of functions, for instance, the set of all possible solutions of an ordinary differential equation. Theses cases include the classical problem of the set-point control, the problem of active suppression of harmonic disturbances of unknown amplitude, phase and even frequency, the synchronization of nonlinear oscillations, and similar others.

In general, a tracking problem of this kind can be cast in the following terms. Consider a finite-dimensional, time-invariant, nonlinear system modeled by equations of the form

$$\begin{aligned} \dot{\boldsymbol{x}} &= f(\boldsymbol{w}, \boldsymbol{x}, \boldsymbol{u}) , \\ \boldsymbol{e} &= h(\boldsymbol{w}, \boldsymbol{x}) , \\ \boldsymbol{y} &= k(\boldsymbol{w}, \boldsymbol{x}) , \end{aligned} \tag{9.61}$$

in which $\boldsymbol{x} \in \mathbb{R}^n$ is a vector of state variables, $\boldsymbol{u} \in \mathbb{R}^m$ is a vector of inputs used for *control* purposes, $\boldsymbol{w} \in \mathbb{R}^s$ is a vector of inputs which cannot be controlled and include *exogenous* commands, exogenous disturbances, and model uncertainties, $\boldsymbol{e} \in \mathbb{R}^p$ is a vector of *regulated* outputs which include tracking errors and any other variable that needs to be steered to 0, and $\boldsymbol{y} \in \mathbb{R}^q$ is a vector of outputs that are available for *measurement* and hence used to feed the device that supplies the control action. The problem is to design a controller, which receives $\boldsymbol{y}(t)$ as input and produces $\boldsymbol{u}(t)$ as output, able to guarantee that, in the resulting closed-loop system, $\boldsymbol{x}(t)$ remains bounded and

$$\lim_{t \to \infty} \boldsymbol{e}(t) = 0 , \tag{9.62}$$

regardless of what the exogenous input $\boldsymbol{w}(t)$ actually is.

As observed at the beginning, $\boldsymbol{w}(t)$ is not available for measurement, nor it is known ahead of time, but it is known *to belong to a fixed family* of functions of time, the family of all solutions obtained from a fixed ordinary differential equation of the form

$$\dot{\boldsymbol{w}} = s(\boldsymbol{w}) \tag{9.63}$$

as the corresponding initial condition $\boldsymbol{w}(0)$ is allowed to vary on a prescribed set. This autonomous system is known as *the exosystem*.

The control law is to be provided by a system modeled by equations of the form

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= \varphi(\boldsymbol{\xi}, \boldsymbol{y}) , \\ \boldsymbol{u} &= \gamma(\boldsymbol{\xi}, \boldsymbol{y}) , \end{aligned} \tag{9.64}$$

with state $\boldsymbol{\xi} \in \mathbb{R}^\nu$. The initial conditions $\boldsymbol{x}(0)$ of the *plant* (9.61), $\boldsymbol{w}(0)$ of the *exosystem* (9.63), and $\boldsymbol{\xi}(0)$ of the *controller* (9.64) are allowed to range over fixed *compact* sets $X \subset \mathbb{R}^n$, $W \subset \mathbb{R}^s$, and $\Xi \subset \mathbb{R}^\nu$, respectively. All maps characterizing the model of the controlled plant, of the exosystem, and of the controller are assumed to be sufficiently differentiable.

The *generalized servomechanism problem* (or *problem of output regulation*) is to design a feedback controller of the form (9.64) so as to obtain a closed-loop system in which all trajectories are bounded and the regulated output $e(t)$ asymptotically decays to 0 as $t \to \infty$. More precisely, it is required that the composition of (9.61), (9.63), and (9.64), that is, the *autonomous* system

$$\begin{aligned} \dot{\boldsymbol{w}} &= s(\boldsymbol{w}) , \\ \dot{\boldsymbol{x}} &= f(\boldsymbol{w}, \boldsymbol{x}, \gamma(\boldsymbol{\xi}, k(\boldsymbol{w}, \boldsymbol{x}))) , \\ \dot{\boldsymbol{\xi}} &= \varphi(\boldsymbol{\xi}, k(\boldsymbol{w}, \boldsymbol{x})) , \end{aligned} \tag{9.65}$$

with output $\boldsymbol{e} = h(\boldsymbol{w}, \boldsymbol{x})$ be such that:

