An Introduction to Model Reference Adaptive Control (MRAC)

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Outline

- Plant and reference models
- Ideal Model Reference controller
- Adaptation law and Model Reference Adaptive Controller (MRAC)
- Stability analysis of the closed-loop system
- Projection operator and its implementation
- Closing the loop
- Convergence of the tracking error to 0
- Estimation of the uncertainty
- Simulations

Model Reference Control

- Plant model

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}(W_f + \Delta W_f)$$

Model Reference Control

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- Model reference

$$\dot{\boldsymbol{x}}_d = \boldsymbol{A}_d \boldsymbol{x}_d + \boldsymbol{b}_d r$$

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- We take $\boldsymbol{b}_d = \boldsymbol{b}$

Model Reference Control

- Plant model

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- Model reference

$$\dot{\boldsymbol{x}}_d = \boldsymbol{A}_d \boldsymbol{x}_d + \boldsymbol{b}_d r$$

- We take $\boldsymbol{b}_d = \boldsymbol{b}$
- Assume that the pair $(\boldsymbol{A}, \boldsymbol{b})$ is stabilizable. Construct $\boldsymbol{k}_d$ so that

$$\boldsymbol{A}_d = \boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}_d$$

is asymptotically stable

Robust controller construction

- Let

$$\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_d(t)$$

Robust controller construction

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- **Our objective**: Construct a MRAC such that

$$\boldsymbol{e}(t) \to 0 \quad \text{as} \quad t \to \infty$$

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- Find $\dot{\boldsymbol{e}}(t)$,

$$\dot{\boldsymbol{e}} = \dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_d$$

Robust controller construction

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- **Our objective**: Construct a MRAC such that

$$\boldsymbol{e}(t) \to 0 \quad \text{as} \quad t \to \infty$$

- Find $\dot{\boldsymbol{e}}(t)$,

$$\begin{aligned} \dot{\boldsymbol{e}} &= \dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_d \\ &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}(W_f + \Delta W_f) - \boldsymbol{A}_d\boldsymbol{x}_d - \boldsymbol{b}r \end{aligned}$$

Reference model tracking error equation

- Add and subtract the term $\boldsymbol{b}\boldsymbol{k}_d\boldsymbol{x}$,

$$\dot{\boldsymbol{e}} = \dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_d$$

Reference model tracking error equation

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&= \boldsymbol{A}\boldsymbol{x} - {\color{red}\boldsymbol{b}\boldsymbol{k}_d\boldsymbol{x}} + {\color{red}\boldsymbol{b}\boldsymbol{k}_d\boldsymbol{x}} + \boldsymbol{b}(W_f + \Delta W_f) - \boldsymbol{A}_d\boldsymbol{x}_d - \boldsymbol{b}r
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&= \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\boldsymbol{k}_d\boldsymbol{x} + \boldsymbol{b}\boldsymbol{k}_d\boldsymbol{x} + \boldsymbol{b}(W_f + \Delta W_f) - \boldsymbol{A}_d\boldsymbol{x}_d - \boldsymbol{b}r \\
&= (\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}_d)\boldsymbol{x} - \boldsymbol{A}_d\boldsymbol{x}_d + \boldsymbol{b}(W_f + \Delta W_f + \boldsymbol{k}_d\boldsymbol{x} - r)
\end{aligned}$$

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\end{aligned}
$$

- Recall that $\boldsymbol{A}_d = \boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}_d$; hence,

$$
\dot{\boldsymbol{e}} = (\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}_d)\boldsymbol{e} + \boldsymbol{b}(W_f + \Delta W_f + \boldsymbol{k}_d\boldsymbol{x} - r),
$$

where W_f is the control input

Ideal MRAC

- Recall the tracking error dynamics,

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- If we knew ΔW_f, we could use the ideal controller,

$$W_f = -\Delta W_f - \boldsymbol{k}_d \boldsymbol{x} + r$$

to obtain

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$$\begin{aligned} \dot{\boldsymbol{e}} &= (\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}_d)\boldsymbol{e} \\ &= \boldsymbol{A}_d\boldsymbol{e} \end{aligned}$$

Ideal MRAC

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$$W_f = -\Delta W_f - \boldsymbol{k}_d\boldsymbol{x} + r$$

  to obtain

- $$\begin{aligned} \dot{\boldsymbol{e}} &= (\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}_d)\boldsymbol{e} \\ &= \boldsymbol{A}_d\boldsymbol{e} \end{aligned}$$

