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SYNCHRONOUS AND ASYNCHRONOUS BRAIN-STATE-IN-A-BOX INFORMATION SYSTEM NEURAL MODELS

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Synchronous and asynchronous information system neural models are proposed that are hybrids of Pawlak's information system and Brain-State-in-a-Box (BSB) neural models. The stability of the proposed models is studied using LaSalle's Invariance Principle. Applications to an analysis of the United Nations activities are presented as examples.

1. Introduction

In 1984, Pawlak[1] proposed a mathematical model of conflicts based on his theory of information systems and rough sets. These concepts can be applied to a broad class of practical problems. Indeed, Pawlak, Słowiński, and Słowiński[2] applied the concept of rough sets to medical treatment of patients after vagotomy of duodenal ulcer, while Żakowski[3] used the theory of information systems and conflicts in his analysis of parliamentary activities. Pawlak's information system was further studied by Żakowski and Koutny[4], Żakowski,[5–11] Muraszkiewicz and Rybiński,[12] Muraszkiewicz, Ostrowski and Rybiński,[13] Łuba, Mochocki and Rybnik,[14] and Skowron and Suraj[15] among others.

In this paper, we propose both synchronous and asynchronous neural information system models. The proposed models, presented in Secs. 3 and 4, are hybrids of Pawlak's information system and a Brain-State-in-a-Box (BSB) neural model. We study the models' stability properties using LaSalle's Invariance Principle. In Sec. 5, the neural information systems are applied to an analysis of the United Nations activities.

In the following section, we present technical results that we will use later in the paper.

2. Background Results

Pawlak's information system, denoted as S, can be viewed as a pair $S = (U, A)$, where $U = \{x_1, x_2, \ldots, x_n\}$ is a nonempty finite set of objects. The elements of U may be interpreted, for example, as concepts, values, goals, political parties, individuals, states, etc. The fixed set U, with respect to which we perform subsequent manipulations, is called the universe. The elements of the set A, denoted as $\mathbf{a}_j$, $j = 1, 2, \ldots, r$, are called the attributes. The attributes are vector valued functions. In this paper, we consider the following attributes:

$$\mathbf{a}_j : U \to \{-1, 0, 1\}^n , \quad j = 1, 2, \ldots, r .$$

Example 1

Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ and $A = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. Then, an information system may have the form shown in Table 1.

The attributes may be interpreted, for example, as issues under negotiation by the members of the universe U. Thus, the second component of $\mathbf{a}_1$ being -1 may mean that x_2 is opposed to the issue $\mathbf{a}_1$, while x_5 supports $\mathbf{a}_1$, etc. Such data can be collected from newspapers, surveys, or experts.

Let $B \subseteq A$. The set B defines the partition of U, denoted as U/B, into equivalence classes such that $x, y \in U$ belong to the same equivalence class of U/B, if for every $a \in B$, $a(x) = a(y)$. If $B = \emptyset$, then $U/B = U$. In terms of negotiations, U/B may be interpreted as the partition of the set U into blocks of negotiators that have the same opinion on all of the issues of B.

Let $V \subseteq U$. Then, the set V defines the partition, denoted as A/V, of the set A into equivalence classes such that $a, b \in A$ belong to the same equivalence class of A/V, if $a(x) = b(x)$ for all $x \in V$. In other words, A/V represents the partition of A into subsets whose elements have the same values for all the elements of V. In terms of negotiations, A/V may be interpreted as the partition of the set of issues A into blocks of issues, such that the members of V have the same opinions on each issue in the block, though different members of V may have different opinions on the same issue.

Example 2

Consider the information system represented by Table 1. Then,

$$U/A = \{\{x_1, x_5\}, \{x_2\}, \{x_3\}, \{x_4\}\} ,$$

and

$$A/U = \{\{\mathbf{a}_1, \mathbf{a}_3\}, \{\mathbf{a}_2\}\} .$$

Let $V = \{x_3\}$ and $B = \{\mathbf{a}_1, \mathbf{a}_2\}$. Then,

$$U/B = \{\{x_1, x_5\}, \{x_2\}, \{x_3\}, \{x_4\}\} ,$$

and

$$A/V = A .$$

Table 1. A simple information system.

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$
x_1	1	0	1
x_2	-1	0	-1
x_3	1	1	1
x_4	0	0	0
x_5	1	0	1

The elements of the universe U are linked with each other. Connections between the elements of the universe U may be characterized, for example, by a function,

$$\varphi : U \times U \to \{-1, 0, 1\} .$$

The pair $C = (U, \varphi)$ is called the configuration of U.

A configuration (U, φ) is called regular if

(i) $\varphi(x, x) = 1$,
(ii) $\varphi(x, y) = \varphi(y, x)$,
(iii) $\varphi(x, y)\varphi(y, z) = \varphi(x, z)$ or $\varphi(x, y) = \varphi(y, z) = 0$.

The matrix $\mathbf{M}(C) = c_{ij}$, where $c_{ij} = \varphi(x_i, x_j)$ is called the configuration, or connection, matrix of the configuration $C = (U, \varphi)$. Żakowski[10] gave the following necessary and sufficient condition for an $n \times n$ matrix to be a matrix of a regular configuration of an information system.

Lemma 1

A matrix $\mathbf{R}$ is the matrix of a regular configuration if and only if

$$r_{ij} \in \{-1, 0, 1\} , \quad r_{ii} = 1 , \quad r_{ij} = r_{ji} , \tag{1}$$

and

$$r_{ij}r_{jk} = r_{ik}, \quad \text{or} \quad r_{ij} = r_{jk} = 0 . \tag{2}$$

An example of a configuration that is not regular is given in Table 2. This configuration satisfies Eq. (1), but it does not satisfy Eq. (2). Indeed, $r_{32}r_{21} = 1 \cdot 0 \neq r_{31} = -1$. An example of a matrix of a regular configuration is the identity matrix $\mathbf{I}_n$.

Table 2. An example of a configuration that is not regular.

	x_1	x_2	x_3
x_1	1	0	-1
x_2	0	1	1
x_3	-1	1	1

Lemma 1 gives us a necessary and sufficient condition for a configuration matrix to be regular. Żakowski[10] also gave a lemma that provides a method for constructing matrices of regular configurations. To proceed, let δ_{ij} denote the Kronecker symbol, i.e., $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$.

Lemma 2

Let $\mathbf{u} \in \{-1, 0, 1\}^n$ and let u_i denote the ith component of $\mathbf{u}$. Then, the matrix $\mathbf{M}(\mathbf{u})$ whose elements are given by:

$$m_{ij} = u_i u_j + (1 - u_i u_j)\delta_{ij}\,, \quad i, j = 1, 2, \ldots, n\,, \tag{3}$$

is a matrix of a regular configuration.

