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Brief paper

Sliding-mode observers for systems with unknown inputs: A high-gain approach[☆]

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ABSTRACT

Sliding-mode observers can be constructed for systems with unknown inputs if the so-called observer matching condition is satisfied. However, most systems do not satisfy this condition. To construct sliding-mode observers for systems that do not satisfy the observer matching condition, auxiliary outputs are generated using high-gain approximate differentiators and then employed in the design of sliding-mode observers. The state estimation error of the proposed high-gain approximate differentiator based sliding-mode observer is shown to be uniformly ultimately bounded with respect to a ball whose radius is a function of design parameters. Finally, the unknown input reconstruction using the proposed observer is analyzed and then illustrated with a numerical example.

1. Introduction

Observers are dynamic systems that can be used to estimate the state of a plant using its input-output measurements; they were first proposed by Luenberger (1966). In some cases, the inputs to the plant are unknown or partially known, which led to the development of the so-called unknown input observer (UIO). Examples of linear UIO architectures for linear systems are analyzed in Chen and Patton (1999), Corless and Tu (1998), Darouach, Zasadzinski and Xu (1994), Hou and Müller (1992) and Hui and Żak (2005). Motivated by the design of sliding-mode controllers, sliding-mode based UIOs have been developed; see, for example, Edwards and Spurgeon (1998), Edwards, Spurgeon and Patton (2000), Hui and Żak (2005), Walcott and Żak (1987), and Żak and Walcott (1990). The main advantage of using sliding-mode observers over their linear counterparts is that, while in sliding, they are insensitive to the matched unknown inputs. Moreover, they can be used to reconstruct unknown inputs which could be a combination of system disturbances, faults or non-linearities. The reconstruction of unknown inputs has found useful applications in fault detection and isolation (Chen & Patton, 1999; Edwards & Spurgeon, 1998; Edwards et al., 2000).

For most of the linear and non-linear unknown input observers proposed thus far, the necessary and sufficient conditions for the construction of such observers is that the invariant zeros of the system must lie in the open left half complex plane, and the observer matching condition is satisfied. However, the second condition seriously limits the applicability of this technique. Recently, high-order sliding-mode based unknown input observers (Floquet & Barbo, 2006; Floquet, Edwards & Spurgeon, 2007) have been developed for systems that do not satisfy the observer matching condition. In Floquet and Barbo (2006), a change of coordinates is used to transform the system into a quasi-block triangular observable form. Then, a second-order sliding-mode observer is constructed for the transformed system. In Floquet et al. (2007), auxiliary outputs are defined such that the conventional unknown input sliding-mode observer proposed in Edwards and Spurgeon (1998) can be developed for systems without the observer matching condition. In order to obtain those auxiliary outputs, high-order sliding-mode observers which act as exact differentiators are constructed using the super-twisting algorithm.

In this paper, we adopt the idea of generating auxiliary outputs from Floquet et al. (2007). High-gain observers are then used as approximate differentiators (Khalil, 1999) to obtain the estimates of these auxiliary outputs. The auxiliary outputs generated by high-gain observers are then used to construct the sliding-mode observer which was first introduced in Walcott and Żak (1987) and later modified for a more general class of systems in Hui and Żak (2005). The major advantage of the proposed high-gain approximate differentiator-based sliding-mode observer is the simplicity

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of the overall observer architecture. The main contribution of this paper is the application of the sliding-mode observer presented in Walcott and Żak (1987) to the state observation for linear systems without the observer matching condition being satisfied. It is also the first time that the high-gain observer is used in such an application.

The remainder of this paper is organized as follows. The system description and the problem statement are given in Section 2. In Section 3, the high-gain observer is first described. Then the high-gain approximate differentiator based sliding-mode observer is proposed and analyzed in Section 4. In Section 5, the application of the proposed sliding-mode observer to reconstruct unknown inputs is discussed. Simulation results are included in Section 6, and conclusions are found in Section 7.

