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Variable Structure, Sliding-Mode Controller Design

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50.1 Introduction and Background

This chapter investigates variable structure control (VSC) as a high-speed switched feedback control resulting in a sliding mode. For example, the gains in each feedback path switch between two values according to a rule that depends on the value of the state at each time instant. The purpose of the switching control law is to drive the plant's state trajectory onto a prespecified (user-chosen) surface in the state space and to maintain the plant's state trajectory on this surface for all subsequent time. This surface is called a *switching surface* and the resulting motion of the state trajectory a *sliding mode*. When the plant state trajectory is "above" the surface, a feedback path has one gain and a different gain if the trajectory drops "below" the surface. This surface defines the rule for proper switching. The surface is also called a *sliding surface* (sliding manifold) because, ideally speaking, once intercepted, the switched control maintains the plant's state trajectory on the surface for all subsequent time and the plant's state trajectory then slides along this surface. The plant dynamics restricted to this surface represent the controlled system's behavior. The first critical phase of a VSC design is to properly construct a switching surface, so that the plant, restricted to the surface, has desired dynamics, such as stability to the origin, tracking, regulation, and so on.

In summary, a VSC control design generally breaks down into two phases. The first phase is to design or choose a sliding manifold/switching surface, so that the plant state restricted to the surface has desired dynamics. The second phase is to design a switched control that will drive the plant state to the switching surface and maintain it on the surface upon interception. A Lyapunov approach is used in this chapter to characterize this second design phase. Here, a generalized Lyapunov function, which characterizes the motion of the state trajectory to the surface, is specified in terms of the surface. For each chosen switched control structure, one chooses the "gains," so that the derivative of this Lyapunov function is negative definite with respect to the sliding surface, thus guaranteeing motion of the state trajectory to the surface.

As an introductory example, consider the first-order system $\dot{x}(t) = u(x, t)$ with control

$$u(x,t) = -\mathrm{sgn}(x) = \begin{cases} -1, & \text{if } x > 0, \\ +1, & \text{if } x < 0. \end{cases}$$

Hence, the system with control satisfies $\dot{x} = -\mathrm{sgn}(x)$ with trajectories plotted in Figure 50.1a. Here the control $u(x, t)$ switches, changing its value between ± 1 around the surface $\sigma(x, t) = x = 0$. Hence, for any initial condition x_0, a finite time t_1 exists for which $x(t) = 0$ for all $t \geq t_1$. Now, suppose $\dot{x}(t) = u(x, t) + v(t)$, where again $u(x, t) = -\mathrm{sgn}(x)$ and $v(t)$ is a bounded disturbance for which $\sup_t |v(t)| < 1$. As before, the control $u(x,t)$ switches its value between ± 1 around the surface $\sigma(x, t) = x = 0$. It follows that if $x(t) > 0$, then $\dot{x}(t) = -\mathrm{sgn}[x(t)] + v(t) < 0$, forcing motion toward the line $\sigma(x, t) = x = 0$, and if $x(t) < 0$, then $\dot{x}(t) = -\mathrm{sgn}[x(t)] + v(t) > 0$, again forcing motion toward the line $\sigma(x, t) = x = 0$. For a positive initial condition, this is illustrated in Figure 50.1b. The rate of convergence to the line depends on the disturbance. Nevertheless, a finite time t_1 exists for which $x(t) = 0$ for all $t \geq t_1$. If the disturbance

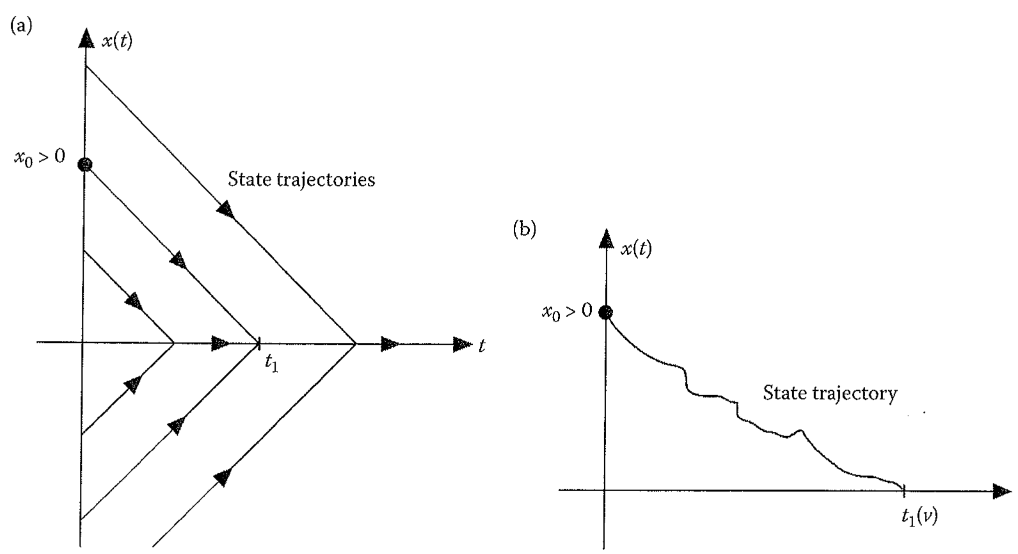

FIGURE 50.1 (a) State trajectories for the system $\dot{x} = -\mathrm{sgn}(x)$; (b) State trajectory for the system $\dot{x}(t) = -\mathrm{sgn}[x(t)] + v(t)$.

magnitude exceeds 1, then the gain can always be adjusted to compensate for the change. Hence, this VSC law is robust in the face of bounded disturbances, illustrating the simplicity and advantage of the VSC technique.

From the above example, one can see that VSC can provide a robust means of controlling (nonlinear) plants with disturbances and parameter uncertainties. Further, the advances in computer technology and high-speed switching circuitry have made the practical implementation of VSC quite feasible and of increasing interest. Indeed, pulse-width modulation control and switched dc-to-dc power converters [1] can be viewed in a VSC framework.

50.2 System Model, Control Structure, and Sliding Modes

50.2.1 System Model

The class of systems investigated herein has a state model nonlinear in the state vector $x(\cdot)$ and linear in the control vector $u(\cdot)$ of the form

$$\dot{x}(t) = F(x, t, u) = f(x, t) + B(x, t)u(x, t), \tag{50.1}$$

where $x(t) \in R^n$, $u(t) \in R^m$, and $B(x, t) \in R^{n\times m}$; further, each entry in $f(x,t)$ and $B(x,t)$ is assumed continuous with a bounded continuous derivative with respect to x. In the linear time-invariant case, Equation 50.1 becomes

$$\dot{x} = Ax + Bu \tag{50.2}$$

with A $n \times n$ and B $n \times m$ being constant matrices.

As mentioned in the previous section, the first phase of VSC or sliding mode control (SMC) design is to choose a manifold $S \subset R^n$, so that the control goal is reached once the state is maintained on S. As such we formally define the $(n - m)$-dimensional switching surface (also called a *discontinuity*, *sliding manifold*, or *equilibrium manifold*), as (the possibly time-varying)

$$S = \left\{x \in R^n | \sigma(x, t) = 0\right\} = \bigcap_{i=1}^{n} \left\{x \in R^n | \sigma_i(x, t) = 0\right\}, \tag{50.3}$$

where

$$\sigma(x, t) = [\sigma_1(x, t), \ldots, \sigma_m(x, t)]^T = 0. \tag{50.4}$$

(We will often refer to S as $\sigma(x, t) = 0$.) When there is no t-dependence, this $(n - m)$-dimensional manifold $S \subset R^n$ is determined as the intersection of m $(n - 1)$-dimensional surfaces $\sigma_i(x, t) = 0$. These surfaces are designed in such a way that the system state trajectory, restricted to $\sigma(x, t) = 0$, has a desired behavior such as stability or tracking. Although general nonlinear time-varying surfaces as in Equation 50.3 are possible, linear ones are more prevalent in design [2–6]. Linear surface design is presented in Section 50.4.

50.2.2 Control Structure

After proper design of the surface, a controller $u(x, t) = [u_1(x, t), \ldots, u_m(x, t)]^T$ is constructed, which generally has a switched form

$$u_i(x, t) = \begin{cases} u_i^+(x, t), & \text{when } \sigma_i(x, t) > 0, \\ u_i^-(x, t), & \text{when } \sigma_i(x, t) < 0. \end{cases} \tag{50.5}$$

Equation 50.5 indicates that the control changes its value depending on the sign of $\sigma(x, t)$. Here we can define that a discontinuity set, D, in the right-hand side is a union of the hypersurfaces defined by $\sigma_i(x,t) = 0$:

$$D = \bigcup_{i=1}^{n} \{x \in R^n | \sigma_i(x,t) = 0\}.$$

Thus, the (possibly t-dependent) hypersurfaces $\{x \in R^n | \sigma_i(x,t) = 0\}$ can be called switching surfaces and the functions $\sigma_i(x,t)$ switching functions. The goal of phase 2 is to stabilize the state to S. Off S, the control values $u_i^{\pm}$ are chosen so that the state trajectory converges to S in finite time, that is, the sliding mode exists on S, but the sliding mode may (or may not) also exist on some of the hypersurfaces $\{x \in R^n | \sigma_i(x,t) = 0\}$ while the state is converging to S.