- Since $\boldsymbol{A}_d$ is asymptotically stable, we have

$$\boldsymbol{e} \to 0 \quad \text{as} \quad t \to \infty$$

Preparing to construct a MRAC

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Preparing to construct a MRAC

- We do not know ΔW_f.
- Construct an estimator for ΔW_f to obtain an estimate $\widetilde{\Delta W_f}$ and use it in the controller implementation

$$W_f = -\widetilde{\Delta W_f} - \boldsymbol{k}_d \boldsymbol{x} + r$$

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$$W_f = -\widetilde{\Delta W_f} - \boldsymbol{k}_d \boldsymbol{x} + r$$

- How do we get $\widetilde{\Delta W_f}$?
- Use the Lyapunov second method and Barbalat's lemma
- Use a dynamic estimator of ΔW_f to obtain $\widetilde{\Delta W_f}$ such that

$$\widetilde{\Delta W_f} \to \Delta W_f \quad \text{as} \quad t \to \infty$$

and then use $\widetilde{\Delta W_f}$ instead of ΔW_f in W_f

Analysis of the ideal tracking error equation

- Consider the ideal error system $\dot{\boldsymbol{e}} = \boldsymbol{A}_d \boldsymbol{e}$ and let

$$V = \boldsymbol{e}^\top \boldsymbol{P} \boldsymbol{e}, \quad \boldsymbol{P} = \boldsymbol{P}^\top \succ 0$$

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- Evaluate $\frac{dV}{dt}$ on the trajectories of $\dot{\boldsymbol{e}} = \boldsymbol{A}_d \boldsymbol{e}$,

$$\frac{dV}{dt} = \frac{d}{dt}(\boldsymbol{e}^\top \boldsymbol{P} \boldsymbol{e})$$

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\frac{dV}{dt} &= \frac{d}{dt}(\boldsymbol{e}^\top \boldsymbol{P} \boldsymbol{e}) \\
&= \frac{d}{dt}(\boldsymbol{e}^\top (\boldsymbol{P} \boldsymbol{e})) \\
&= \dot{\boldsymbol{e}}^\top (\boldsymbol{P} \boldsymbol{e}) + \boldsymbol{e}^\top \boldsymbol{P} \dot{\boldsymbol{e}}
\end{aligned}$$

- Substitute $\dot{\boldsymbol{e}} = \boldsymbol{A}_d \boldsymbol{e}$ and $\dot{\boldsymbol{e}}^\top = \boldsymbol{e}^\top \boldsymbol{A}_d^\top$ into the above

Analysis of the ideal tracking error equation—contd.

- We obtain

$$\frac{d}{dt}(\boldsymbol{e}^\top \boldsymbol{P}\boldsymbol{e}) = \boldsymbol{e}^\top \boldsymbol{A}_d^\top \boldsymbol{P}\boldsymbol{e} + \boldsymbol{e}^\top \boldsymbol{P}\boldsymbol{A}_d \boldsymbol{e}$$

Analysis of the ideal tracking error equation—contd.

- We obtain

$$\begin{aligned} \frac{d}{dt}(\boldsymbol{e}^\top \boldsymbol{P}\boldsymbol{e}) &= \boldsymbol{e}^\top \boldsymbol{A}_d^\top \boldsymbol{P}\boldsymbol{e} + \boldsymbol{e}^\top \boldsymbol{P}\boldsymbol{A}_d \boldsymbol{e} \\ &= \boldsymbol{e}^\top \left(\boldsymbol{A}_d^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_d \right) \boldsymbol{e} \end{aligned}$$

Analysis of the ideal tracking error equation—contd.

- We obtain

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where

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- Note that we could have used the chain rule to obtain

$$\frac{d}{dt}(\boldsymbol{e}^\top \boldsymbol{P}\boldsymbol{e}) = 2\boldsymbol{e}^\top \boldsymbol{P}\dot{\boldsymbol{e}}$$

Analysis of the ideal tracking error equation—contd.