- The positive orbit of $W \times X \times \Xi$ is bounded, i. e., there exists a bounded subset S of $\mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}^\nu$ such that, for any $(\boldsymbol{w}_0, \boldsymbol{x}_0, \boldsymbol{\xi}_0) \in W \times X \times \Xi$, the integral curve $(\boldsymbol{w}(t), \boldsymbol{x}(t), \boldsymbol{\xi}(t))$ of (9.65) passing through $(\boldsymbol{w}_0, \boldsymbol{x}_0, \boldsymbol{\xi}_0)$ at time $t = 0$ remains in S for all $t \geq 0$.
- $\lim_{t \to \infty} e(t) = 0$, uniformly in the initial condition; i. e., for every $\varepsilon > 0$ there exists a time $\bar{t}$, depending only on ε and *not on* $(\boldsymbol{w}_0, \boldsymbol{x}_0, \boldsymbol{\xi}_0)$ such that the integral curve $(\boldsymbol{w}(t), \boldsymbol{x}(t), \boldsymbol{\xi}(t))$ of (9.65) passing through $(\boldsymbol{w}_0, \boldsymbol{x}_0, \boldsymbol{\xi}_0)$ at time $t = 0$ satisfies $\|\boldsymbol{e}(t)\| \leq \varepsilon$ for all $t \geq \bar{t}$.

9.7.2 Tracking and Regulation for Linear Systems

We show in this section how the servomechanism problem is treated in the case of linear systems. Let system (9.61) and exosystem (9.63) be linear systems, modeled by equations of the form

$$\begin{aligned} \dot{\boldsymbol{w}} &= \mathbf{S}\boldsymbol{w}\,, \\ \dot{\boldsymbol{x}} &= \mathbf{P}\boldsymbol{w} + \mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u}\,, \\ \boldsymbol{e} &= \mathbf{Q}\boldsymbol{w} + \mathbf{C}\boldsymbol{x}\,, \end{aligned} \tag{9.66}$$

and suppose that $\boldsymbol{y} = \boldsymbol{e}$, i. e., that regulated and measured variables coincide. We also consider, for simplicity, the case in which $m = 1$ and $p = 1$. Without loss of generality, it is assumed that all eigenvalues of $\mathbf{S}$ are simple and are on the imaginary axis.

A convenient point of departure for the analysis is the identification of conditions for the existence of a solution of the design problem. To this end, consider a dynamic, output-feedback controller

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= \mathbf{F}\boldsymbol{\xi} + \mathbf{G}e\,, \\ \boldsymbol{u} &= \mathbf{H}\boldsymbol{\xi} \end{aligned} \tag{9.67}$$

and the associated closed-loop system

$$\begin{aligned} \dot{\boldsymbol{w}} &= \mathbf{S}\boldsymbol{w}\,, \\ \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} &= \begin{pmatrix} \mathbf{P} \\ \mathbf{GQ} \end{pmatrix} \boldsymbol{w} + \begin{pmatrix} \mathbf{A} & \mathbf{BH} \\ \mathbf{GC} & \mathbf{F} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \xi \end{pmatrix} . \end{aligned} \tag{9.68}$$

If the controller solves the problem at issue, all trajectories are bounded and $\boldsymbol{e}(t)$ asymptotically decays to zero. Boundedness of all trajectories implies that all eigenvalues of

$$\begin{pmatrix} \mathbf{A} & \mathbf{BH} \\ \mathbf{GC} & \mathbf{F} \end{pmatrix} \tag{9.69}$$

have nonpositive real part. However, if some of the eigenvalues of this matrix were on the imaginary axis, the property of boundedness of trajectories could be lost as a result of infinitesimal variations in the parameters of (9.66) and/or (9.67). Thus, only the case in which the eigenvalues of (9.69) have negative real part is of interest. If the controller is such that this happens, then *necessarily* the pair of matrices $(\mathbf{A}, \mathbf{B})$ is stabilizable and the pair of matrices $(\mathbf{A}, \mathbf{C})$ is detectable. Observe now that, if the matrix (9.69) has all eigenvalues with negative real part, system (9.68) has a well-defined steady state, which takes place on an invariant subspace (the steady-state locus). The latter, as shown earlier, is necessarily the graph of a linear map, which expresses the $\boldsymbol{x}$ and $\boldsymbol{\xi}$ components of the state vector as functions of the $\boldsymbol{w}$ component. In other terms, the steady-state locus is the set of all triplets $(\boldsymbol{w}, \boldsymbol{x}, \boldsymbol{\xi})$ in which $\boldsymbol{w}$ is arbitrary, while $\boldsymbol{x}$ and $\boldsymbol{\xi}$ are expressed as