50.2.3 Sliding Modes

The control $u(x,t)$ is designed in such a way that the system state trajectory is attracted to S and, once having intercepted S, remains there for all subsequent time; thus, the state trajectory can be viewed as sliding along S meaning that the system is in a sliding mode. A sliding mode exists if, in the vicinity of the switching surface, S, the tangent or velocity vectors of the state trajectory point toward the switching surface. If the state trajectory intersects the sliding surface, the value of the state trajectory or "representative point" remains within an ε-neighborhood of S. If a sliding mode exists on S, then S, or more commonly $\sigma(x,t) = 0$, is also termed a sliding surface. Note that interception of the surface $\sigma_i(x,t) = 0$ does not guarantee sliding on the surface for all subsequent time as illustrated in Figure 50.2, although this is possible.

An *ideal sliding mode* exists only when the state trajectory $x(t)$ of the controlled plant satisfies $\sigma(x(t),t) = 0$ at every $t \geq t_1$ for some t_1. This may require infinitely fast switching. In real systems, a switched controller has imperfections, such as delay, hysteresis, and so on, which limit switching to a finite frequency. The representative point then oscillates within a neighborhood of the switching surface. This oscillation, called *chattering* (discussed in a later section), is also illustrated in Figure 50.2. If the frequency of the switching is very high relative to the dynamic response of the system, the imperfections and the finite switching frequencies are often but not always negligible. The subsequent development focuses primarily on ideal sliding modes.

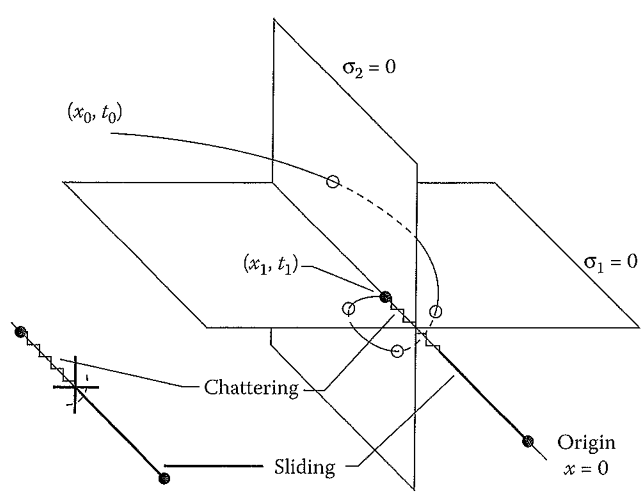

FIGURE 50.2 A situation in which a sliding mode exists on the intersection of the two indicated surfaces for $t \geq t_1$.

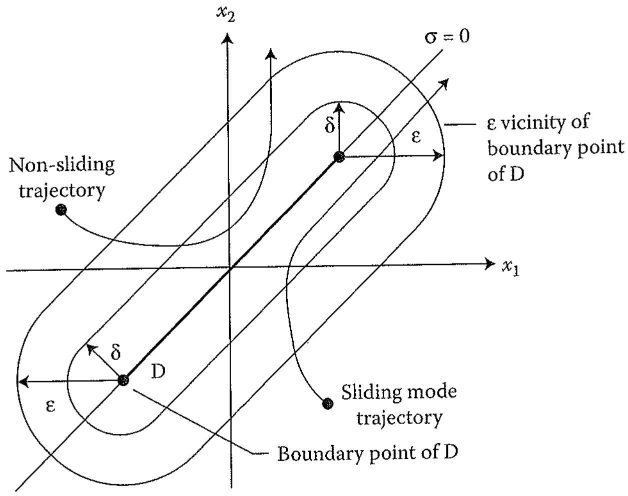

FIGURE 50.3 Two-dimensional illustration of the domain of a sliding mode.

50.2.4 Conditions for the Existence of a Sliding Mode

The existence of a sliding mode [2,5,6] requires stability of the state trajectory to the switching surface $\sigma(x,t) = 0$, that is, after some finite time t_1, the system representative point, $x(t)$, must be in some suitable neighborhood, $\{x| \, \|\sigma(x,t)\| < \varepsilon\}$, of S for suitable $\varepsilon > 0$. A domain, D, of dimension $n - m$ in the manifold, S, is a sliding-mode domain if, for each $\varepsilon > 0$, there is a $\delta > 0$, so that any motion starting within an n-dimensional δ-vicinity of D may leave the n-dimensional ε-vicinity of D only through the n-dimensional ε-vicinity of the boundary of D as illustrated in Figure 50.3.

The *region of attraction* is the largest subset of the state space from which sliding is achievable. A sliding mode is globally reachable if the domain of attraction is the entire state space. The second method of Lyapunov provides the natural setting for a controller design leading to a sliding mode. In this effort one uses a generalized Lyapunov function, $V(t,x,\sigma)$, that is positive definite with a negative time derivative in the region of attraction.

Theorem 50.1: [5, p. 83]:

For the $(n-m)$-dimensional domain D to be the domain of a sliding mode, it is sufficient that in some n-dimensional domain $\Omega \supset D$, a function $V(t,x,\sigma)$ exists, continuously differentiable with respect to all of its arguments and satisfying the following conditions:

1. *$V(t,x,\sigma)$ is positive definite with respect to σ, that is, for arbitrary t and x, $V(t,x,\sigma) > 0$, when $\sigma \neq 0$ and $V(t,x,0) = 0$; on the sphere $||\sigma|| \leq \rho > 0$, for all $x \in \Omega$ and any t, the relationships*

$$\inf_{||\sigma||=\rho} V(t,x,\sigma) = h_\rho, \;\; h_\rho > 0 \;\; \textit{and} \;\; \sup_{||\sigma||=\rho} V(t,x,\sigma) = H_\rho, \;\; H_\rho > 0$$

   hold, where h_ρ and H_ρ depend only on ρ with $h_\rho \neq 0$ if $\rho \neq 0$.
2. *The total time derivative of $V(t,x,\sigma)$ on the trajectories of the system of Equation 50.1 has a negative supremum for all $x \in \Omega$ except for x on the switching surface where the control inputs are undefined and the derivative of $V(t,x,\sigma)$ does not exist.*

In summary, two phases underlie VSC design. The first phase is to construct a switching surface $\sigma(x,t)=0$, so that the system restricted to the surface has a desired global behavior, such as stability, tracking, regulation, and so on. The second phase is to design a (switched) controller $u(x,t)$, so that away from the surface $\sigma(x,t)=0$, the tangent vectors of the state trajectories point toward the surface, that is, there is stability to the switching surface. This second phase is achieved by defining an appropriate Lyapunov function $V(t,x,\sigma)$, differentiating this function so that the control $u(x,t)$ becomes explicit, and adjusting controller gains so that the derivative is negative definite. The choice of $V(t,x,\sigma)$ determines the allowable controller structures. Conversely, a workable control structure has a set of possible Lyapunov functions to verify its viability. A later section discusses general control structures.

50.2.5 An Illustrative Example

To conclude this section, we present an illustrative example for a single pendulum system,

$$\dot{x} = A(x)x + Bu(x) = \begin{bmatrix} 0 & 1 \\ -\dfrac{\sin(x_1)}{x_1} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(x),$$

with a standard feedback control structure, $u(x) = k_1(x)x_1 + k_2(x)x_2$, having nonlinear feedback gains switched according to the rule

$$k_i(x) = \begin{cases} \alpha_i(x), & \text{if } \sigma(x)x_i > 0, \\ \beta_i(x), & \text{if } \sigma(x)x_i < 0, \end{cases}$$

which depend on the linear switching surface $\left(\sigma(x) = [s_1\ s_2]x\right)$. Without loss of generality, assume $s_2 > 0$. For such single-input systems it is ordinarily convenient to choose a Lyapunov function of the form $V(t,x,\sigma) = 0.5\sigma^2(x)$. To determine the gains necessary to drive the system state to the surface $\sigma(x) = 0$, they may be chosen so that

$$\begin{aligned} \dot{V}(t,x,\sigma) &= 0.5\frac{d\sigma^2}{dt} = \sigma(x)\frac{d\sigma(x)}{dt} = \sigma(x)[s_1\ s_2]\dot{x} \\ &= \sigma(x)x_1\left[s_2\left(k_1(x) - \frac{\sin(x_1)}{x_1}\right)\right] + \sigma(x)x_2\,[s_1 + s_2k_2(x)] < 0. \end{aligned}$$

This is satisfied whenever

$$\begin{aligned} \alpha_1(x) = \alpha_1 &< \min_{x_1}\left[\frac{\sin(x_1)}{x_1}\right] = -1, \\ \beta_1(x) = \beta_1 &> \max_{x_1}\left[\frac{\sin(x_1)}{x_1}\right] = 1, \end{aligned}$$

$\alpha_2 < -(s_1/s_2)$ and $\beta_2 > -(s_1/s_2)$. Thus, for properly chosen s_1 and s_2, the controller gains are readily computed.