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$$
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&= \boldsymbol{e}^\top \left(\boldsymbol{A}_d^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_d\right)\boldsymbol{e} \\
&= -\boldsymbol{e}^\top \boldsymbol{Q}\boldsymbol{e},
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\end{aligned}
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Analysis of the tracking error equation

- Take $V = \boldsymbol{e}^\top \boldsymbol{P}\boldsymbol{e}$, $\boldsymbol{P} = \boldsymbol{P}^\top \succ 0$, and evaluate $\frac{dV}{dt}$ on the trajectories of

$$\dot{\boldsymbol{e}} = \boldsymbol{A}_d \boldsymbol{e} + \boldsymbol{b}(W_f + \Delta W_f + \boldsymbol{k}_d \boldsymbol{x} - r)$$

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&= 2\boldsymbol{e}^\top \boldsymbol{P}\boldsymbol{A}_d \boldsymbol{e} \\
&\quad + 2\boldsymbol{e}^\top \boldsymbol{P}\boldsymbol{b}(W_f + \Delta W_f + \boldsymbol{k}_d \boldsymbol{x} - r)
\end{aligned}$$

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&= 2\boldsymbol{e}^\top \boldsymbol{P}\boldsymbol{A}_d\boldsymbol{e} \\
&\quad +2\boldsymbol{e}^\top \boldsymbol{P}\boldsymbol{b}(W_f + \Delta W_f + \boldsymbol{k}_d\boldsymbol{x} - r)
\end{aligned}$$

- Substitute into the above $W_f = -\widetilde{\Delta W_f} - \boldsymbol{k}_d\boldsymbol{x} + r$

Analysis of the tracking error equation—contd.

- We obtain

$$\begin{aligned}\frac{d}{dt}(\boldsymbol{e}^\top \boldsymbol{P}\boldsymbol{e}) &= -\boldsymbol{e}^\top \boldsymbol{Q}\boldsymbol{e} \\ &\quad +2\boldsymbol{e}^\top \boldsymbol{P}\boldsymbol{b}(-\widetilde{\Delta W_f} - \boldsymbol{k}_d\boldsymbol{x} + r + \Delta W_f + \boldsymbol{k}_d\boldsymbol{x} - r)\end{aligned}$$

Analysis of the tracking error equation—contd.

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- We assume that $\Delta W_f = \text{const}$

Analysis of the tracking error equation—contd.

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- We assume that $\Delta W_f = \text{const}$
- Consider the augmented Lyapunov function candidate

$$V_a = \boldsymbol{e}^\top \boldsymbol{P}\boldsymbol{e} + \frac{1}{\Gamma}\left(\Delta W_f - \widetilde{\Delta W_f}\right)^2$$

Analysis of the tracking error equation—contd.

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- Consider the augmented Lyapunov function candidate

$$V_a = \boldsymbol{e}^\top \boldsymbol{P}\boldsymbol{e} + \frac{1}{\Gamma}\left(\Delta W_f - \widetilde{\Delta W_f}\right)^2$$

- $$\frac{dV_a}{dt} = ?$$

Evaluating $\frac{dV_a}{dt}$ on $\dot{\boldsymbol{e}} = \boldsymbol{A}_d\boldsymbol{e} + \boldsymbol{b}(\Delta W_f - \widetilde{\Delta W_f})$

- Evaluate

$$\frac{dV_a}{dt} = \frac{d}{dt}\left(\boldsymbol{e}^\top \boldsymbol{P}\boldsymbol{e} + \frac{1}{\Gamma}\left(\Delta W_f - \widetilde{\Delta W_f}\right)^2\right) =?$$

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- The first term,

$$\frac{d}{dt}\left(\boldsymbol{e}^\top \boldsymbol{P}\boldsymbol{e}\right) = -\boldsymbol{e}^\top \boldsymbol{Q}\boldsymbol{e} + 2\boldsymbol{e}^\top \boldsymbol{P}\boldsymbol{b}(\Delta W_f - \widetilde{\Delta W_f})$$

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- The second term

$$\frac{d}{dt}\left(\frac{1}{\Gamma}\left(\Delta W_f - \widetilde{\Delta W_f}\right)^2\right) = -\frac{2}{\Gamma}\left(\Delta W_f - \widetilde{\Delta W_f}\right)\dot{\widetilde{\Delta W_f}}$$

Manipulate $\frac{dV_a}{dt}$ on $\dot{\boldsymbol{e}} = \boldsymbol{A}_d\boldsymbol{e} + \boldsymbol{b}(\Delta W_f - \widetilde{\Delta W_f})$

- Let

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Evaluating $\frac{dV_a}{dt}$ on $\dot{\boldsymbol{e}} = \boldsymbol{A}_d\boldsymbol{e} + \boldsymbol{b}(\Delta W_f - \widetilde{\Delta W_f})$—contd.