Using a matrix notation, we can represent Eq. (3) as:

$$\mathbf{M}(\mathbf{u}) = \mathbf{u}\mathbf{u}^T + \mathbf{D}(\mathbf{u})\,, \tag{4}$$

where $\mathbf{D}(\mathbf{u})$ is a diagonal matrix whose diagonal elements are:

$$d_{ii} = \begin{cases} 0 & \text{if } u_i = \pm 1 \\ 1 & \text{if } u_i = 0\,. \end{cases} \tag{5}$$

Note that the matrix $\mathbf{D}(\mathbf{u})$ is symmetric and positive semi-definite. This fact will be used in the proof of Lemma 5. Furthermore,

$$\mathbf{M}(\mathbf{u})\mathbf{u} = \|\mathbf{u}\|^2 \mathbf{u}\,, \tag{6}$$

which means that the matrix $\mathbf{M}(\mathbf{u})$ possesses the same property as the outer product Hebbian learning rule for an autoassociative system, described, for example, by Anderson (Ref. 16, pp. 159–170). It follows from Lemma 2 that if we use the vector $\mathbf{a}_j$, $j = 1, 2, \ldots, r$, representing the jth attribute, to form the matrix $\mathbf{M}(\mathbf{a}_j)$ using Eq. (3) or Eq. (4), then we obtain a matrix of a regular configuration of U.

Let $\emptyset \neq B \subseteq A$ and let $\mathrm{card}(B)$ denote the number of attributes in the set B, i.e., the cardinality of the set B. The matrix,

$$\mathbf{M}(B) = \frac{1}{\mathrm{card}(B)} \sum_{\mathbf{a} \in B} \mathbf{M}(\mathbf{a}) \tag{7}$$

is called the connection matrix of U with respect to B — see Żakowski.[10] Observe that the matrix $\mathbf{M}(B)$ has similar properties to the outer product Hebbian rule for storing multiple autoassociations in the linear associator, as discussed, for example, by Anderson (Ref. 16, pp. 167–170).

Let m_{ij} be the ijth element of the matrix $\mathbf{M}(B)$. Note that:

$$m_{ij} \in [-1, 1]\,, \quad m_{ii} = 1\,, \quad \text{and} \quad m_{ij} = m_{ji}\,. \tag{8}$$

Following Żakowski,[10] we can use a conflict theoretic interpretation of the sign and magnitude of the elements of the matrix $\mathbf{M}(B)$. If $m_{ij} > 0$, then we can say that x_i and x_j are in alliance to degree m_{ij} with respect to the set of issues represented by B. If $m_{ij} < 0$, then we say that x_i and x_j are in conflict to degree $|m_{ij}|$. Finally, if $m_{ij} = 0$, then we say that x_i and x_j are neutral.

The configuration of Pawlak's information system can be viewed as a cognitive map of Axelrod,[17] which consists of points, or nodes, and directed links between the nodes. The nodes correspond to concepts. The cognitive map was introduced by Axelrod to study "...complex policy alternatives in terms of the consequences a particular choice would cause, and ultimately of what the sum of all these effects would be" (Ref. 17, p. 5). Cognitive maps can be used to analyze decision-making processes and assertions of a particular person as well as decision-making processes in various different settings. For example, Roberts (in Ref. 17, Chapter 7) uses cognitive maps to study decision problems involving transportation and energy, while Hart (in Ref. 17, Chapter 8) uses cognitive maps to analyze international cooperation in the exploitation and conservation of ocean resources.

Pawlak's information system as well as Axelrod's cognitive maps are static in that they are used to analyze data of the events that took place in the past. Neither Pawlak's information system nor Axelrod's cognitive map accounts for the time evolution of events. To model the time evolution of events, Kosko (Ref. 18, Chapter 15) introduces fuzzy cognitive maps (FCMs). The FCMs are nonlinear dynamical systems acting as neural networks with unipolar activation nonlinearity, i.e., the trajectories of the FCMs reside in the unit hypercube $[0, 1]^n$, where n is the dimension of the state vector of the underlying neural network. Note that n is also the number of nodes in the FCM. The reason for referring to this type of dynamical systems as fuzzy cognitive maps is that the nodes of FCMs can be viewed as fuzzy sets. Kosko (Ref. 18, Chapter 15) considers only synchronous FCMs, i.e., the FCMs whose state vectors are updated in parallel. Our proposed neural information system is also a nonlinear dynamical system acting as a neural network. However, we use a bipolar activation nonlinearity and we analyze both synchronous and asynchronous models. The advantage of using bipolar activation nonlinearity is that we include negative concept node values, which is consistent with the interpretation of the attributes.

In the following sections, we present two types of neural information systems.

3. Synchronous Neural Information System

To model a synchronous neural information system, we use the Brain-State-in-a-Box (BSB) neural model, which was proposed by Anderson, Silverstein, Ritz, and Jones.[19] The BSB neural model can be viewed as a discrete dynamical system. The dynamics of the BSB model can be described by the difference equation,

$$\mathbf{x}(k+1) = \mathbf{g}(\mathbf{x}(k) + \alpha\mathbf{W}\mathbf{x}(k)) \tag{9}$$

subject to the initial condition $\mathbf{x}(0) = \mathbf{x}_0$, where $\mathbf{x}(k) \in \mathbb{R}^n$ is the state of the system at time k, $\alpha > 0$ is the step size, and $\mathbf{W} \in \mathbb{R}^{n\times n}$ is a symmetric weight matrix determined during the training procedure. The function $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ is a vector valued function whose ith component $(\mathbf{g})_i$ is defined as:

$$(\mathbf{g}(\mathbf{y}))_i = \begin{cases} \operatorname{sign}(y_i) & \text{if } |y_i| \geq 1 \\ y_i & \text{if } -1 < y_i < 1\,. \end{cases} \tag{10}$$

Clearly, the function $\mathbf{g}$ is continuous. The BSB model gets its name from the fact that the system trajectories are constrained to the hypercube $[-1, 1]^n$. The BSB model was used by Anderson *et al.*[19] in their study of human cognitive computation. The dynamics of the BSB model and its variants were further analyzed by Golden,[20] Greenberg,[21] Grossberg,[22] Hui and Żak,[23] Anderson,[24] Hui, Lillo and Żak,[25] Golden,[26] and Hassoun (Ref. 27, Chapter 7). A modified form of the BSB neural model, which we refer to as the generalized Brain-State-in-a-Box (gBSB) neural model, was proposed and studied by Hui and Żak.[23] Its dynamics are described by the equation,

$$\boxed{\mathbf{x}(k+1) = \mathbf{g}((\mathbf{I}_n + \alpha\mathbf{W})\mathbf{x}(k) + \alpha\mathbf{b})} \tag{11}$$

where the bias vector $\mathbf{b} \in \mathbb{R}^n$, and the weight matrix $\mathbf{W} \in \mathbb{R}^{n\times n}$ need not be symmetric as in the BSB model. Thus, the original BSB model can be viewed as a special case of the gBSB model when the interconnection matrix $\mathbf{W}$ is symmetric and $\mathbf{b} = \mathbf{0}$. The two main reasons for which the BSB model was modified, by adding the bias vector $\mathbf{b}$, were:

- the presence of $\alpha\mathbf{b}$ in the gBSB model allows us to better control the extent of the basins of attraction of asymptotically stable equilibrium points of model (11);
- the BSB model behavior on the boundary regions of the hypercube $[-1, 1]^n$ is reduced to gBSB type models rather than of the BSB type. On the other hand, the dynamic behavior of the gBSB model on the boundary regions of the hypercube $[-1, 1]^n$ can be described using reduced-order gBSB models.