This example proposed no methodology for choosing s_1 and s_2, that is, for designing the switching surface. Section 50.4 presents this topic. Further, this example was only single input. For the multi-input case, ease of computation of the control gains depends on a properly chosen Lyapunov function. For most cases, a quadratic Lyapunov function is adequate. This topic is discussed in Section 50.5.

50.3 Existence and Uniqueness of Solutions to VSC Systems

VSC produces system dynamics with discontinuous right-hand sides owing to the switching action of the controller. Thus they fail to satisfy conventional existence and uniqueness results of differential

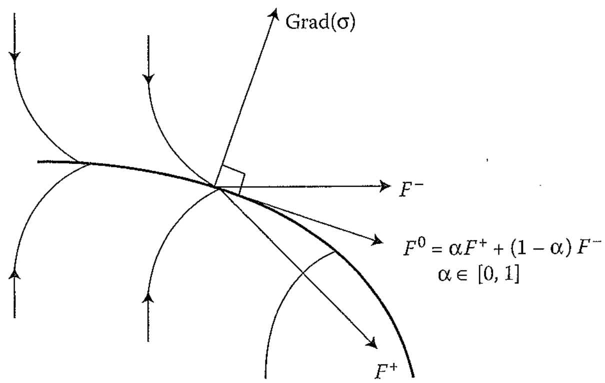

FIGURE 50.4 Illustration of the Filippov method of determining the desired velocity vector F^0 for the motion of the state trajectory on the sliding surface as per Equation 50.6.

equation theory. Nevertheless, an important aspect of VSC is the presumption that the plant behaves uniquely when restricted to $\sigma(x,t)=0$. One of the earliest and conceptually straightforward approaches addressing existence and uniqueness is the method of Filippov [7]. The following briefly reviews this method in the two-dimensional, single-input case.

From Equation 50.1, $\dot{x}(t)=F(x,t,u)$ and the control $u(x,t)$ satisfy Equation 50.5. Filippov's work shows that the state trajectories of Equation 50.1 with control Equation 50.5 on the switching manifold Equation 50.3 solve the equation

$$\dot{x}(t)=\alpha F^{+}+(1-\alpha)F^{-}=F^{0},\quad 0\leq\alpha\leq 1. \tag{50.6}$$

This is illustrated in Figure 50.4, where $F^{+}=F(x,t,u^{+})$, $F^{-}=F(x,t,u^{-})$, and F^0 is the resulting velocity vector of the state trajectory in a sliding mode.

The problem is to determine α, which follows from solving the equation $\langle \text{grad}(\sigma), F^0\rangle = 0$, where the notation $\langle a, b\rangle$ denotes the inner product of a and b, that is,

$$\alpha=\frac{\langle \text{grad}(\sigma), F^{-}\rangle}{\langle \text{grad}(\sigma), (F^{-}-F^{+})\rangle},$$

provided:

1. $\langle \text{grad}(\sigma), (F^{-}-F^{+})\rangle \neq 0$.
2. $\langle \text{grad}(\sigma), F^{+}\rangle \leq 0$.
3. $\langle \text{grad}(\sigma), F^{-}\rangle \geq 0$.

Here, F^0 represents the "average" velocity, $\dot{x}(t)$ of the state trajectory restricted to $\sigma(x,t)=0$. On average, the solution to Equation 50.1 with control Equation 50.5 exists and is uniquely defined on the switching surface S. This technique can also be used to determine the plant behavior in a sliding mode.

50.4 Switching-Surface Design

Switching-surface design is predicated based on the knowledge of the system behavior in a sliding mode. This behavior depends on the parameters of the switching surface. Nonlinear switching surfaces are nontrivial to design. In the linear case, the switching-surface design problem can be converted into an equivalent state feedback design problem. In any case, achieving a switching-surface design requires analytically specifying the motion of the state trajectory in a sliding mode. The so-called method of equivalent control is essential to this specification.

50.4.1 Equivalent Control

Equivalent control constitutes a control input which, when exciting the system, produces the motion of the system on the sliding surface whenever the initial state is on the surface. Suppose at t_1 the plant's state trajectory intercepts the switching surface and a sliding mode exists. The existence of a sliding mode implies that for all $t \geq t_1, \sigma(x(t), t) = 0$, and hence $\dot{\sigma}(x(t), t) = 0$. Using the chain rule, we define the equivalent control u_{eq} for systems of the form of Equation 50.1 as the input, satisfying

$$\dot{\sigma} = \frac{\partial \sigma}{\partial t} + \frac{\partial \sigma}{\partial x}\dot{x} = \frac{\partial \sigma}{\partial t} + \frac{\partial \sigma}{\partial x} f(x,t) + \frac{\partial \sigma}{\partial x} B(x,t) u_{eq} = 0.$$

Assuming that the matrix product $(\partial\sigma/\partial x)B(x,t)$ is nonsingular for all t and x, one can compute u_{eq} as

$$u_{eq} = -\left[\frac{\partial \sigma}{\partial x} B(x,t)\right]^{-1} \left(\frac{\partial \sigma}{\partial t} + \frac{\partial \sigma}{\partial x} f(x,t)\right). \tag{50.7}$$

Therefore, given that $\sigma(x(t_1), t_1) = 0$, then, for all $t \geq t_1$, the dynamics of the system on the switching surface will satisfy

$$\dot{x}(t) = \left[I - B(x,t)\left[\frac{\partial \sigma}{\partial x} B(x,t)\right]^{-1} \frac{\partial \sigma}{\partial x}\right] f(x,t) - B(x,t)\left[\frac{\partial \sigma}{\partial x} B(x,t)\right]^{-1} \frac{\partial \sigma}{\partial t}. \tag{50.8}$$

This equation represents the *equivalent system dynamics* on the sliding surface. The driving term is present when some form of tracking or regulation is required of the controlled system, for example, when

$$\sigma(x,t) = Sx + r(t) = 0$$

with $r(t)$ serving as a "reference" signal [4].

The $(n-m)$-dimensional switching surface, $\sigma(x,t) = 0$, imposes m constraints on the plant dynamics in a sliding mode. Hence, m of the state variables can be eliminated, resulting in an equivalent reduced-order system whose dynamics represent the motion of the state trajectory in a sliding mode. Unfortunately, the structure of Equation 50.8 does not allow convenient exploiting of this fact in switching-surface design. To set forth a clearer switching-surface design algorithm, we first convert the plant dynamics to the so-called regular form.

50.4.2 Regular Form of the Plant Dynamics

The *regular form* of the dynamics of Equation 50.1 is

$$\begin{aligned} \dot{z}_1 &= \hat{f}_1(z,t), \\ \dot{z}_2 &= \hat{f}_2(z,t) + \hat{B}_2(z,t)u(z,t), \end{aligned} \tag{50.9}$$

where $z_1 \in R^{n-m}$, $z_2 \in R^m$. This form can often be constructed through a linear state transformation, $z(t) = Tx(t)$, where T has the property

$$TB(x,t) = TB(T^{-1}z,t) = \begin{bmatrix} 0 \\ \hat{B}_2(z,t) \end{bmatrix},$$

and $\hat{B}_2(z,t)$ is an $(m \times m)$ nonsingular mapping for all t and z. In general, computing the regular form requires the nonlinear transformation,

$$z(t) = T(x,t) = \begin{bmatrix} T_1(x,t) \\ T_2(x,t) \end{bmatrix},$$

where

1. $T(x,t)$ is a diffeomorphic transformation, that is, a continuous differentiable inverse mapping $\tilde{T}(z,t) = x$ exists, satisfying $\tilde{T}(0,t) = 0$ for all t.
2. $T_1(\cdot,\cdot) : R^n \times R \to R^{n-m}$ and $T_2(\cdot,\cdot) : R^n \times R \to R^m$.
3. $T(x,t)$ has the property that

$$\frac{\partial T}{\partial x}B(x,t) = \begin{bmatrix} \dfrac{\partial T_1}{\partial x} \\ \dfrac{\partial T_2}{\partial x} \end{bmatrix} B(\tilde{T}(z,t),t) = \begin{bmatrix} 0 \\ \hat{B}_2(z,t) \end{bmatrix}.$$

This partial differential equation has a solution only if the so-called Frobenius condition is satisfied [8]. The resulting nonlinear regular form of the plant dynamics has the structure,

$$\begin{aligned} \dot{z}_1 &= \frac{\partial T_1}{\partial x} f(\tilde{T}(z,t),t) + \frac{\partial T_1}{\partial t} \triangleq \hat{f}_1(z,t), \\ \dot{z}_2 &= \frac{\partial T_2}{\partial x} f(\tilde{T}(z,t),t) + \frac{\partial T_2}{\partial t} + \frac{\partial T_2}{\partial x} B(\tilde{T}(z,t),t) \\ &\triangleq \hat{f}_2(z,t) + \hat{B}_2(z,t)u. \end{aligned} \tag{50.10}$$

Sometimes all nonlinearities in the plant model can be moved to $\hat{f}_2(z,t)$ so that

$$\dot{z}_1 = \hat{f}_1(z,t) = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \tag{50.11}$$

which solves the sliding-surface design problem with linear techniques (to be shown). If the original system model is linear, the regular form is given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u, \tag{50.12}$$

where $z_1 \in R^{n-m}$ and $z_2 \in R^m$ are as above.