$$\begin{aligned} \boldsymbol{x} &= \Pi \boldsymbol{w}\,, \\ \boldsymbol{\xi} &= \Sigma \boldsymbol{w}\,, \end{aligned}$$

for some Π and Σ. These matrices, in turn, are solutions of the Sylvester equation

$$\begin{pmatrix} \Pi \\ \Sigma \end{pmatrix} \mathbf{S} = \begin{pmatrix} \mathbf{P} \\ \mathbf{GQ} \end{pmatrix} + \begin{pmatrix} \mathbf{A} & \mathbf{BH} \\ \mathbf{GC} & \mathbf{F} \end{pmatrix} \begin{pmatrix} \Pi \\ \Sigma \end{pmatrix} . \tag{9.70}$$

All trajectories asymptotically converge to the steady state. Thus, in view of the expression thus found for the steady-state locus, it follows that

$$\begin{aligned} &\lim_{t\to\infty} [\boldsymbol{x}(t) - \Pi \boldsymbol{w}(t)] = 0\,, \\ &\lim_{t\to\infty} [\boldsymbol{\xi}(t) - \Sigma \boldsymbol{w}(t)] = 0\,. \end{aligned}$$

In particular, it is seen from this that

$$\lim_{t\to\infty} \boldsymbol{e}(t) = \lim_{t\to\infty} [\mathbf{C}\Pi + \mathbf{Q}]\boldsymbol{w}(t)\,.$$

Since $\boldsymbol{w}(t)$ is a persistent function (none of the eigenvalues of $\mathbf{S}$ has negative real part), it is concluded that the regulated variable $\boldsymbol{e}(t)$ converges to 0 as $t \to \infty$ only if the map $e = \mathbf{C}\boldsymbol{x} + \mathbf{Q}\boldsymbol{w}$ is zero on the steady-state locus, i. e., if

$$0 = \mathbf{C}\Pi + \mathbf{Q}\,. \tag{9.71}$$

Note that the Sylvester equation (9.70) can be split into two equations, the former of which

$$\Pi \mathbf{S} = \mathbf{P} + \mathbf{A}\Pi + \mathbf{BH}\Sigma\,,$$

having set $\Gamma := \mathbf{H}\Sigma$, can be rewritten as

$$\Pi \mathbf{S} = \mathbf{A}\Pi + \mathbf{B}\Gamma + \mathbf{P}\,,$$

while the second one, bearing in mind the constraint (9.71), reduces to

$$\Sigma \mathbf{S} = \mathbf{F}\Sigma\,.$$

These arguments have proven – *in particular* – that, if there exists a controller that controller solves the problem, necessarily there exists a pair of matrices Π, Γ such that

$$\begin{aligned} \Pi \mathbf{S} &= \mathbf{A}\Pi + \mathbf{B}\Gamma + \mathbf{P} \\ 0 &= \mathbf{C}\Pi + \mathbf{Q}\,. \end{aligned} \tag{9.72}$$

The (linear) equations thus found are known as the *regulator equations* [9.14]. If, as observed above, the controller is required to solve the problem is spite of arbitrary (small) variations of the parameters of (9.66), the existence of solutions (9.72) is required to hold independently of the specific values of **P** and **Q**. This occurs if and only if none of the eigenvalues of **S** is a root of

$$\det\begin{pmatrix}\mathbf{A}-\lambda\mathbf{I} & \mathbf{B}\\ \mathbf{C} & \mathbf{0}\end{pmatrix}=0\,. \tag{9.73}$$

This condition is usually referred to as the *nonresonance* condition.

In summary, it has been shown that, if there exists a controller that solves the servomechanism problem, necessarily the controlled plant (with $\boldsymbol{w}=0$) is *stabilizable and detectable* and *none of the eigenvalues of* **S** *is a root of* (9.73). These *necessary* conditions turn out to be also *sufficient* for the existence of a controller that solves the servomechanism problem.