A slightly different generalization of the BSB neural model was proposed by Golden.[26] Both Hui and Żak[23] and Golden[26] have analyses that are applicable to Eq. (11) when the weight matrix $\mathbf{W}$ is symmetric.

We now propose a model of a neural information system (NIS). This model is a hybrid of the Pawlak's information system and the gBSB neural model. It is described by the difference equation,

$$\boxed{\mathbf{x}(k+1) = \mathbf{g}(\mathbf{M}(B)\mathbf{x}(k) + \mathbf{b})} \tag{12}$$

The above model can be viewed as a special case of the model (11), where:

$$\mathbf{W} = \mathbf{M}(B) - \mathbf{I}_n \quad \text{and} \quad \alpha = 1\,. \tag{13}$$

The bias vector $\mathbf{b}$ in Eq. (12), which is a design parameter, can be used to direct the model's trajectory towards a state whose components have the same sign as that of $\mathbf{b}$. To see this, suppose that we have $|b_i| > |(\mathbf{M}(B)\mathbf{x}(k))_i|$. Then $x_i(k+1)$ will have the same sign as b_i. In fact, it is simple to show that if $|b_i| > n+1$, then for $k = 0, 1, 2, \ldots,$

$$x_i(k+1) = \begin{cases} 1 & \text{if } b_i \text{ is positive} \\ -1 & \text{if } b_i \text{ is negative.} \end{cases} \tag{14}$$

In the neural network literature, the models given by Eqs. (9), (11), and (12) are referred to as synchronous models. The reason for this label is that all the components of the state vector of the above models are updated at the same time in a parallel fashion. Following this terminology, we will refer to the model (12) as the synchronous neural information system.

We can interpret the sign and magnitude of the elements of $\mathbf{x}(k)$ as follows. If $x_i(k) > 0$, the member x_i of the universe U supports the issue under investigation to degree $x_i(k)$ at time k. If $x_i(k) < 0$, the member x_i opposes the issue to degree $|x_i(k)|$. And if $x_i(k) = 0$, the member x_i neither supports nor opposes the issue. An event is a collection of actions or values that are represented by the components of the state vector $\mathbf{x}$. The synchronous NIS models the time evolution of events. An event at time k is modeled by the state $\mathbf{x}(k)$. An event $\mathbf{x}(k)$ causes an event $\mathbf{x}(k+1)$.

We will next prove a theorem which concerns with the dynamics of the model (11). We will use this theorem in our analysis of the information system (12). To proceed, we need the following definition.

Definition 1

A point $\mathbf{x}^*$ is an equilibrium point of Eq. (11) if

$$\mathbf{x}^* = \mathbf{g}((\mathbf{I}_n + \alpha\mathbf{W})\mathbf{x}^* + \alpha\mathbf{b})\,.$$

In addition, we need the following lemmas.

Lemma 3

Let $L(\mathbf{x}(k)) = (\mathbf{I}_n + \alpha\mathbf{W})\mathbf{x}(k) + \alpha\mathbf{b}$, and $\Delta x_i(k) = x_i(k+1) - x_i(k)$. Then,

$$(L(\mathbf{x}(k)) - \mathbf{x}(k))_i \Delta x_i(k) \geq (\Delta x_i(k))^2 \text{ for all } i = 1, \ldots, n\,.$$

Proof

Clearly, $\Delta x_i(k) = 0$ satisfies the above inequality. For $\Delta x_i(k) > 0$, i.e., $x_i(k+1) > x_i(k)$, we have

$$x_i(k+1) = (\mathbf{g}(L(\mathbf{x}(k))))_i > x_i(k)\,.$$

By the definition of the activation nonlinearity, $\mathbf{g}$, given by Eq. (10), we have $x_i(k) \geq -1$. Hence, $(L(\mathbf{x}(k)))_i > -1$ since by assumption, $x_i(k+1) > x_i(k)$. It then follows from Eq. (10) that $(\mathbf{g}(L(\mathbf{x}(k))))_i \leq (L(\mathbf{x}(k)))_i$. Thus,

$$\begin{aligned}(L(\mathbf{x}(k)))_i - x_i(k) &\geq (\mathbf{g}(L(\mathbf{x}(k))))_i - x_i(k) \\ &= \Delta x_i(k) > 0\,. \end{aligned} \tag{15}$$

Similarly, if $\Delta x_i(k) < 0$, then

$$\begin{aligned}(L(\mathbf{x}(k)))_i - x_i(k) &\leq (\mathbf{g}(L(\mathbf{x}(k))))_i - x_i(k) \\ &= \Delta x_i(k) < 0\,. \end{aligned} \tag{16}$$

Multiplying Eq. (15) and Eq. (16) by $\Delta x_i(k)$ yields

$$(L(\mathbf{x}(k)) - \mathbf{x}(k))_i \Delta x_i(k) \geq (\Delta x_i)^2\,,$$

which completes the proof of the lemma. □

Lemma 4

$$(L(\mathbf{x}(k)) - \mathbf{x}(k))^T \Delta\mathbf{x}(k) \geq \Delta\mathbf{x}(k)^T \Delta\mathbf{x}(k)$$

Proof

Using Lemma 3, we obtain

$$\begin{aligned}(L(\mathbf{x}(k)) - \mathbf{x}(k))^T \Delta\mathbf{x}(k) &= \sum_{i=1}^{n} (L(\mathbf{x}) - \mathbf{x}(k))_i \Delta x_i(k) \\ &\geq \sum_{i=1}^{n} (\Delta x_i)^2 \\ &= \Delta\mathbf{x}(k)^T \Delta\mathbf{x}(k)\,,\end{aligned}$$

which completes the proof. □

We say that a function $E : \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function for the model (11) if (i) E is continuous on $\mathbb{R}^n$; and (ii) $\Delta E \leq 0$ on the trajectories of (11).

A subset M of $\mathbb{R}^n$ is called an invariant set of the model (11) if $\mathbf{x}(k) \in M$, then $\mathbf{x}(k+1) = \mathbf{g}(L(\mathbf{x}(k))) \in M$.

Definition 2

A sequence $\{\mathbf{x}(k)\}$ is said to converge to a set $\mathcal{A}$ if

$$\lim_{k\to\infty}\inf\{\|\mathbf{x}(k)-\mathbf{y}\| : \mathbf{y}\in\mathcal{A}\}=0\,.$$

We now state LaSalle's Invariance Principle (Ref. 28, p. 9) that we will use in the proofs of the main results of this paper. Our statement of the Invariance Principle is specifically worded for the purpose of this paper.

Theorem 1

If E is a Lyapunov function for the model (11), whose solutions are bounded, then there is a number c such that $\mathbf{x}(k)$ converges to $M\cap E^{-1}(c)$, where M is the largest invariant set contained in the set $S=\{\mathbf{x}\in\mathbb{R}^n : \Delta E=0\}$.