50.4.3 Equivalent System Dynamics via Regular Form

The regular form of the equivalent state dynamics is convenient for analysis and switching-surface design. To simplify the development we make three assumptions: (1) the sliding surface is given in terms of the states of the regular form; (2) the surface has the linear time-varying structure,

$$\sigma(z,t) = Sz + r(t) = [S_1 \ \ S_2] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + r(t) = 0,$$

where the matrix S_2 is chosen to be nonsingular; and (3) the system is in a sliding mode, that is, for some t_1, $\sigma(x(t),t) = 0$ for all $t \geq t_1$. With these three assumptions, one can solve for $z_2(t)$ as

$$z_2(t) = -S_2^{-1}S_1 z_1(t) - S_2^{-1}r(t). \tag{50.13}$$

Substituting Equation 50.13 into the nonlinear regular form of Equation 50.10 yields

$$\dot{z}_1 = \hat{f}_1(z_1,z_2,t) = \hat{f}_1\left(z_1, -S_2^{-1}S_1 z_1 - S_2^{-1}r(t), t\right).$$

The goal then is to choose S_1 and S_2 to achieve a desired behavior of this nonlinear system.

If this system is linear, that is, if Equation 50.11 is satisfied, then, using Equation 50.13, the reduced-order dynamics are

$$\dot{z}_1 = \left(A_{11} - A_{12}S_2^{-1}S_1\right) z_1 - A_{12}S_2^{-1}r(t). \tag{50.14}$$

50.4.4 Analysis of the State Feedback Structure of Reduced-Order Linear Dynamics

The equivalent reduced-order dynamics of Equation 50.14 exhibit a state feedback structure in which $F = S_2^{-1}S_1$ is a state feedback map and A_{12} represents an "input" matrix. Under the conditions that the original (linear) system is controllable, the following well-known theorem applies.

Theorem 50.2: [9]:

If the linear regular form of the state model (Equation 50.12) is controllable, then the pair (A_{11}, A_{12}) of the reduced-order equivalent system of Equation 50.14 is controllable.

This theorem leads to a wealth of switching-surface design mechanisms. First, it permits setting the poles of $A_{11} - A_{12}S_2^{-1}S_1$, for stabilizing the state trajectory to zero when $r(t) = 0$ or to a prescribed rate of tracking, otherwise. Alternatively, one can determine S_1 and S_2 to solve the LQR (linear quadratic regulator) problem when $r(t) = 0$.

As an example, suppose a system has the regular form of Equation 50.12 except that A_{21} and A_{22} are time-varying and nonlinear,

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{array}\right] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] u,$$

where $a_{ij} = a_{ij}(t,x)$ and $a_{ij}^{\min} \le a_{ij}(t,x) \le a_{ij}^{\max}$. Let the switching surface be given by

$$\sigma(z) = [S_1 \;\; S_2] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0.$$

The pertinent matrices of the reduced-order equivalent system matrices are

$$A_{11} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_{12} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

To stabilize the system, suppose that the goal is to find F so that the equivalent system has eigenvalues at {−1,−2,−3}. Using MATLAB®'s Control System's Toolbox yields

$$F = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = S_2^{-1}S_1.$$

Choosing $S_2 = I$ leaves $S_1 = F$. This then specifies the switching-surface matrix $S = [F \quad I]$.

Alternatively, suppose that the objective is to find the control that minimizes the performance index

$$J = \int_0^{\infty} (z_1^T Q z_1 + \hat{u}^T R \hat{u})\, dt,$$

where the lower limit of integration refers to the initiation of sliding. This is associated with the equivalent reduced-order system

$$\dot{z}_1 = A_{11}z_1 - A_{12}\hat{u},$$

where

$$\hat{u} = S_2^{-1} S_1 z_1 \equiv F z_1.$$

Suppose weighting matrices are taken as

$$Q = \begin{bmatrix} 1.0 & 0.5 & 1.0 \\ 0.5 & 2.0 & 1.0 \\ 1.0 & 1.0 & 3.0 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Using MATLAB Control Systems Toolbox, the optimal feedback is

$$F = \begin{bmatrix} 0.6420 & 1.4780 & 0.2230 \\ 0.4190 & 0.4461 & 1.7031 \end{bmatrix}.$$

Again, choosing $S_2 = I$, the switching-surface matrix is given by $S = [F \quad I]$. Here, the poles of the system in sliding are $\{-1.7742, -0.7034 \pm j\,0.2623\}$.

50.5 Controller Design

50.5.1 Stability to Equilibrium Manifold

As mentioned, in VSC a Lyapunov approach is used for deriving conditions on the control $u(x,t)$ that will drive the state trajectory to the equilibrium manifold. Ordinarily, it is sufficient to consider only quadratic Lyapunov function candidates of the form

$$V(t,x,\sigma) = \sigma^T(x,t) W \sigma(x,t), \tag{50.15}$$

where W is a symmetric positive-definite matrix. The control $u(x,t)$ must be chosen so that the time derivative of $V(t,x,\sigma)$ is negative definite for $\sigma \neq 0$. To this end, consider

$$\dot{V}(t,x,\sigma) = \dot{\sigma}^T W \sigma + \sigma^T W \dot{\sigma} = 2\sigma^T W \dot{\sigma}, \tag{50.16}$$

where we have suppressed specific x and t dependencies. Recalling Equation 50.1, it follows that

$$\dot{\sigma} = \frac{\partial \sigma}{\partial t} + \frac{\partial \sigma}{\partial x}\dot{x} = \frac{\partial \sigma}{\partial t} + \frac{\partial \sigma}{\partial x} f + \frac{\partial \sigma}{\partial x} Bu. \tag{50.17}$$

Substituting Equation 50.17 into Equation 50.16 leads to a Lyapunov-like theorem.

Theorem 50.3:

A sufficient condition for the equilibrium manifold (Equation 50.3) to be globally attractive is that the control $u(x,t)$ be chosen so that

$$\dot{V} = 2\sigma^T W \frac{\partial \sigma}{\partial t} + 2\sigma^T W \frac{\partial \sigma}{\partial x} f + 2\sigma^T W \frac{\partial \sigma}{\partial x} Bu < 0 \tag{50.18}$$

for $\sigma \neq 0$, that is, $\dot{V}(t,x,\sigma)$ is negative definite.

Observe that Equation 50.18 is linear in the control. Virtually all control structures for VSC are chosen so that this inequality is satisfied for appropriate W. Some control laws utilize an x- and t-dependent W requiring that the derivation above be generalized.

50.5.2 Various Control Structures

To make the needed control structures more transparent, recall the equivalent control of Equation 50.7,

$$u_{eq}(x,t) = -\left[\frac{\partial\sigma}{\partial x}B(x,t)\right]^{-1}\left(\frac{\partial\sigma}{\partial t} + \frac{\partial\sigma}{\partial x}f(x,t)\right)$$

computed assuming that the matrix product $\frac{\partial\sigma}{\partial x}B(x,t)$ is nonsingular for all t and x. We can now decompose the general control structure as

$$u(x,t) = u_{eq}(x,t) + u_N(x,t), \tag{50.19}$$

where $u_N(x,t)$ is as yet an unspecified substructure. Substituting the above into Equation 50.18 produces the following sufficient condition for stability to the switching surface: Choose $u_N(x,t)$ so that

$$\dot{V}(t,x,\sigma) = 2\sigma^T W \frac{\partial\sigma}{\partial x}B(x,t)u_N(x,t) < 0. \tag{50.20}$$

Because $\frac{\partial\sigma}{\partial x}B(x,t)$ is assumed to be nonsingular for all t and x, it is convenient to set

$$u_N(x,t) = \left[\frac{\partial\sigma}{\partial x}B(x,t)\right]^{-1}\hat{u}_N(x,t). \tag{50.21}$$

Often a switching surface $\sigma(x,t)$ can be designed to achieve a desired system behavior in sliding and, at the same time, to satisfy the constraint $\frac{\partial\sigma}{\partial x}B = I$ in which case $u_N = \hat{u}_N$. Without loss of generality, we make one last simplifying assumption, $W = I$, because $W > 0$, W is nonsingular, and can be compensated for in the control structure. Hence, stability on the surface reduces to finding $\hat{u}_N(x,t)$ such that

$$\dot{V} = 2\sigma^T W\left[\frac{\partial\sigma}{\partial x}B\right]\left[\frac{\partial\sigma}{\partial x}B\right]^{-1}\hat{u}_N = 2\sigma^T\hat{u}_N < 0. \tag{50.22}$$

These simplifications allow us to specify five common controller structures:

1. Relays with constant gains: $\hat{u}_N(x,t)$ is chosen so that

$$\hat{u}_N = \alpha\ \mathrm{sgn}(\sigma(x,t))$$

with $\alpha = [\alpha_{ij}]$ an $m \times m$ matrix, and sgn $(\sigma(x,t))$ is defined componentwise. Stability to the surface is achieved if $\alpha = [\alpha_{ij}]$ is chosen diagonally dominant with negative diagonal entries [5]. Alternatively, if α is chosen to be diagonal with negative diagonal entries, then the control can be represented as

$$\hat{u}_{Ni} = \alpha_{ii}\mathrm{sgn}(\sigma_i(x,t))$$

and, for $\sigma_i \neq 0$,

$$2\sigma_i\hat{u}_{Ni} = 2\alpha_{ii}\sigma_i\mathrm{sgn}(\sigma_i) = 2\alpha_{ii}|\sigma_i| < 0,$$

which guarantees stability to the surface, given the Lyapunov function, $V(t,x,\sigma) = \sigma^T(x,t)\sigma(x,t)$.