2. System description and problem statement

We consider a class of linear time-invariant systems with unknown inputs,

$$\left.\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{B}_2\boldsymbol{u}_2\\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x},\end{aligned}\right\} \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{y} \in \mathbb{R}^p$, $\boldsymbol{u}_1 \in \mathbb{R}^{m_1}$ and $\boldsymbol{u}_2 \in \mathbb{R}^{m_2}$ are the state, output, known and unknown input vectors with $m_2 \le p$, and $\boldsymbol{B}_2$ and $\boldsymbol{C}$ are of full rank. We assume that $\|\boldsymbol{u}_2\| \le \rho$ for $\rho > 0$. We also assume that the invariant zeros of the system model given by the triple $(\boldsymbol{A}, \boldsymbol{B}_2, \boldsymbol{C})$ are in the open left-hand complex plane, that is,

$$\operatorname{rank}\begin{bmatrix} s\boldsymbol{I}_n - \boldsymbol{A} & \boldsymbol{B}_2\\ \boldsymbol{C} & \boldsymbol{O}_{p\times m_2}\end{bmatrix} = n + m_2 \tag{2}$$

for all s such that $\Re(s) \ge 0$. It follows from Walcott and Żak (1987) that if the so-called observer matching condition is also satisfied for the system modeled by (1), that is,

$$\operatorname{rank}\boldsymbol{B}_2 = \operatorname{rank}(\boldsymbol{C}\boldsymbol{B}_2) = m_2, \tag{3}$$

then we can construct the following Walcott–Żak sliding-mode observer,

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{L}\left(\boldsymbol{y} - \hat{\boldsymbol{y}}\right) + \boldsymbol{B}_2\boldsymbol{E} \tag{4}$$

with $\hat{\boldsymbol{y}} = \boldsymbol{C}\hat{\boldsymbol{x}}$ and

$$\boldsymbol{E} = \begin{cases}\eta\dfrac{\boldsymbol{F}\left(\boldsymbol{y} - \hat{\boldsymbol{y}}\right)}{\left\|\boldsymbol{F}\left(\boldsymbol{y} - \hat{\boldsymbol{y}}\right)\right\|} & \text{if } \boldsymbol{F}\left(\boldsymbol{y} - \hat{\boldsymbol{y}}\right) \ne \boldsymbol{0}\\ \boldsymbol{0} & \text{if } \boldsymbol{F}\left(\boldsymbol{y} - \hat{\boldsymbol{y}}\right) = \boldsymbol{0},\end{cases} \tag{5}$$

where η is a positive design parameter, and $\boldsymbol{L}$ and $\boldsymbol{F}$ are matrices such that $(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})^\top\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C}) = -2\boldsymbol{Q}$ and $\boldsymbol{F}\boldsymbol{C} = \boldsymbol{B}_2^\top\boldsymbol{P}$ for some symmetric positive definite $\boldsymbol{P}$ and $\boldsymbol{Q}$. A design algorithm for the above observer can be found in Hui and Żak (2005).

For many physical systems modeled by (1), the observer matching condition (3) is not satisfied. To overcome the restriction imposed by this condition, the method of using auxiliary outputs to construct the sliding-mode observer (Edwards and Spurgeon (1998)) has been proposed in Floquet et al. (2007). Let $\boldsymbol{c}_i$ be the i-th row of the output matrix $\boldsymbol{C}$. Recall that the relative degree of the i-th output y_i with respect to the unknown input $\boldsymbol{u}_2$ is defined to be the smallest positive integer r_i such that

$$\boldsymbol{c}_i\boldsymbol{A}^k\boldsymbol{B}_2 = \boldsymbol{0}, \quad \text{for } k = 0, \dots, r_i - 2$$
$$\boldsymbol{c}_i\boldsymbol{A}^{r_i-1}\boldsymbol{B}_2 \ne \boldsymbol{0}.$$

We can choose integers γ_i $(1 \le \gamma_i \le r_i)$ such that

$$\boldsymbol{C}_a = \begin{bmatrix}\boldsymbol{c}_1\\ \vdots\\ \boldsymbol{c}_1\boldsymbol{A}^{\gamma_1-1}\\ \vdots\\ \boldsymbol{c}_p\\ \vdots\\ \boldsymbol{c}_p\boldsymbol{A}^{\gamma_p-1}\end{bmatrix}$$

is of full rank with $\operatorname{rank}(\boldsymbol{C}_a\boldsymbol{B}_2) = \operatorname{rank}\boldsymbol{B}_2$. It is proved in Floquet et al. (2007) that the system zeros of the system model given by the triple $(\boldsymbol{A}, \boldsymbol{B}_2, \boldsymbol{C}_a)$ are in the open left-hand complex plane if the triple $(\boldsymbol{A}, \boldsymbol{B}_2, \boldsymbol{C})$ satisfies (2). Thus, we can construct the sliding-mode observer of the form (4) for the following system model,

$$\left.\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{B}_2\boldsymbol{u}_2\\ \boldsymbol{y}_a &= \boldsymbol{C}_a\boldsymbol{x},\end{aligned}\right\}$$

if the output $\boldsymbol{y}_a = \boldsymbol{C}_a\boldsymbol{x}$ is available. Because those auxiliary outputs in $\boldsymbol{y}_a$ are not available, high-order sliding-mode exact differentiators have been employed in Floquet et al. (2007) to obtain them. In this paper, we develop an alternative approach of using high-gain approximate differentiators of simpler architecture to estimate the auxiliary outputs instead.