We are now ready to prove a theorem concerning the gBSB model given by (11). This theorem is very similar to Golden's BSB Asymptotic Stability Theorem (Ref. 20, p. 78), which in turn is a special case of Golden's Generalized BSB Theorem (Ref. 26, p. 292). For reader's convenience, we include our proof of this result using a method that is similar to Golden's approach.

Theorem 2

Every trajectory of the model (11) converges to a set of equilibrium points if the matrix $(2\mathbf{I}_n+\alpha\mathbf{W})$ is symmetric and positive definite.

Proof

Consider the Lyapunov function candidate,

$$E(\mathbf{x}(k))=-2\alpha\mathbf{x}(k)^T\mathbf{b}-\alpha\mathbf{x}(k)^T\mathbf{W}\mathbf{x}(k)\,. \qquad (17)$$

Clearly, E is continuous on $\mathbb{R}^n$. Observe that:

$$\begin{aligned}\Delta E(k)&=E(\mathbf{x}(k+1))-E(\mathbf{x}(k))\\&=-2\alpha(\mathbf{x}(k)+\Delta\mathbf{x}(k))^T\mathbf{b}-\alpha(\mathbf{x}(k)\\&\quad+\Delta\mathbf{x}(k))^T\mathbf{W}(\mathbf{x}(k)+\Delta\mathbf{x}(k))\\&\quad-(-2\alpha\mathbf{x}(k)^T\mathbf{b}-\alpha\mathbf{x}(k)^T\mathbf{W}\mathbf{x}(k))\\&=-2\alpha\Delta\mathbf{x}(k)^T\mathbf{b}-\alpha\Delta\mathbf{x}(k)^T\mathbf{W}\Delta\mathbf{x}(k)\\&\quad-\alpha\Delta\mathbf{x}(k)^T(\mathbf{W}+\mathbf{W}^T)\mathbf{x}(k).\end{aligned}$$

Note that $\mathbf{W}=\mathbf{W}^T$, since $(I_n+\alpha\mathbf{W})$ is symmetric. Therefore,

$$\begin{aligned}\Delta E(k)&=-2\alpha\Delta\mathbf{x}(k)^T\mathbf{b}-\alpha\Delta\mathbf{x}(k)^T\mathbf{W}\Delta\mathbf{x}(k)\\&\quad-2\alpha\Delta\mathbf{x}(k)^T\mathbf{W}\mathbf{x}(k)\\&=-2\Delta\mathbf{x}(k)^T(\alpha\mathbf{W}\mathbf{x}(k)+\alpha\mathbf{b})\\&\quad-\alpha\Delta\mathbf{x}(k)^T\mathbf{W}\Delta\mathbf{x}(k)\\&=-2\Delta\mathbf{x}(k)^T(L(\mathbf{x}(k))-\mathbf{x}(k))\\&\quad-\alpha\Delta\mathbf{x}(k)^T\mathbf{W}\Delta\mathbf{x}(k)\,.\end{aligned} \qquad (18)$$

Applying Lemma 4 to the above yields

$$\begin{aligned}\Delta E(k)&\le-2\Delta\mathbf{x}(k)^T\Delta\mathbf{x}(k)-\alpha\Delta\mathbf{x}(k)^T\mathbf{W}\Delta\mathbf{x}(k)\\&=-\Delta\mathbf{x}(k)^T(2\mathbf{I}_n+\alpha\mathbf{W})\Delta\mathbf{x}(k)\,.\end{aligned} \qquad (19)$$

Since $(2I_n+\alpha\mathbf{W})$ is positive definite, we have

$$\Delta E(k)\le 0\,,$$

which means that the function E is a Lyapunov function for the model (11). Note that the trajectory $\mathbf{x}(k)\in[-1,1]^n$, and hence is bounded. By Theorem 1, there is a number c such that $\mathbf{x}(k)$ converges to $M\cap E^{-1}(c)$, where M is the largest invariant set within $S=\{\mathbf{x}\in\mathbb{R}^n : \Delta E=0\}$. From Eq. (18), $\Delta E=0$ if $\Delta\mathbf{x}=\mathbf{0}$. And from Eq. (19), $\Delta\mathbf{x}=\mathbf{0}$ if $\Delta E=0$. Thus, $\Delta E=0$ if and only if $\Delta\mathbf{x}=\mathbf{0}$. It follows that S consists of the equilibrium points of (11) only, and therefore $M=S$. Thus, the trajectory $\mathbf{x}(k)$ converges to a set of equilibrium points. □

We now use the above results to examine the stability of the information system modeled by (12). Note that a point $\mathbf{x}^*$ is an equilibrium point of (12) if

$$\mathbf{x}^*=\mathbf{g}(\mathbf{M}(B)\mathbf{x}^*+\mathbf{b})\,.$$

In order to prove our next theorem, we need the following lemma.

Lemma 5

The matrix $\mathbf{M}(B)$ given by Eq. (7) is symmetric and positive semi-definite.

Proof

Indeed, for any $\mathbf{a} \in B$, we have

$$\mathbf{M}(\mathbf{a}) = \mathbf{a}\mathbf{a}^T + \mathbf{D}(\mathbf{a}) = (\mathbf{a}\mathbf{a}^T)^T + \mathbf{D}^T(\mathbf{a}) = \mathbf{M}^T(\mathbf{a}) . \tag{20}$$

Hence,

$$\begin{aligned} \mathbf{M}(B) &= \frac{1}{\mathrm{card}(B)} \sum_{\mathbf{a}\in B} \mathbf{M}(\mathbf{a}) \\ &= \frac{1}{\mathrm{card}(B)} \sum_{\mathbf{a}\in B} \mathbf{M}^T(\mathbf{a}) = \mathbf{M}^T(B) . \end{aligned} \tag{21}$$

It remains to show that $\mathbf{M}(B)$ is positive semidefinite. Since the matrix $\mathbf{D}(\mathbf{a})$ is positive semidefinite, then for any $\mathbf{z} \in \mathbb{R}^n$ we have

$$\mathbf{z}^T\mathbf{M}(\mathbf{a})\mathbf{z} = \mathbf{z}^T\mathbf{a}\mathbf{a}^T\mathbf{z} + \mathbf{z}^T\mathbf{D}(\mathbf{a})\mathbf{z} \geq (\mathbf{a}^T\mathbf{z})^2 \geq 0 . \tag{22}$$

Hence,

$$\mathbf{z}^T\mathbf{M}(B)\mathbf{z} = \frac{1}{\mathrm{card}(B)} \textstyle\sum_{\mathbf{a}\in B} \mathbf{z}^T\mathbf{M}(\mathbf{a})\mathbf{z} \geq 0 . \quad \square$$

Theorem 3

Every trajectory $\mathbf{x}(k)$ of (12) converges to a set of equilibrium points.