2. Relays with state-dependent gains: Each entry of $\hat{u}_N(x,t)$ is chosen so that

$$\hat{u}_{Ni} = \alpha_{ii}(x,t)\mathrm{sgn}(\sigma_i(x,t)), \quad \alpha_{ii}(x,t) < 0.$$

The condition for stability to the surface is that

$$2\sigma_i\hat{u}_N = 2\alpha_{ii}(x,t)\sigma_i\mathrm{sgn}(\sigma_i) = 2\alpha_{ii}(x,t)|\sigma_i| < 0 \quad \text{for } \sigma_i \neq 0.$$

An adequate choice for $\alpha_{ii}(x,t)$ is to choose $\beta_i < 0$, $\gamma_i > 0$, and k a natural number with

$$\alpha_{ii}(x) = \beta_i(\sigma_i^{2k}(x,t) + \gamma_i).$$

3. Linear state feedback with switched gains: Here $\hat{u}_N(x,t)$ is chosen so that

$$\hat{u}_N = \Psi x; \quad \Psi = [\Psi_{ij}]; \quad \Psi_{ij} = \begin{cases} \alpha_{ij} < 0, & \sigma_i x_j > 0, \\ \beta_{ij} > 0, & \sigma_i x_j < 0. \end{cases}$$

   To guarantee stability, it is sufficient to choose α_{ij} and β_{ij} so that

$$\sigma_i \hat{u}_{Ni} = \sigma_i \left(\Psi_{i1}x_1 + \Psi_{i2}x_2 + \cdots + \Psi_{in}x_n\right) = \Psi_{i1}\sigma_i x_1 + \Psi_{i2}\sigma_i x_2 + \cdots + \Psi_{in}\sigma_i x_n < 0.$$

4. Linear continuous feedback: Choose

$$\hat{u}_N = -P\sigma(x,t), \quad P = P^T > 0,$$

   that is, $P \in R^{m\times m}$ is a symmetric positive-definite constant matrix. Stability is achieved because

$$\sigma^T \hat{u}_N = -\sigma^T P \sigma < 0,$$

   where P is often chosen as a diagonal matrix with positive diagonal entries.
5. Univector nonlinearity with scale factor: In this case, choose

$$\hat{u}_N = \begin{cases} \dfrac{\sigma(x,t)}{||\sigma(x,t)||}\rho, & \rho < 0 \quad \text{and} \quad \sigma \neq 0, \\ 0, & \sigma = 0. \end{cases}.$$

   Stability to the surface is guaranteed because, for $\sigma \neq 0$,

$$\sigma^T \hat{u}_N = \frac{\sigma^T\sigma}{||\sigma||}\rho = ||\sigma||\rho < 0.$$

   Of course, it is possible to make ρ time dependent, if necessary, for certain tracking problems. This concludes our discussion of control structures to achieve stability to the sliding surface.

50.6 Design Examples

This section presents two design examples illustrating typical VSC strategies.

Design Example 50.1:

In this example, we illustrate a constant gain relay control with nonlinear sliding surface design for a single-link robotic manipulator driven by a *dc* armature-controlled motor modeled by the normalized (i.e., scaled) simplified equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ \sin(x_1) + x_3 \\ x_2 + x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \equiv f(x) + Bu$$

in the regular form.

To determine the structure of an appropriate sliding surface, recall the assumption that $\frac{\partial\sigma}{\partial x}B(x,t)$ is nonsingular. Because $B = [0\ 0\ 1]^T$, it follows that $\frac{\partial\sigma}{\partial x_3}$ must be nonzero. Without losing generality, we set $\frac{\partial \sigma}{\partial x_3} = 1$. Hence, it is sufficient to consider sliding surfaces of the form

$$\sigma(x) = \sigma(x_1, x_2, x_3) = \sigma_1(x_1, x_2) + x_3 = 0. \tag{50.23}$$

Our design presumes that the reduced-order dynamics have a second-order response represented by the reduced-order state dynamics,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -a_1 x_1 - a_2 x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This form allows us to specify the characteristic polynomial of the dynamics and thus the eigenvalues, that is, $\pi_A(\lambda) = \lambda^2 + a_2\lambda + a_1$. Proper choice of a_1 and a_2 leads to proper rise time, settling time, overshoot, gain margin, and so on.

The switching-surface structure of Equation 50.23 implies that, in a sliding mode,

$$x_3 = -\sigma_1(x_1, x_2).$$

Substituting the above equation into the given system model, the reduced-order dynamics become

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \sin(x_1) - \sigma_1(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_2 \\ -a_1 x_1 - a_2 x_2 \end{bmatrix}.$$

Hence the switching-surface design is completed by setting

$$\sigma_1(x_1, x_2) = \sin(x_1) + a_1 x_1 + a_2 x_2.$$

To complete the controller design, we first compute the equivalent control,

$$u_{eq} = -\begin{bmatrix} \frac{\partial \sigma_1}{\partial x_1} & \frac{\partial \sigma_1}{\partial x_2} & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ \sin(x_1) + x_3 \\ x_2 + x_3 \end{bmatrix}.$$

For the constant gain relay control structure (Equation 50.19),

$$u_N = \alpha \, \mathrm{sgn}(\sigma(x)).$$

Stability to the switching surface results whenever $\alpha < 0$ as

$$\sigma\dot{\sigma} = \alpha\sigma \, \mathrm{sgn}(\sigma) = \alpha|\sigma| < 0.$$

Design Example 50.2:

Consider the fourth-order (linear) model of a mass–spring system that could represent a simplified model of a flexible structure in space with two-dimensional control (Figure 50.5).

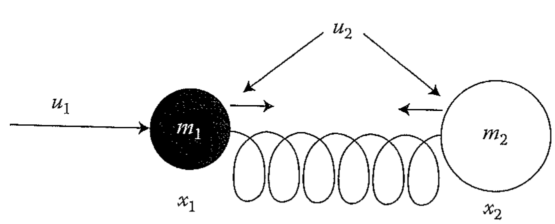

FIGURE 50.5 A mass spring system for design Example 50.2.

Here, x_1 is the position of m_1, x_2 the position of m_2, u_1 the force applied to m_1, and u_2 the force applied between m_1 and m_2. The differential equation model has the form

$$m_1\ddot{x}_1 + k(x_1 - x_2) = u_1 + u_2,$$
$$m_2\ddot{x}_2 + k(x_2 - x_1) = -u_2,$$

where k is the spring constant. Given that $x_3 = \dot{x}_1$ and $x_4 = \dot{x}_2$, the resulting state model in regular form is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ -\dfrac{k}{m_1}x_1 + \dfrac{k}{m_1}x_2 \\ \dfrac{k}{m_2}x_1 - \dfrac{k}{m_2}x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{m_1}u_1 + \dfrac{1}{m_1}u_2 \\ -\dfrac{1}{m_2}u_2 \end{bmatrix}.$$

There are two simultaneous control objectives:

1. Stabilize oscillations, that is, $x_1 = x_2$.
2. Track a desired trajectory, $x_2(t) = z_{ref}(t)$.

These goals are achieved if the following relationships are maintained for $c_1, c_2 > 0$:

$$\dot{x}_1 - \dot{x}_2 + c_1(x_1 - x_2) = 0 \implies x_1 - x_2 \longrightarrow 0$$

and

$$\dot{x}_2 - \dot{z}_{\text{ref}} + c_2(x_2 - z_{\text{ref}}) = 0 \implies x_2 - z_{\text{ref}} \longrightarrow 0.$$

The first step is to determine the appropriate sliding surface. To achieve the first control objective, set

$$\sigma_1(x,t) = x_3 - x_4 + c_1(x_1 - x_2) = 0,$$

and to achieve the desired tracking, set

$$\sigma_2(x,t) = x_4 - \dot{z}_{ref} + c_2(x_2 - z_{ref}) = 0.$$

The next step is to design a VSC law to drive the state trajectory to the intersection of these switching surfaces. In this effort, we illustrate two controller designs. The first is a hierarchical structure [2] so that, for $\sigma \neq 0$,

$$u_1 = \alpha_1 \text{sgn}(\sigma_1),$$
$$u_2 = \alpha_2 \text{sgn}(\sigma_2)$$

with the sign of $\alpha_1, \alpha_2 \neq 0$ to be determined.