3. High-gain approximate differentiator

In this section, we construct the high-gain approximate differentiator. Let $y_{ij} = \boldsymbol{c}_i\boldsymbol{A}^{j-1}\boldsymbol{x}$ for $i = 1, \dots, p$ and $j = 1, \dots, \gamma_i$. Thus, we have $\boldsymbol{y}_a = [\boldsymbol{y}_{a1}^\top \cdots \boldsymbol{y}_{ap}^\top]^\top$, where $\boldsymbol{y}_{ai} = [y_{i1} \ \cdots \ y_{i\gamma_i}]^\top$. If $\gamma_i > 1$, the dynamics of $\boldsymbol{y}_{ai}, j = 1, \dots, \gamma_i$, are given by

$$\left.\begin{aligned}\dot{\boldsymbol{y}}_{ai} &= \bar{\boldsymbol{A}}_i\boldsymbol{y}_{ai} + \bar{\boldsymbol{b}}_{i1}f_i(\boldsymbol{x}, \boldsymbol{u}_2) + \bar{\boldsymbol{b}}_{i2}\boldsymbol{u}_1\\ y_{i1} &= \bar{\boldsymbol{c}}_i\boldsymbol{y}_{ai},\end{aligned}\right\} \tag{6}$$

where the pair $(\bar{\boldsymbol{A}}_i, \bar{\boldsymbol{b}}_{i1})$ is in canonical controllable form representing the chain of γ_i integrators, $f_i(\boldsymbol{x}, \boldsymbol{u}_2) = \boldsymbol{c}_i\boldsymbol{A}^{\gamma_i}\boldsymbol{x} + \boldsymbol{c}_i\boldsymbol{A}^{\gamma_i-1}\boldsymbol{B}_1\boldsymbol{u}_2$, $\bar{\boldsymbol{b}}_{i2} = [\boldsymbol{c}_i\boldsymbol{B}_1 \ \cdots \ \boldsymbol{c}_i\boldsymbol{A}^{\gamma_i-1}\boldsymbol{B}_1]^\top$ and $\bar{\boldsymbol{c}}_i = [1 \ 0 \ \cdots \ 0]$. We assume, as in Floquet et al. (2007), that $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ are bounded and that y_{ij} satisfies $\left|y_{ij}\right| \le d_{ij}$. If $\gamma_i > 1$, we construct the following high-gain observer to estimate $\boldsymbol{y}_{ai}$,

$$\dot{\boldsymbol{y}}_{hi} = \bar{\boldsymbol{A}}_i\boldsymbol{y}_{hi} + \boldsymbol{l}_i\bar{\boldsymbol{c}}_i\left(\boldsymbol{y}_{ai} - \boldsymbol{y}_{hi}\right) + \bar{\boldsymbol{b}}_{i2}\boldsymbol{u}_1 \tag{7}$$

with $\boldsymbol{y}_{hi} = [\hat{y}_{i1} \ \cdots \ \hat{y}_{i\gamma_i}]^\top$ and

$$\boldsymbol{l}_i = \left[\frac{\alpha_{i1}}{\epsilon} \ \cdots \ \frac{\alpha_{i\gamma_i}}{\epsilon^{\gamma_i}}\right]^\top,$$

where $\epsilon \in (0, 1)$ is a design parameter and $\alpha_{ij}, j = 1, \dots, \gamma_i$, are selected so that the roots of the equation, $s^{\gamma_i} + \alpha_{i1}s^{\gamma_i-1} + \cdots + \alpha_{i(\gamma_i-1)}s + \alpha_{i\gamma_i} = 0$, have negative real parts. If $\gamma_i = 1$, we do not need to construct the above high-gain observer (7) because y_{i1} are already available. In this case, we have $\boldsymbol{y}_{hi} = \boldsymbol{y}_{ai} = y_{i1}$. To proceed, let $\zeta_i = 0$ if $\gamma_i = 1$ and let $\zeta_i = [\zeta_{i1} \ \cdots \ \zeta_{i\gamma_i}]^\top$ if $\gamma_i > 1$, where

$$\zeta_{ij} = \frac{y_{ij} - \hat{y}_{ij}}{\epsilon^{\gamma_i - j}}, \quad j = 1, \dots, \gamma_i. \tag{8}$$