Proof

If we set $\mathbf{W} = \mathbf{M}(B) - \mathbf{I}_n$ and $\alpha = 1$ in Eq. (11), then we obtain Eq. (12). By Lemma 5, the matrix $(\mathbf{I}_n + \alpha\mathbf{W}) = \mathbf{M}(B)$ is symmetric and positive semidefinite, which implies that the matrix $(2\mathbf{I}_n + \alpha\mathbf{W})$ is positive definite. Hence, by Theorem 2, any trajectory $\mathbf{x}(k)$ of Eq. (12) converges to a set of equilibrium points. □

We now investigate the conditions under which an equilibrium point of the information system (12) is asymptotically stable, i.e., we are interested in finding conditions for which a trajectory $\mathbf{x}(k)$ of the information system converges to one of the equilibrium points. Hui *et al.*[25] present a general method for locating all the equilibrium points of the model (11). Using these results and Eq. (13), we can obtain the equilibrium points of the information system. For the case where an equilibrium point is a vertex of the hypercube $[-1, 1]^n$, we can use asymptotic stability conditions reported in Refs. 25, 29 and 30. In our further analysis, we will use the following lemma that comes from Hui *et al.* (Ref. 25, p. 219).

Lemma 6

Let $\mathbf{v}$ be a vertex of the hypercube $[-1, 1]^n$. If

$$(L(\mathbf{v}))_i v_i > 1 , \quad i = 1, \ldots, n ,$$

then $\mathbf{v}$ is an asymptotically stable equilibrium point of model (11).

We will also use a sufficiency condition for a vertex of hypercube $[-1, 1]^n$ not to be an equilibrium point of the gBSB model. Such a result can be found in Żak *et al.* (Ref. 30, p. 847).

Lemma 7

A vertex $\mathbf{v}$ is not an equilibrium point of model (11) if and only if

$$(L(\mathbf{v}))_i v_i < 1 \text{ for some } i = 1, \ldots, n .$$

Applying Lemma 6 to the information system model (12) yields the following proposition.

Proposition 1

If

$$\sum_{\substack{j=1\\ j\neq i}}^{n} (\mathbf{M}(B))_{ij} v_i v_j + b_i v_i > 0 , \quad i = 1, \ldots, n ,$$

then the vertex $\mathbf{v}$ of the hypercube $[-1, 1]^n$ is an asymptotically stable equilibrium point of the information system (12).

Applying Lemma 7 to the information system model (12), we obtain the following result.

Proposition 2

A vertex $\mathbf{v}$ is not an equilibrium point of the information system (12) if and only if

$$\sum_{\substack{j=1\\ j\neq i}}^{n} (\mathbf{M}(B))_{ij} v_i v_j + b_i v_i < 0 \text{ for some } i = 1, \ldots, n .$$

Using the above two propositions, we can study the time evolution of the decision making process starting from initial assertions represented by the initial condition vector $\mathbf{x}(0)$ for a particular relationship between the elements of the universe represented by the connection matrix $\mathbf{M}(B)$.

4. Asynchronous Neural Information System

In the previous section, we modeled the time evolution of events that occur in a synchronous fashion, which means that all coordinates, $x_1, x_2, \ldots, x_n$, of an event are updated simultaneously. In this section, we study asynchronous version of the neural information system (NIS) model that we presented in Sec. 3. In the asynchronous mode of the evolution of the trajectory of the NIS, only one component of the state vector $\mathbf{x}(k)$ is allowed to change its value as the time changes; the remaining components are fixed. Thus, the asynchronous evolution of the trajectory of the gBSB neural model can be described as:

$$\boxed{\begin{array}{l} x_i(k+1) = (\mathbf{g}((\mathbf{I}_n + \alpha\mathbf{W})\mathbf{x}(k) + \alpha\mathbf{b}))_i \\ x_j(k+1) = x_j(k)\,, \quad j \neq i\,, \end{array}} \tag{23}$$

where the index i of the component that is to be updated at time k can be chosen randomly or using other selection scheme. Note that the operator defined by Eq. (23) is continuous. In our further discussion, we consider a general asynchronous update case. Let F_i be a continuous operator that operates on the state vector of a dynamical system where only the ith component of the state vector is affected. An example of such an operator is the update scheme described by Eq. (23). In a general case of the process of asynchronous evolution of the trajectory, at any instance of time one operator from the set $\{F_1, F_2, \ldots, F_n\}$ acts upon the current state to obtain the next state, i.e.,

$$\mathbf{x}(k+1) = T_k(\mathbf{x}(k))\,, \quad \mathbf{x}(0) = \mathbf{x}^0\,, \tag{24}$$

where

$$T_k \in \{F_1, F_2, \ldots, F_n\}\,. \tag{25}$$

A solution to the initial value problem (24) will be denoted as:

$$\mathbf{x}(k) = \mathbf{x}^k = T^{(k)}(\mathbf{x}^0)\,,$$

where

$$T^{(k)} = T_k \cdots T_1 T_0\,.$$

Similarly, the asynchronous evolution of the trajectory of the NIS model can be described as:

$$\boxed{\begin{array}{l} x_i(k+1) = (\mathbf{g}(\mathbf{M}(B)\mathbf{x}(k) + \mathbf{b}))_i \\ x_j(k+1) = x_j(k)\,, \quad j \neq i\,. \end{array}} \tag{26}$$

Note that in the synchronous mode, $x_i(k+1)$ could be different from $x_i(k)$ for all values of $i = 1, \ldots, n$, while in the asynchronous mode, the state vector can differ from its previous value in one component only.

We will now analyze the dynamic behavior of the asynchronous NIS, and in particular, the stability of its equilibrium points. The definitions of equilibrium points are the same for synchronous and asynchronous cases.

As in the synchronous case, we say that a function $E : \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov function for the model (23) if (i) E is continuous on $\mathbb{R}^n$; and (ii) $\Delta E \leq 0$ on the trajectories of (23).

A subset M of $\mathbb{R}^n$ is called an invariant set of the model (23) if $\mathbf{x}(k) \in M$, then $\mathbf{x}(k+1) \in M$.

In our subsequent discussion, we let J be the set of indices i corresponding to the coordinates of the state vector $\mathbf{x}$ that are being updated infinitely many times. Thus, given any $\tilde{k}$, there exists $k > \tilde{k}$ such that Eq. (23), or Eq. (26), is executed if and only if $i \in J$.

Consider a particular solution $T^{(k)}(\mathbf{x}^0)$ to the initial value problem (24). A point $\mathbf{y} \in \mathbb{R}^n$ is a limit point of $T^{(k)}(\mathbf{x}^0)$ if there is a subsequence $\{k_j\}$ such that:

$$T^{(k_j)}(\mathbf{x}^0) \to \mathbf{y}\,.$$

The set of all limits points of $T^{(k)}(\mathbf{x}^0)$ is called the limit set and is denoted by $\Omega(\mathbf{x}^0)$. Having defined the relevant notation, we now state and prove a new result that can be used to prove a generalized version of the LaSalle Invariance Principle.