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then

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Convergence of the tracking error to 0

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- For asymptotic stability we need $\frac{dV_a}{dt} < 0$ in the $\begin{bmatrix} \boldsymbol{e} \\ \Phi_f \end{bmatrix}$-space
- Can we though conclude that $\boldsymbol{e} \to 0$ as $t \to \infty$?

Using the Lyapunov-like lemma to show $\boldsymbol{e} \to 0$

1. $$V_a = V_a(\boldsymbol{e}(t), \Phi_f(t)) = \boldsymbol{e}^\top \boldsymbol{P} \boldsymbol{e} + \frac{1}{\Gamma}\Phi_f^2$$

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3. To conclude that
   $$\lim_{t\to\infty} \dot{V}_a(\boldsymbol{e}(t), \Phi_f(t)) = 0$$
   need to show that
   $$\dot{V}_a(\boldsymbol{e}(t), \Phi_f(t))$$
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$\dot{V}_a(\boldsymbol{e}(t), \Phi_f(t))$ uniformly continuous in t and so $\boldsymbol{e}(t) \to 0$!

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- The above implies that

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Practical implementation of the adaptation law

- Ensure boundedness of the estimates

$$\dot{\widetilde{\Delta W_f}} = \begin{cases} 0 & \text{if} \quad \widetilde{\Delta W_f} \geq \overline{\widetilde{\Delta W_f}} \quad \text{and} \quad \Gamma\sigma > 0 \\ 0 & \text{if} \quad \widetilde{\Delta W_f} \leq \underline{\widetilde{\Delta W_f}} \quad \text{and} \quad \Gamma\sigma < 0 \\ \Gamma\sigma & \text{if} \quad \text{otherwise} \end{cases}$$

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- That is,

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Stability analysis when $\dot{\widetilde{\Delta W_f}} = \text{Proj}_{\widetilde{\Delta W_f}}(\Gamma\sigma)$

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$$\frac{dV_a}{dt} \leq -\boldsymbol{e}^\top \boldsymbol{Q}\boldsymbol{e}$$

- Use the Lyapunov-like lemma to conclude that we have, as before,

$$\lim_{t\to\infty} \boldsymbol{e}(t) = 0$$

Estimation of the uncertainty ΔW_f

- We showed that $\boldsymbol{e} \to \mathbf{0}$ as $t \to \infty$

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- If $\boldsymbol{b}$ is full column rank, then the above is satisfied for

$$\Delta W_f - \widetilde{\Delta W_f} = 0,$$

that is, in steady-state,

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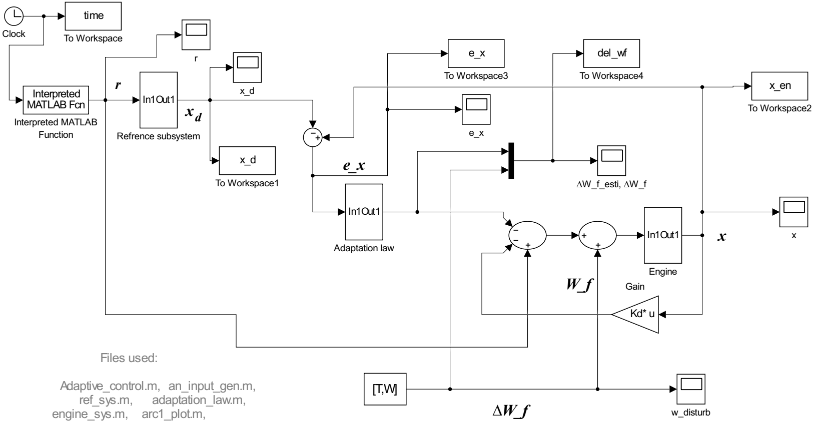
MRAC controlled system

Adaptive controller for a two-spool turbofan

Adaptive robust control with unknown disturbance ∆W_f



Files used:

Adaptive_control.m, an_input_gen.m, ref_sys.m, adaptation_law.m, engine_sys.m, arc1_plot.m, ref_signal_gen.m

Last updated January 19, 2016 by Stan Żak