For stability to the surface, it is sufficient to have $\sigma_1\dot{\sigma}_1 < 0$ and $\sigma_2\dot{\sigma}_2 < 0$, as can be seen from Equation 50.16, with $W = I$. Observe that

$$\dot{\sigma}_1 = \dot{x}_3 - \dot{x}_4 + c_1(\dot{x}_1 - \dot{x}_2) = \dot{x}_3 - \dot{x}_4 + c_1(x_3 - x_4)$$

and

$$\dot{\sigma}_2 = \dot{x}_4 - \ddot{z}_{ref} + c_2(\dot{x}_2 - \dot{z}_{ref}) = \dot{x}_4 - \ddot{z}_{ref} + c_2(x_4 - \dot{z}_{ref}).$$

Substituting for the derivatives of x_3 and x_4 leads to

$$\begin{bmatrix} \dot{\sigma}_1 \\ \dot{\sigma}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{m_1} & \dfrac{1}{m_1} + \dfrac{1}{m_2} \\ 0 & -\dfrac{1}{m_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{m_1} & \dfrac{1}{m_1} + \dfrac{1}{m_2} \\ 0 & -\dfrac{1}{m_2} \end{bmatrix} \begin{bmatrix} \alpha_1 \text{sgn}(\sigma_1) \\ \alpha_2 \text{sgn}(\sigma_2) \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \quad (50.24)$$

where

$$h_1 = -\frac{k}{m_1}x_1 + \frac{k}{m_1}x_2 - \frac{k}{m_2}x_1 + \frac{k}{m_2}x_2 + c_1x_3 - c_1x_4$$

and

$$h_2 = \frac{k}{m_2}x_1 - \frac{k}{m_2}x_2 - \ddot{z}_{\text{ref}} + c_2x_4 - c_2\dot{z}_{\text{ref}}.$$

Taking a brute force approach to the computation of the control gains, stability to the switching surface is achieved, provided

$$\sigma_2\dot{\sigma}_2 = \frac{-\alpha_2}{m_2}\sigma_2\text{sgn}(\sigma_2) + \sigma_2 h_2 < 0,$$

which is satisfied if

$$\alpha_2 > m_2|h_2|(> 0),$$

and provided

$$\sigma_1\dot{\sigma}_1 = \frac{\alpha_1}{m_1}\sigma_1\text{sgn}(\sigma_1) + \sigma_1\left[\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\alpha_2\text{sgn}(\sigma_2) + h_1\right] < 0$$

which is satisfied if

$$\alpha_1 < -m_1|h_1| - \left(1 + \frac{m_1}{m_2}\right)\alpha_2.$$

In a second controller design, we recall Equation 50.24. For $\sigma_1 \neq 0$ and $\sigma_2 \neq 0$, it is convenient to define the controller as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{m_1} & \frac{1}{m_1} + \frac{1}{m_2} \\ 0 & -\frac{1}{m_2} \end{bmatrix}^{-1} \begin{bmatrix} \beta_1\text{sgn}(\sigma_1) \\ \beta_2\text{sgn}(\sigma_2) \end{bmatrix},$$

where β_1 and β_2 are to be determined. It follows that

$$\begin{bmatrix} \dot{\sigma}_1 \\ \dot{\sigma}_2 \end{bmatrix} = \begin{bmatrix} \beta_1\text{sgn}(\sigma_1) \\ \beta_2\text{sgn}(\sigma_2) \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}.$$

As in the first controller design, the state trajectory will intercept the sliding surface in finite time and sliding will occur for β_1 and β_2 sufficiently negative, thereby achieving the desired control objective.

A unifying characterization of sliding mode controllers that drive a trajectory to the sliding manifold is given in [10].

50.7 Chattering

The VSC controllers developed earlier assure the desired behavior of the closed-loop system. These controllers, however, require an infinitely (in the ideal case) fast switching mechanism. The phenomenon of nonideal but fast switching was labeled as chattering (actually, the word stems from the noise generated by the switching element). The high-frequency components of the chattering are undesirable because they may excite unmodeled high-frequency plant dynamics resulting in unforeseen instabilities. To reduce chatter, define a so-called boundary layer as

$$\{x|\ \|\sigma(x)\| \leq \varepsilon, \varepsilon > 0\}, \tag{50.25}$$

whose thickness is 2ε. Now, modify the control law of Equation 50.26 (suppressing t and x arguments) to

$$u = \begin{cases} u_{eq} + u_N, & \|\sigma\| \geq \varepsilon, \\ u_{eq} + p(\sigma, x), & \|\sigma\| \leq \varepsilon, \end{cases} \tag{50.26}$$

where $p(\sigma, x)$ is any continuous function satisfying $p(0, x) = 0$ and $p(\sigma, x) = u_N(x)$ when $\|\sigma(x)\| = \varepsilon$. This control guarantees that trajectories are attracted to the boundary layer. Inside the boundary layer, Equation 50.26 offers a continuous approximation to the usual discontinuous control action. Similar to Corless and Leitmann [11], asymptotic stability is not guaranteed but ultimate boundedness of trajectories to within an ε-dependent neighborhood of the origin is assured.

50.8 Robustness to Matched Disturbances and Parameter Variations

To explore the robustness of VSC to disturbances and parameter variations, one modifies Equation 50.1 to

$$\dot{x}(t) = [f(x,t) + \Delta f(x,t,q(t))] + [B(x,t) + \Delta B(x,t,q(t))]u(x,t) + d(t), \tag{50.27}$$

where $q(t)$ is a vector function representing parameter uncertainties, Δf and ΔB represent the cumulative effects of all plant uncertainties, and $d(t)$ denotes an external (deterministic) disturbance. The first critical assumption in our development is that all uncertainties and external disturbances satisfy the so-called *matching condition*, that is, Δf, ΔB, and $d(t)$ lie in the image of $B(x,t)$ for all x and t. As such they can all be lumped into a single vector function $\xi(x,t,q,d,u)$, so that

$$\dot{x}(t) = f(x,t) + B(x,t)u(x,t) + B(x,t)\xi(x,t,q,d,u). \tag{50.28}$$

The second critical assumption is that a positive continuous bounded function $\rho(x,t)$ exists, satisfying

$$||\xi(x,t,q,d,u)|| \leq \rho(x,t). \tag{50.29}$$

To incorporate robustness into a VSC design, we utilize the control structure of Equation 50.19, $u(x,t) = u_{eq}(x,t) + u_N(x,t)$, where $u_{eq}(x,t)$ is given by Equation 50.7. Given the plant and disturbance model of Equation 50.28, then, as per Equation 50.20, it is necessary to choose $u_N(x,t)$, so that

$$\dot{V}(t,x,\sigma) = 2\sigma^T W \frac{\partial \sigma}{\partial x} B(x,t)\left[u_N(x,t) + \xi(x,t,q,d,u)\right] < 0.$$

Choosing any one of the control structures outlined in Section 50.5, a choice of sufficiently "high" gains will produce a negative-definite $\dot{V}(t,x,\sigma)$. Alternatively, one can use a control structure [2],

$$u_N(x,t) = \begin{cases} -\dfrac{B^T\left[\frac{\partial\sigma}{\partial x}\right]^T\sigma}{\left\|B^T\left[\frac{\partial\sigma}{\partial x}\right]^T\sigma\right\|}[\rho(x,t) + \alpha(x,t)] & \text{for } \sigma(x,t) \neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{50.30}$$

where $\alpha(x,t)$ is to be determined. Assuming $W = I$, it follows that, for $\sigma \neq 0$,

$$\begin{aligned} \dot{V}(t,x,\sigma) &= -2\sigma^T \frac{\frac{\partial\sigma}{\partial x}BB^T\left[\frac{\partial\sigma}{\partial x}\right]^T\sigma}{\left\|B^T\left[\frac{\partial\sigma}{\partial x}\right]^T\sigma\right\|} \times [\rho(x,t) + \alpha(x,t)] + 2\sigma^T\frac{\partial\sigma}{\partial x}B\xi \\ &\leq -2\left\|B^T\left[\frac{\partial\sigma}{\partial x}\right]^T\sigma\right\|\alpha(x,t). \end{aligned}$$

Choosing $\alpha(x,t) = \alpha > 0$ leads to the stability of the state trajectory to the equilibrium manifold despite matched disturbances and parameter variations, demonstrating the robustness property of a VSC law.