Theorem 4

The limit set $\Omega(\mathbf{x}^0)$ generated by the family of continuous operators

$$\{F_1, F_2, \ldots, F_n\}$$

via the recurrence formula (24) is invariant under the F_q's that appear infinitely often. Furthermore, all bounded subsequences of $T^{(k)}(\mathbf{x}^0)$ converge to $\Omega(\mathbf{x}^0)$ as $k \to \infty$.

Proof

Let $\mathbf{y} \in \Omega(\mathbf{x}^0)$. Then, there is a subsequence $\{k_j\}$ such that:

$$\mathbf{x}^{k_j} \to \mathbf{y}\,.$$

Let

$$\mathcal{S}_q = \{k_j : T_{k_j} = F_q\}\,. \tag{27}$$

The set $\mathcal{S}_q$ is the subset of $\{k_j\}$ corresponding to instances when the operator F_q is applied. We enumerate the elements of $\mathcal{S}_q$ by:

$$k_1^q < k_2^q < \cdots$$

Since each operator F_i is continuous, we have

$$\mathbf{x}^{k_j^q+1} = F_q(\mathbf{x}^{k_j^q}) \to F_q(\mathbf{y})$$

as $j \to \infty$. Thus, if $\mathbf{y}$ is in $\Omega(\mathbf{x}^0)$ then so is $F_q(\mathbf{y})$, which implies that the limit set $\Omega(\mathbf{x}^0)$ is invariant.

We will now show, by contradiction, that $T^{(k)}(\mathbf{x}^0) \to \Omega(\mathbf{x}^0)$ as $k \to \infty$. Recall that, by Definition 2, a sequence $\{\mathbf{x}^k\}$ is said to converge a set $\mathcal{A}$ if

$$\rho(\mathbf{x}^k, \mathcal{A}) = \inf\{\|\mathbf{x}^k - \mathbf{y}\| : \mathbf{y} \in \mathcal{A}\} \to 0\,.$$

Let $\{\mathbf{x}^{k_j}\}$ be a bounded subsequence. If $\rho(\mathbf{x}^{k_j}, \Omega(\mathbf{x}^0))$ does not converge to zero, then there exists a subsequence $\{\mathbf{z}^j\}$ of $\{\mathbf{x}^{k_j}\}$ such that $\mathbf{z}^j \to \mathbf{y}$ and $\rho(\mathbf{z}^j, \Omega(\mathbf{x}^0)) \geq a > 0$ for all j. But since $\mathbf{y} \in \Omega(\mathbf{x}^0)$,

$$\rho(\mathbf{z}^j, \Omega(\mathbf{x}^0)) \leq \|\mathbf{z}^j - \mathbf{y}\| \to 0\,,$$

and thus $\rho(\mathbf{z}^j, \Omega(\mathbf{x}^0)) \to 0$, which is a contradiction. The proof of the theorem is complete. □

A generalized version of the Invariance Principle that does not require that the update operator modeling a dynamical systems be continuous follows from Theorem 4. In the asynchronous case, the update operator is discontinuous. However, it is generated by a family of continuous operators given by Eq. (25). The statement of this generalized Invariance Principle is identical to the one stated in Theorem 1 except that now we refer to the model given by Eq. (24).

Theorem 5 (Generalized Invariance Principle)

If E is a Lyapunov function for the model,

$$\mathbf{x}(k+1) = T_k(\mathbf{x}(k))\,, \quad \mathbf{x}(0) = \mathbf{x}^0\,,$$

with bounded solutions, where

$$T_k \in \{F_1, F_2, \ldots, F_n\}$$

is the set whose elements are continuous operators, then there is a number c such that $\mathbf{x}(k)$ converges to $M \cap E^{-1}(c)$, where M is the largest invariant set contained in the set $S = \{\mathbf{x} \in \mathbb{R}^n : \Delta E = 0\}$.

Proof

To prove the theorem, it is enough to use Theorem 4 and arguments similar to those of LaSalle (Ref. 28, p. 10). □

Using the above result, we will now give a sufficient condition for the trajectories of the asynchronous gBSB neural model (23), to converge to a set that contains the set of the equilibrium points of the model. This result is later used to show that the trajectories of the asynchronous information system neural model converge, under certain conditions, to a set of the equilibrium points of the model.

Theorem 6

The state $\mathbf{x}(k)$ of Eq. (23) converges to a subset of

$$G = \{\mathbf{x} : x_i = (\mathbf{g}((\mathbf{I}_n + \alpha\mathbf{W})\mathbf{x} + \alpha\mathbf{b}))_i,\ i \in J\}$$

if the weight matrix $\mathbf{W}$ is symmetric and

$$w_{ii} > -\frac{2}{\alpha}\,, \quad i \in J\,.$$

Proof

Note that there exists some finite $k' \geq 0$ such that for all $k \geq k'$, Eq. (23) is executed only for $i \in J$. It suffices to show that $\mathbf{x}(k)$, $k \geq k'$, converges to a subset of G. For this, we consider a Lyapunov function candidate,

$$E(\mathbf{x}(k)) = -2\alpha\mathbf{x}(k)^T\mathbf{b} - \alpha\mathbf{x}(k)^T\mathbf{W}\mathbf{x}(k)\,.$$

We now show that $E(\mathbf{x}(k))$ evaluated on the trajectories of Eq. (23) is monotonically decreasing. Without loss of generality, we assume that at time k, the pth component of $\mathbf{x}$ is updated. Thus,

$$\Delta\mathbf{x}(k) = [0 \cdots 0\ \Delta x_p(k)\ 0 \cdots 0]^T\,.$$

It follows that:

$$\begin{aligned}\Delta E(k) &= E(\mathbf{x}(k+1)) - E(\mathbf{x}(k)) \\ &= -2\alpha(\mathbf{x}(k)+\Delta\mathbf{x}(k))^T\mathbf{b} - \alpha(\mathbf{x}(k) \\ &\quad + \Delta\mathbf{x}(k))^T\mathbf{W}(\mathbf{x}(k)+\Delta\mathbf{x}(k)) \\ &\quad - (-2\alpha\mathbf{x}(k)^T\mathbf{b} - \alpha\mathbf{x}(k)^T\mathbf{W}\mathbf{x}(k)) \\ &= -2\alpha\Delta\mathbf{x}(k)^T\mathbf{b} - \alpha\Delta\mathbf{x}(k)^T\mathbf{W}\Delta\mathbf{x}(k) \\ &\quad - \alpha\Delta\mathbf{x}(k)^T(\mathbf{W}+\mathbf{W}^T)\mathbf{x}(k) \\ &= -2\alpha\Delta\mathbf{x}(k)^T\mathbf{b} - \alpha\Delta\mathbf{x}(k)^T\mathbf{W}\Delta\mathbf{x}(k) \\ &\quad - 2\alpha\Delta\mathbf{x}(k)^T\mathbf{W}\mathbf{x}(k) \\ &= -2\alpha\Delta x_p(k)b_p - \alpha w_{pp}(\Delta x_p(k))^2 \\ &\quad - 2\alpha\Delta x_p(k)(\mathbf{W}\mathbf{x}(k))_p \\ &= -\alpha w_{pp}(\Delta x_p(k))^2 \\ &\quad - 2\Delta x_p(k)((L(\mathbf{x}(k)))_p - x_p(k)), \qquad (28)\end{aligned}$$

since $\mathbf{W} = \mathbf{W}^T$ and $L(\mathbf{x}(k)) - \mathbf{x}(k) = \alpha(\mathbf{W}\mathbf{x}(k) + \mathbf{b})$. Applying Lemma 3 to Eq. (28) yields

$$\begin{aligned}\Delta E(k) &\le -\alpha w_{pp}(\Delta x_p(k))^2 - 2(\Delta x_p(k))^2 \\ &= -(\alpha w_{pp}+2)(\Delta x_p(k))^2 . \qquad (29)\end{aligned}$$