It follows from (6) and (7) that, if $\gamma_i > 1$, we have

$$\epsilon\dot{\zeta}_i = \bar{\boldsymbol{A}}_{ci}\zeta_i + \epsilon\bar{\boldsymbol{b}}_{i1}f_i(\boldsymbol{x}, \boldsymbol{u}_2) \tag{9}$$

with

$$\bar{\boldsymbol{A}}_{ci} = \begin{bmatrix}-\alpha_i^\top & \boldsymbol{I}_{\gamma_i-1}\\ -\alpha_{i\gamma_i} & \boldsymbol{0}_{\gamma_i-1}^\top\end{bmatrix},$$

where $\alpha_i = [\alpha_{i1} \ \cdots \ \alpha_{i(\gamma_i-1)}]$ is Hurwitz. Using the arguments in Mahmoud and Khalil (1996), we can prove the following lemma that we use in further development.

Lemma 1. *For the high-gain observer (7), there exist a positive constant β_i and a finite time $T_i(\epsilon)$ such that $\|\zeta_i(t)\| \le \beta_i\epsilon$ for $t \ge t_0 + T_i(\epsilon)$. Moreover, $\lim_{\epsilon\to 0^+} T_i(\epsilon) = 0$.*

It follows from (8) that $\boldsymbol{y}_{ai} - \boldsymbol{y}_{hi} = \boldsymbol{D}_i\zeta_i$, where $\boldsymbol{D}_i = \text{diag}[\epsilon^{\gamma_i-1}\ \epsilon^{\gamma_i-2}\ \cdots\ 1]$. Let $\boldsymbol{y}_h = [\boldsymbol{y}_{h1}^\top\ \cdots\ \boldsymbol{y}_{hp}^\top]^\top$, $\boldsymbol{D} = \text{diag}[\boldsymbol{D}_1\ \cdots\ \boldsymbol{D}_p]$ and $\zeta = [\zeta_1^\top\ \cdots\ \zeta_p^\top]^\top$. We have

$$\boldsymbol{y}_a - \boldsymbol{y}_h = \boldsymbol{D}\zeta. \tag{10}$$

Note that the induced Euclidean norm of $\boldsymbol{D}$ is 1; that is, $\|\boldsymbol{D}\| = 1$. Let $\beta_i = 0$ and $T_i(\epsilon) = 0$ if $\gamma_i = 1$. Thus, it follows from Lemma 1 that $\|\zeta\| \le \beta\epsilon$, where $\beta = (\sum_{i=1}^p \beta_i^2)^{\frac{1}{2}}$, after a finite time $T(\epsilon) = \max_{1\le i\le p} T_i(\epsilon)$, and $\lim_{\epsilon\to 0} T(\epsilon) = 0$.

4. Sliding-mode observer construction

In this section, we proceed to construct the sliding-mode observer. To eliminate the peaking phenomenon of the high-gain observer (Esfandiari & Khalil, 1992), we introduce saturation on the signal $\boldsymbol{y}_h$ such that $\boldsymbol{y}_h^s = [\boldsymbol{y}_{h1}^{s\ \top}\ \cdots\ \boldsymbol{y}_{hp}^{s\ \top}]^\top$, where $\boldsymbol{y}_{hi}^s = \boldsymbol{y}_{ai} = y_{i1}$ if $\gamma_i = 1$ and

$$\boldsymbol{y}_{hi}^s = \left[S_{i1}\text{sat}\left(\frac{\hat{y}_{i1}}{S_{i1}}\right)\ \cdots\ S_{i\gamma_i}\text{sat}\left(\frac{\hat{y}_{i\gamma_i}}{S_{i\gamma_i}}\right)\right]^\top$$

with $S_{ij} > d_{ij}$ if $\gamma_i > 1$. Then we construct the following sliding-mode observer,