50.9 Observer Design

"Observers" can be viewed as software algorithms that allow online estimation of the current state of a dynamic system when only the output and the input of the system can be measured. In the case of a linear system, we have

$$\dot{x} = Ax + Bu + B\xi,$$
$$y = Cx, \tag{50.31}$$

where $C \in R^{p\times n}$, and we assume that the pair (C, A) is observable. The observer design problem is to construct a dynamic system that estimates the system state based on knowledge of the input and the output measurement. This results in the so-called Luenberger observer when $\xi = 0$,

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}), \tag{50.32}$$

The estimation error $e(t) = x(t) - \hat{x}(t)$ satisfies: $\dot{e}(t) = (A - LC)e(t)$. Since (C, A) is observable, the eigenvalues of $A - LC$ can be assigned arbitrarily by a choice of the gain matrix L, although in practice this is limited by the bandwidth of the system.

The sliding mode concept can be used for designing an observer by replacing $L(y - C\hat{x})$ in Equation 50.32 with a discontinuous function $E_d(y, \hat{x})$ of and $\hat{x}$ yielding

$$\dot{\hat{x}} = A\hat{x} + Bu + E_d(y, \hat{y}),$$
$$\hat{y} = C\hat{x}, \tag{50.33}$$

where E_d is a user-chosen function to insure convergence in the presence of uncertainties modeled by nonzero ξ in Equation 50.31.

One possibility is to choose $E_d(y, \hat{y}) = L(y - C\hat{x}) + BE(y, \hat{y})$, where L is chosen so that A-LC is a stability matrix (eigenvalues in the open left-half-complex plane) and

$$E(y, \hat{y}) = \eta \frac{F(y - \hat{y})}{\|F(y - \hat{y})\|}, \tag{50.34}$$

where η is a design parameter satisfying $\eta > ||\xi||$. Now, $L, F \in R^{m\times p}$, $p \geq$ m, and a matrix $P = P^T > 0$ must simultaneously satisfy:

1. $\text{eig}(A - LC) \subset C^-$
2. $FC = B^T P$, and
3. $(A - LC)^T P + P(A - LC) = -Q$

for an appropriate $Q = Q^T > 0$, if it exists. A solution for (L, F, P) exists if and only if

1. $\text{rank}(B) = \text{rank}(CB) = r$ and
2. $\text{rank}\begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} = n + r, \text{Re}[s] \geq 0$

With the estimation error as $e(t) = x(t) - \hat{x}(t)$, the error dynamics become

$$\dot{e}(t) = (A - LC)e(t) - BE(y, \hat{y}) + B\xi. \tag{50.35}$$

It follows that

$$\frac{d}{dt}(e^T Pe) = -e^T Qe - 2\eta\, \|FCe\| + 2e^T PB\xi$$
$$\leq -e^T Qe - 2\eta\, \|FCe\| + 2\, \|FCe\|\, \|\xi\| \leq -e^T Qe,$$

which implies $\lim_{t\to\infty} e(t) = 0$. For further analysis see [3,12,13]. For an alternate sliding mode observer structure, see [14–16].

50.9.1 Observer Design 2 [17,18]

Now consider $E_d(y, \hat{x}) = L\,\text{sign}(y - C\hat{x})$ resulting in the observer dynamics

$$\dot{\hat{x}} = A\hat{x} + Bu + L\,\text{sign}(y - C\hat{x}). \tag{50.36}$$

For the deterministic case ($\xi = 0$) the observation error satisfies $\dot{e} = Ae - L\,\text{sgn}(Ce)$. For such a system, a sliding mode is possible on the manifold $Ce = 0$. In order to describe the choice of the observer gain L and analyze the error dynamics let us consider a nonsingular transformation of the state x into a new set of coordinates such that the first p coordinates correspond to the observed vector y:

$$\begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} C \\ M \end{bmatrix} x.$$

The transformed plant dynamics are

$$\begin{bmatrix} \dot{y} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u. \tag{50.37}$$

The observer in the new coordinates is

$$\dot{\hat{y}} = A_{11}\hat{y} + A_{12}\hat{w} + B_1 u + L_1\,\text{sgn}(y - \hat{y}), \tag{50.38a}$$

$$\dot{\hat{w}} = A_{21}\hat{y} + A_{22}\hat{w} + B_2 u + L_2\,\text{sgn}(y - \hat{y}). \tag{50.38b}$$

Denoting $e_1(t) = y(t) - \hat{y}(t)$ and $e_2(t) = w(t) - \hat{w}(t)$, the error dynamics for the first subsystem is

$$\dot{e}_1 = A_{11}e_1 + A_{12}e_2 - L_1\,\text{sgn}(e_1), \tag{50.39a}$$

$$\dot{e}_2 = A_{21}e_1 + A_{22}e_2 - L_2\,\text{sgn}(e_1). \tag{50.39b}$$

By choosing an appropriate nonsingular gain matrix L_1 (it is a square matrix), we can enforce sliding regime in the first equation along the manifold $e_1(t) = 0$. Indeed, the equivalent control is obtained from Equation 50.39a under the condition, $\dot{e}_1(t) = 0$ as

$$\left[\text{sgn}(e_1)\right]_{eq} = L_1^{-1}A_{12}e_2. \tag{50.40}$$

The dynamics of the system in a sliding mode ($\dot{e}_1 = 0$) can be obtained by substituting this value into Equation 50.39b, to obtain the linear equation

$$\dot{e}_2 = (A_{22} - L_2L_1^{-1}A_{12})e_2. \tag{50.41}$$

Let us note that the observability of the original pair (C, A) implies observability of the pair (A_{12}, A_{22}) in the system (Equation 50.37). Using this fact it follows that we can assign any eigenvalues in this system by appropriate choice of L_2; thus, guaranteeing convergence $e_2(t) \to 0$ with any desired exponential rate. The dimension of the system or Equation 50.41 is n–p. The case when the output is corrupted by measurement noise was also considered in [18]. Similar observer structures and explanations can be found in [6,14–16]. An application of such an observer structure to state estimation of a magnetic bearing is considered in [19].

In [20], Drakunov proposed a sliding mode observer structure

$$\dot{\hat{x}} = \left[\frac{\partial H(\hat{x})}{\partial x}\right]^{-1} M(\hat{x}, t)\,\text{sgn}[V - H(\hat{x})] \tag{50.42}$$

that can be used for a nonlinear system of the form

$$\dot{x} = f(x),$$
$$y = h(x),$$

where the measurement map $h: R^n \to R$ is a scalar and where $H(x) = [h_1(x)\; h_2(x)\; \cdots\; h_n(x)]^T$ has entries defined using repeated Lie derivatives: $h_1(x) = h(x)$, $h_2(x) = L_f h(x)$, $h_3(x) =$

$L_f^2h(x), \ldots, h_n(x) = L_f^{n-1}h(x)$; $M(\hat{x}, t) = diag(m_1(\hat{x}, t), \ldots, m_n(\hat{x}, t))$ is a diagonal gain matrix and the vector $V = [v_1\ v_2\ \cdots\ v_n]^T$ has components defined recursively: $v_1(t) = y(t)$, $v_{i+1}(t) = \left[m_i(\hat{x}, t)\text{sgn}\left(v_i(t) - h_i(\hat{x})\right)\right]_{eq}$. The equivalent values can be obtained using an equivalent control filter such as a low-pass filter, although a first-order low-pass filter may not be sufficient; more complicated even nonlinear digital filters may need to be employed.

Example 50.3:

To illustrate the above nonlinear observer design, consider the nonlinear state model

$$\dot{x}_1 = (1 - 2x_1 + 2x_2^2)x_2,$$
$$\dot{x}_2 = -x_1 + x_2^2$$

with the output $y = x_1 - x_2^2$. In this case, we have $h(x) = h_1(x_1, x_2) = x_1 - x_2^2$, and since $n = 2$ we need only the first Lie derivative: $h_2(x_1, x_2) = L_f h(x) = x_2$. Therefore, the corresponding map H and its Jacobian matrix are

$$H(x) = \begin{bmatrix} x_1 - x_2^2 \\ x_2 \end{bmatrix}, \quad \frac{\partial H}{\partial x} = \begin{bmatrix} 1 & -2x_2 \\ 0 & 1 \end{bmatrix} \Rightarrow \left(\frac{\partial H}{\partial x}\right)^{-1} = \begin{bmatrix} 1 & 2x_2 \\ 0 & 1 \end{bmatrix}.$$

The observer of Equation 50.42 is

$$\dot{\hat{x}}_1 = m_1\ \text{sgn}(y - \hat{x}_1 + \hat{x}_2^2) + 2m_2\hat{x}_2\ \text{sgn}(v - \hat{x}_2),$$
$$\dot{\hat{x}}_2 = m_2\ \text{sgn}(v - \hat{x}_2),$$

where $v = \{m_1\ \text{sgn}(y - \hat{x}_1 + \hat{x}_2^2)\}_{eq}$. The second-order observer converges as long as the observer gains are sufficiently large, which means that $m_1 \geq |x_2|, m_2 \geq \left|x_1 - x_2^2\right|$. If the region of initial conditions and system trajectories are bounded, then the gains can be chosen to be constant. In general, the gains depend on $(\hat{x}_1, \hat{x}_2)$. The equivalent value operator $\{\cdots\}_{eq}$ can be implemented in different