A procedure for the design of a controller is described below. Let

$$\psi(\lambda)=\lambda^s+d_{s-1}\lambda^{s-1}+\cdots+d_1\lambda+d_0$$

denote the minimal polynomial of **S**. Set

$$\Phi=\begin{pmatrix}0&1&0&\cdots&0&0\\0&0&1&\cdots&0&0\\\cdot&\cdot&\cdot&\cdots&\cdot&\cdot\\0&0&0&\cdots&0&1\\-d_0&-d_1&-d_2&\cdots&-d_{s-2}&-d_{s-1}\end{pmatrix},$$

$$\boldsymbol{G}=\begin{pmatrix}0\\0\\\cdots\\0\\1\end{pmatrix},$$

$$\boldsymbol{H}=\begin{pmatrix}1&0&0&\cdots&0&0\end{pmatrix}.$$

Let Π, Γ be a solution pair of (9.72) and note that the matrix

$$\Upsilon=\begin{pmatrix}\Gamma\\\Gamma\mathbf{S}\\\cdots\\\Gamma\mathbf{S}^{s-1}\end{pmatrix}$$

satisfies

$$\Upsilon\mathbf{S}=\Phi\Upsilon\,,\qquad\Gamma=\mathbf{H}\Upsilon\,. \tag{9.74}$$

Define a controller as follows:

$$\begin{aligned}\dot{\boldsymbol{\xi}}&=\Phi\boldsymbol{\xi}+\mathbf{G}\boldsymbol{e}\,,\\\dot{\boldsymbol{\eta}}&=\mathbf{K}\boldsymbol{\eta}+\mathbf{L}\boldsymbol{e}\,,\\\boldsymbol{u}&=\mathbf{H}\boldsymbol{\xi}+\mathbf{M}\boldsymbol{\eta}\,,\end{aligned} \tag{9.75}$$

in which the matrices Φ, **G**, **H** are those defined before and **K**, **L**, **M** are matrices to be determined. Consider now the associated closed-loop system, which can be written in the form

$$\begin{aligned}\dot{\boldsymbol{w}}&=\mathbf{S}\boldsymbol{w}\,,\\\begin{pmatrix}\dot{\boldsymbol{x}}\\\dot{\boldsymbol{\xi}}\\\dot{\boldsymbol{\eta}}\end{pmatrix}&=\begin{pmatrix}\mathbf{P}\\\mathbf{GQ}\\\mathbf{LQ}\end{pmatrix}\boldsymbol{w}+\begin{pmatrix}\mathbf{A}&\mathbf{BH}&\mathbf{BM}\\\mathbf{GC}&\Phi&0\\\mathbf{LC}&0&\mathbf{K}\end{pmatrix}\begin{pmatrix}\boldsymbol{x}\\\boldsymbol{\xi}\\\boldsymbol{\eta}\end{pmatrix}.\end{aligned} \tag{9.76}$$

By assumption, the pair of matrices (**A**, **B**) is stabilizable, the pair of matrices (**A**, **C**) is detectable, and none of the eigenvalues of **S** is a root of (9.73). As a consequence, in view of the special structure of Φ, **G**, **H**, also the pair

$$\begin{pmatrix}\mathbf{A}&\mathbf{BH}\\\mathbf{GC}&\Phi\end{pmatrix},\ \begin{pmatrix}\mathbf{B}\\0\end{pmatrix}$$

is stabilizable and the pair

$$\begin{pmatrix}\mathbf{A}&\mathbf{BH}\\\mathbf{GC}&\Phi\end{pmatrix},\ \begin{pmatrix}\mathbf{C}&0\end{pmatrix}$$

is detectable. This being the case, it is possible to pick **K**, **L**, **M** in such a way that all eigenvalues of

$$\begin{pmatrix}\mathbf{A}&\mathbf{BH}&\mathbf{BM}\\\mathbf{GC}&\Phi&0\\\mathbf{LC}&0&\mathbf{K}\end{pmatrix}$$

have negative real part.

As a result, all trajectories of (9.76) are bounded. Using (9.72) and (9.74) it is easy to check that the graph of the mapping

$$\pi:\boldsymbol{w}\to\begin{pmatrix}\Pi\\\Upsilon\\0\end{pmatrix}\boldsymbol{w}$$

is invariant for (9.76). This subspace is actually the steady-state locus of (9.76) and $\boldsymbol{e}=\mathbf{C}\boldsymbol{x}+\mathbf{Q}\boldsymbol{w}$ is zero on this subspace. Hence all trajectories of (9.76) are such that $\boldsymbol{e}(t)$ converges to 0 as $t\to\infty$.

The construction described above is insensitive to small arbitrary variations of the parameters, except for the case of parameter variations in the exosystem. The case of parameter variations in the exosystem requires a different design, as explained e.g., in [9.15]. A state-of-the-art discussion of the servomechanism problem for suitable classes of nonlinear systems can be found in [9.16].

9.8 Conclusion

This chapter has reviewed the fundamental methods and models of control theory as applied to automation. The following two chapters address further advancements in this area of automation theory.

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