Thus, as long as

$$w_{ii} > -\frac{2}{\alpha}, \quad i \in J,$$

we have

$$\Delta E(k) \le 0$$

for all $k \ge k'$ no matter which component, x_p, $p \in J$, is being updated. Since E is continuous on $\mathbb{R}^n$ and monotonically decreasing, E is a Lyapunov function of the asynchronous NIS. Furthermore, $\mathbf{x}(k)$ is bounded. By Theorem 5, there exists a number c such that $\mathbf{x}(k)$ converges to $M \cap E^{-1}(c)$, where M is the largest invariant set within $S = \{\mathbf{x} \in \mathbb{R}^n : \Delta E = 0\}$. From Eqs. (28) and (29), $\Delta E(k) = 0$ for all $k \ge k'$ if and only if $\Delta x_p = 0$ for all $p \in J$. This implies that $M = S = G$. Therefore, $\mathbf{x}(k)$ converges to $G \cap E^{-1}(c)$, a subset of G. □

We now examine the stability of the information system modeled by Eq. (26).

Theorem 7

The state $\mathbf{x}(k)$ of Eq. (26) converges to a subset of

$$\{\mathbf{x} : x_i = (\mathbf{g}(\mathbf{M}(B)\mathbf{x} + \mathbf{b}))_i,\ i \in J\}.$$

Proof

Let $\mathbf{W} = \mathbf{M}(B) - \mathbf{I}_n$ and $\alpha = 1$. Then, Eq. (23) becomes Eq. (26). By Lemma 5, $\mathbf{W} = \mathbf{M}^T(B) - \mathbf{I}_n = \mathbf{W}^T$. From Eq. (8), we have $m_{ii} = 1$, $i = 1, \ldots, n$. Thus, for all $i \in J$,

$$w_{ii} = (\mathbf{M}(B) - \mathbf{I}_n)_{ii} = m_{ii} - 1 = 0 > -2/\alpha .$$

It follows from Theorem 6 that $\mathbf{x}(k)$ converges to a subset of:

$$G = \{\mathbf{x} : x_i = (\mathbf{g}(\mathbf{M}(B)\mathbf{x} + \mathbf{b}))_i,\ i \in J\}. \quad \square$$

Corollary 1

Suppose that all the components of $\mathbf{x}$ in Eq. (26) are updated infinitely many times. Then, the state $\mathbf{x}(k)$ of Eq. (26) converges to a set of equilibrium points.

The above results will be illustrated with numerical examples in the next section.

5. Examples

In this section, we apply the neural information systems described in Secs. 3 and 4 to an analysis of some of the United Nations (UN) activities.

Example 3

Suppose that we have a universe U that consists of the following seven trading countries: United States, Canada, Japan, Mexico, China, Germany, and the United Kingdom. Using available document (Ref. 31, pp. 370–387], we have voting records of each of the above countries on the following resolutions:

- $\mathbf{a}_1$: Human rights and scientific and technological developments (A/RES/43/110)
- $\mathbf{a}_2$: Indivisibility and interdependence of economic, social, cultural, civil and political rights (A/RES/43/113)
- $\mathbf{a}_3$: Alternative approaches and ways and means within the UN system for improving the effective enjoyment of human rights and fundamental freedoms (A/RES/43/126)

$\mathbf{a}_4$: Preparation of an international development strategy for the 4th UN Development Decade (A/RES/43/182)

$\mathbf{a}_5$: International conference on money and finance (A/RES/43/187).

Knowledge of the data allows us to construct the information system presented in Table 3, where x_1 is a label attached to the United States, x_2 to Canada, x_3 to Japan, x_4 to Mexico, x_5 to China, x_6 to Germany, and x_7 to the United Kingdom. Thus, $A = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_5\}$. Applying Eq. (7) to the information system given in Table 3, we obtain the following connection matrix, $\mathbf{M}(A)$, between the members of the universe U,

$$\mathbf{M}(A) = \begin{bmatrix} 1 & 0.2 & 0.2 & -0.4 & -0.4 & 0.2 & 0.2 \\ 0.2 & 1 & 0.6 & -0.2 & -0.2 & 0.6 & 0.6 \\ 0.2 & 0.6 & 1 & -0.2 & -0.2 & 0.6 & 0.6 \\ -0.4 & -0.2 & -0.2 & 1 & 1 & -0.2 & -0.2 \\ -0.4 & -0.2 & -0.2 & 1 & 1 & -0.2 & -0.2 \\ 0.2 & 0.6 & 0.6 & -0.2 & -0.2 & 1 & 0.6 \\ 0.2 & 0.6 & 0.6 & -0.2 & -0.2 & 0.6 & 1 \end{bmatrix}.$$

Table 3. The information system considered in Examples 3 and 4.

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$
x_1	0	−1	0	0	−1
x_2	0	0	−1	1	−1
x_3	0	0	−1	1	−1
x_4	1	1	1	1	1
x_5	1	1	1	1	1
x_6	0	0	−1	1	−1
x_7	0	0	−1	1	−1

The connection matrix $\mathbf{M}(A)$ can be viewed as a cognitive map in the sense of Axelrod.[17] Using the above, we wish to infer about the positions that the members of the universe U take when presented with a new resolution to vote upon. We first analyze the case when the members of the universe are allowed to vote in a synchronous fashion. Our simulation is somewhat simplified in that we do not account for the presence of many other members of the UN Organization. We assume that the seven members of the universe U will meet many times to discuss the resolution. In addition, during each meeting, each member will inform the remaining members about its current position on the resolution.