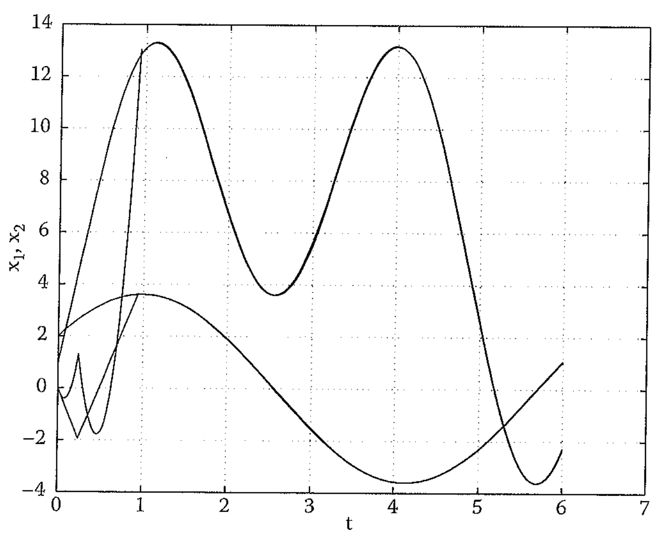

FIGURE 50.6 Nonlinear observer convergence.

ways, the easiest of which is a first-order low-pass filter $\dot{v} = -\lambda v + \lambda m_1 \operatorname{sgn}(y - \hat{x}_1 + \hat{x}_2^2)$ for an appropriate value of $\lambda > 0$. The resulting solution to the filtering equation is $v = \{m_1 \operatorname{sgn}(y - \hat{x}_1 + \hat{x}_2^2)\}_{eq}$. The simulation results are shown in Figure 50.6.

For further work on VSC systems and sliding mode observers we refer the reader to [16,20–22].

50.10 Concluding Remarks

This chapter has summarized the salient results of sliding mode control theory and illustrated the design procedures with various examples. A wealth of literature exists on the subject that cannot be included because of space limitations. In particular, the literature is replete with realistic applications [1,23], extensions to output feedback [24], extensions to decentralized control [25], and extensions to discrete-time systems. Additionally, there is some work, old and new, on higher-order sliding modes [16] and [26]. For extensions of the above methods to time delay systems, see [18,27,29]. The reader is encouraged to search the literature for many papers in this area.

50.11 Defining Terms

Chattering: The phenomenon of nonideal but fast switching. The term stems from the noise generated by a switching element.

Equilibrium (discontinuity) manifold: A specified, user-chosen manifold in the state space to which a system's trajectory is driven and maintained for all time subsequent to intersection of the manifold by a discontinuous control that is a function of the system's states, and hence, discontinuity manifold. Other terms commonly used are sliding surface and switching surface.

Equivalent control: The solution to the algebraic equation involving the derivative of the equation of the switching surface and the plant's dynamic model. The equivalent control is used to determine the system's dynamics on the sliding surface.

Equivalent system dynamics: The system dynamics obtained after substituting the equivalent control into the plant's dynamic model. It characterizes state motion parallel to the sliding surface if the system's initial state is off the surface and state motion is on the sliding surface if the initial state is on the surface.

Ideal sliding mode: Motion of a system's state trajectory along a switching surface when switching in the control law is infinitely fast.

Matching condition: The condition requiring the plant's uncertainties to lie in the image of the input matrix, that is, the uncertainties can affect the plant dynamics only through the same channels as the plant's input.

Region of attraction: A set of initial states in the state space from which sliding is achievable.

Regular form: A particular form of the state-space description of a dynamic system obtained by a suitable transformation of the system's state variables.

Sliding surface: See equilibrium manifold.

Switching surface: See equilibrium manifold.

References

1. Sira-Ramirez, H., Nonlinear P-I controller design for switchmode dc-to-dc power converters, *IEEE Trans. Circuits Systems*, 38(4), 410–417, 1991.

2. DeCarlo, R.A., Żak, S.H., and Matthews, G.P., Variable structure control of nonlinear multivariable systems: A tutorial, *Proc. IEEE*, 76(3), 212–232, 1988.
3. Hui, S. and Żak, S.H., Robust control synthesis for uncertain/nonlinear dynamical systems, *Automatica*, 28(2), 289–298, 1992.
4. Matthews, G.P. and DeCarlo, R.A., Decentralized tracking for a class of interconnected nonlinear systems using variable structure control, *Automatica*, 24(2), 187–193, 1988.
5. Utkin, V.I., *Sliding Modes and Their Application in Variable Structure Control*, Mir, Moscow, 1978.
6. Utkin, V.I., *Sliding Modes in Control and Optimization*, Springer, Berlin, 1992.
7. Filippov, A.F., *Differential Equations with Discontinuous Righthand Sides*, Kluwer Academic, Dordrecht, The Netherlands, 1988.
8. Hunt, L.R., Su, R., and Meyer, G., Global transformations of nonlinear systems, *IEEE Trans. Automat. Control*, AC-28(1), 24–31, 1983.
9. Young, K.-K.D., Kokotović, P.V., and Utkin, V.I., A singular perturbation analysis of high-gain feedback systems, *IEEE Trans. Automat. Control*, AC-22(6), 931–938, 1977.
10. DeCarlo, R.A., Drakunov, S., and Li, Xiaoqui, A unifying characterization of sliding mode control: A Lyapunov approach, *ASME J. Dynamic Systems, Measurement, Control*, special issue on Variable Structure Systems, 122(4), 708–718, 2000.
11. Corless, M.J. and Leitmann, G., Continuous state feedback guaranteeing uniform ultimate boundedness for uncertain dynamic systems, *IEEE Trans. Automat. Control*, AC-26(5), 1139–1144, 1981.
12. Edwards, C. and Spurgeon, S.K., On the development of discontinuous observers, *Int. J. Control*, 59(5), 1211–1229, 1994.
13. Hui, S. and Żak, S.H., Observer design for systems with unknown inputs, *Int. J. Appl. Math. Comput. Sci.*, 15(4), 431–446, 2005.
14. Perruquetti, W. and Barbot, J.P., *Sliding Mode Control in Engineering*, Marcel Dekker, New York, 2002.
15. Edwards, C. and Spurgeon, S.K., *Sliding Mode Control: Theory and Applications*, Taylor & Francis, London, UK, 1998.
16. Sabanovic, A., Fridman, L., and Spurgeon S. (Eds), *Variable Structure Systems: From Principles to Implementation*, IEE Control Engineering Series, The Institute of Electrical Engineers, UK, 2004.
17. Drakunov, S.V., Izosimov, D.B., Lukyanov, A. G., Utkin, V. A., and Utkin, V.I., The block control principle, Part II, *Autom. Remote Control*, 51(6), 737–746, 1990.
18. Drakunov, S. V., Adaptive quasioptimal filters with discontinuous parameters, *Autom. Remote Control*, 44(9), 1167–1175, 1984.
19. Rundell, A., Drakunov, S., and DeCarlo, R., A sliding mode observer and controller for stabilization of rotational motion of a vertical shaft magnetic bearing, *IEEE Trans. Control Systems Technol.*, 4(5), 598–608, 1996.
20. Drakunov, S.V., Sliding-mode observers based on equivalent control method, *Proc. 31st IEEE Conf. Decision Control*, Tucson, AZ, pp. 2368–2369, 1992.
21. Barbot, J.P., Boukhobza, T., and Djemai, M., Sliding mode observer for triangular input form. *Proc. 35th IEEE* CDC, Kobe, Japan, 1996.
22. Fridman, L., Levant, A., and Davila, J., Observation of linear systems with unknown inputs via high-order sliding modes, *Int. J. Syst. Sci.*, 38(10), 773–791, 2007.
23. Utkin, V., Guldner, J., and Shi, J., *Sliding Model Control in Electromechanical Systems*, Taylor & Francis, London, 1999.
24. El-Khazali, R. and DeCarlo, R.A., Output feedback variable structure control design, *Automatica*, 31(5), 805–816, 1995.
25. Matthews, G. and DeCarlo, R.A., Decentralized variable structure control of interconnected multi-input/multi-output nonlinear systems, *Circuits Syst. Signal Process.*, 6(3), 363–387, 1987.
26. Emelyanov, S.V., Korovin, S.K., and Levantovsky, L.V., Second-order sliding modes in controlling uncertain systems, *Sov. J. Comput. Syst. Sci.*, 24(4), 63–68, 1986.
27. Li, X. and DeCarlo, R.A., Robust sliding mode control of uncertain time-delay systems, *Int. J. Control*, 76(13), 1296–1305, 2003.
28. Walcott, B.L. and Żak, S.H., State observation of nonlinear/uncertain dynamical systems, *IEEE Trans. Automat. Control*, AC-32(2), 166–170, 1987.
29. Bengea, S., Li, X., and DeCarlo, R.A., Combined controller–observer design for time delay systems with application to engine idle speed control, *ASME J. Dyn. Sys., Meas. Control*, 126, 2004.