We further assume that the United States will support the resolution while Japan will oppose the resolution to degree one, irrespective of the other members' position on the issue. The above assumption is translated into the bias vector $\mathbf{b}$ that is constructed using Eq. (14),

$$\mathbf{b} = [8\ 0\ -8\ 0\ 0\ 0\ 0]^T .$$

We next assume that the resolution was put forward by the United States. Hence, we have the following initial condition:

$$\mathbf{x}(0) = [1\ 0\ 0\ 0\ 0\ 0\ 0]^T . \qquad (30)$$

Using the above data, we construct the synchronous NIS. The time evolution of the trajectory of the synchronous NIS is presented in Table 4. As can be seen from Table 4, the system settles down in an equilibrium point after five iterations. The equilibrium point can be interpreted as follows: the United States, Canada, Germany, and the United Kingdom support the resolution to degree one, while Japan, Mexico, and China oppose the resolution to degree one. These positions were reached after five meetings between the members of the universe U. The time evolution of the alliances or coalitions between the members of the universe can be examined using the scheme described in Sec. 2. In particular, we have

$$U/\{\mathbf{x}(k)\} = \begin{cases} \{\{x_1\}, \{x_2, x_3, x_4, x_5, x_6, x_7\}\}, & k = 0 \\ \{\{x_1\}, \{x_2, x_6, x_7\}, \{x_3\}, \{x_4, x_5\}\}, & k = 1 \\ \{\{x_1\}, \{x_2, x_6, x_7\}, \{x_3, x_4, x_5\}\}, & 2 \le k \le 4 \\ \{\{x_1, x_2, x_6, x_7\}, \{x_3, x_4, x_5\}\}, & k \ge 5 . \end{cases}$$

Table 4. Time evolutions of events in Examples 3.

k	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	1	0	0	0	0	0	0
1	1	0.2	−1	−0.4	−0.4	0.2	0.2
2	1	0.2	−1	−1	−1	0.2	0.2
3	1	0.44	−1	−1	−1	0.44	0.44
4	1	0.97	−1	−1	−1	0.97	0.97
5	1	1	−1	−1	−1	1	1
6	1	1	−1	−1	−1	1	1

It is interesting that the equilibrium state is an asymptotically stable vertex of the hypercube $[-1, 1]^n$, as it satisfies the stability conditions presented in Proposition 1. Only three other vertices satisfy the same conditions. They are:

$$[1\ -1\ -1\ -1\ -1\ -1\ -1]^T,$$

$$[1\ -1\ -1\ 1\ 1\ -1\ -1]^T,$$

and

$$[1\ 1\ -1\ 1\ 1\ 1\ 1]^T.$$

In the following example, we analyze the asynchronous NIS using the above data, and we will see that the update sequences of the state vector play a significant role in the convergence of the trajectories of the asynchronous NIS.

Example 4

We consider the same problem as in Example 3, with the exception that the positions of the members of the universe U are now solicited using the roll-call method. This scenario is modeled by us using the asynchronous NIS. We assume that the roll-call is taken in alphabetical order. We consider two calls represented by two update sequences. One sequence begins with Canada, which is x_2, and then proceeds in alphabetical order to China (x_5), Germany (x_6), Japan (x_3), etc. Using the subscripts of the symbols representing the countries, the update sequence can be expressed as:

$$\{2, 5, 6, 3, 4, 7, 1, 2, 5, 6, 3, 4, 7, 1, 2, \ldots\}. \quad (31)$$

The other update sequence is obtained by taking call starting with Germany (x_6), and then proceeding in an alphabetical order as before. This update sequence can be expressed as:

$$\{6, 3, 4, 7, 1, 2, 5, 6, 3, 4, 7, 1, 2, 5, 6, \ldots\}. \quad (32)$$

Using the same initial condition as before, we obtain the time evolution of the trajectories of the asynchronous NIS that models the above scenario. In Table 5, the time evolution corresponding to the sequence (31) is shown. The time evolution of the asynchronous NIS with the update sequence (32) is

Table 5. Time evolution of the asynchronous NIS trajectory resulting from the update sequence (31).

k	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	1	0	0	0	0	0	0
1	1	0.2	0	0	0	0	0
2	1	0.2	0	0	−0.44	0	0
3	1	0.2	0	0	−0.44	0.41	0
4	1	0.2	−1	0	−0.44	0.41	0
5	1	0.2	−1	−0.76	−0.44	0.41	0
6	1	0.2	−1	−0.76	−0.44	0.41	0.21
7	1	0.2	−1	−0.76	−0.44	0.41	0.21
8	1	0.41	−1	−0.76	−0.44	0.41	0.21
9	1	0.41	−1	−0.76	−1	0.41	0.21
10	1	0.41	−1	−0.76	−1	0.73	0.21
11	1	0.41	−1	−0.76	−1	0.73	0.21
12	1	0.41	−1	−1	−1	0.73	0.21
13	1	0.41	−1	−1	−1	0.73	0.89
14	1	0.41	−1	−1	−1	0.73	0.89
15	1	1	−1	−1	−1	0.73	0.89
16	1	1	−1	−1	−1	0.73	0.89
17	1	1	−1	−1	−1	1	0.89
18	1	1	−1	−1	−1	1	0.89
19	1	1	−1	−1	−1	1	0.89
20	1	1	−1	−1	−1	1	1

Table 6. Time evolution of the asynchronous NIS trajectory resulting from the update sequence (32).

k	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	1	0	0	0	0	0	0
1	1	0	0	0	0	0.2	0
2	1	0	−1	0	0	0.2	0
3	1	0	−1	−0.24	0	0.2	0
4	1	0	−1	−0.24	0	0.2	−0.23
5	1	0	−1	−0.24	0	0.2	−0.23
6	1	−0.37	−1	−0.24	0	0.2	−0.23
7	1	−0.37	−1	−0.24	−0.36	0.2	−0.23
8	1	−0.37	−1	−0.24	−0.36	−0.44	−0.23
9	1	−0.37	−1	−0.24	−0.36	−0.44	−0.23
10	1	−0.37	−1	−0.59	−0.36	−0.44	−0.23
11	1	−0.37	−1	−0.59	−0.36	−0.44	−0.93
12	1	−0.37	−1	−0.59	−0.36	−0.44	−0.93
13	1	−1	−1	−0.59	−0.36	−0.44	−0.93
14	1	−1	−1	−0.59	−0.68	−0.44	−0.93
15	1	−1	−1	−0.59	−0.68	−1	−0.93
16	1	−1	−1	−0.59	−0.68	−1	−0.93
17	1	−1	−1	−0.88	−0.68	−1	−0.93
18	1	−1	−1	−0.88	−0.68	−1	−1
19	1	−1	−1	−0.88	−0.68	−1	−1
20	1	−1	−1	−0.88	−0.68	−1	−1
21	1	−1	−1	−0.88	−1	−1	−1
22	1	−1	−1	−0.88	−1	−1	−1
23	1	−1	−1	−0.88	−1	−1	−1
24	1	−1	−1	−1	−1	−1	−1

shown in Table 6. In both cases, the trajectories of both sequences converge to the asymptotically stable vertices that were listed in Example 3. The vertices the trajectories converge to, however, are different, despite the same initial condition. This illustrates that the update sequence does indeed influence the dynamical behavior of the asynchronous NIS.

It is interesting to note that the second update scheme which corresponds to the roll-call method of soliciting positions of the negotiating nations, starting with Germany and then proceeding in an alphabetical order, leads to the asynchronous equilibrium point which corresponds to all other nations opposing the resolution put forward by the United States. However, the same resolution received the support of three other nations when the negotiating positions were solicited starting with Canada. The above example seems to indicate that even a slight change in order in which viewpoints of the negotiators are announced, may result in different final votes on the issue under negotiation.

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