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}_1\boldsymbol{u}_1 + \boldsymbol{L}_a\left(\boldsymbol{y}_h^s - \hat{\boldsymbol{y}}_a\right) + \boldsymbol{B}_2\boldsymbol{E}_a, \tag{11}$$

with $\hat{\boldsymbol{y}}_a = \boldsymbol{C}_a\hat{\boldsymbol{x}}$ and

$$\boldsymbol{E}_a = \begin{cases} \eta\dfrac{\boldsymbol{F}_a\left(\boldsymbol{y}_h^s - \hat{\boldsymbol{y}}_a\right)}{\left\|\boldsymbol{F}_a\left(\boldsymbol{y}_h^s - \hat{\boldsymbol{y}}_a\right)\right\|} & \text{if } \boldsymbol{F}_a\left(\boldsymbol{y}_h^s - \hat{\boldsymbol{y}}_a\right) \ne \boldsymbol{0} \\ \boldsymbol{0} & \text{if } \boldsymbol{F}_a\left(\boldsymbol{y}_h^s - \hat{\boldsymbol{y}}_a\right) = \boldsymbol{0}, \end{cases}$$

where $\boldsymbol{L}_a$ and $\boldsymbol{F}_a$ are matrices such that

$$(\boldsymbol{A} - \boldsymbol{L}_a\boldsymbol{C}_a)^\top\boldsymbol{P}_a + \boldsymbol{P}_a(\boldsymbol{A} - \boldsymbol{L}_a\boldsymbol{C}_a) = -2\boldsymbol{Q}_a$$

and

$$\boldsymbol{F}_a\boldsymbol{C}_a = \boldsymbol{B}_2^\top\boldsymbol{P}_a \tag{12}$$

for some symmetric positive definite $\boldsymbol{P}_a$ and $\boldsymbol{Q}_a$. Let $\boldsymbol{e} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ be the state observation error. Then it follows from (1) and (11) that

$$\dot{\boldsymbol{e}} = \boldsymbol{A}\boldsymbol{e} - \boldsymbol{L}_a\left(\boldsymbol{y}_h^s - \hat{\boldsymbol{y}}_a\right) + \boldsymbol{B}_2\boldsymbol{u}_2 - \boldsymbol{B}_2\boldsymbol{E}_a. \tag{13}$$

We now state and prove a theorem concerned with the proposed observer performance.

Theorem 1. *For the dynamic system (1) and the associated sliding-mode observer (11) with high-gain approximate differentiators (7), there exists a constant $\epsilon^* \in (0, 1)$ such that, if $\epsilon \in (0, \epsilon^*)$ and $\eta \ge \rho$, then the state estimation error $\boldsymbol{e}(t)$ is uniformly ultimately bounded. Specifically, we have $\|\boldsymbol{e}(t)\| \le \kappa(\epsilon)$ for $t \ge T_f(\epsilon)$, where $T_f(\epsilon)$ is a finite time and*

$$\kappa(\epsilon) = \frac{\kappa_1\epsilon + \sqrt{\kappa_1^2\epsilon^2 + 4\mu_a\kappa_2\epsilon}}{2\mu_a}\sqrt{\frac{2}{\lambda_{\min}(\boldsymbol{P}_a)}}$$

with positive constants μ_a, κ_1 and κ_2.

Proof. It follows from Lemma 1 that $\|\zeta(t)\| \le \beta\epsilon$ for $t \ge t_0 + T(\epsilon)$. Then, it follows from (10) that $\|\boldsymbol{y}_a(t) - \boldsymbol{y}_h(t)\| \le \beta\epsilon$ for $t \ge t_0 + T(\epsilon)$. There exists a constant $\bar{\epsilon}$ such that, if $\|\boldsymbol{y}_a(t) - \boldsymbol{y}_h(t)\| \le \beta\bar{\epsilon}$, then $\boldsymbol{y}_h^s(t)$ is not saturated; that is, $\boldsymbol{y}_h^s(t) = \boldsymbol{y}_h(t)$. Thus, we can choose $\epsilon^* = \min\{\bar{\epsilon}, 1\}$ such that, if $\epsilon \in (0, \epsilon^*)$, then $\|\zeta(t)\| \le \beta\epsilon$ and $\boldsymbol{y}_h^s(t) = \boldsymbol{y}_h(t)$ for $t \ge t_0 + T(\epsilon)$.

For $t_0 \le t \le t_0 + T(\epsilon)$, it is guaranteed that the observer state vector $\hat{\boldsymbol{x}}(t)$ in (11) is bounded because $\boldsymbol{u}_1$, $\boldsymbol{y}_h^s$ and $\boldsymbol{E}_a$ are bounded and $\boldsymbol{A}-\boldsymbol{L}_a\boldsymbol{C}_a$ is Hurwitz. Thus, $\boldsymbol{e}(t)$ is bounded for $t_0 \le t \le t_0+T(\epsilon)$. For $t \ge t_0 + T(\epsilon)$, because $\boldsymbol{y}_h^s(t) = \boldsymbol{y}_h(t)$ and $\boldsymbol{y}_h = \boldsymbol{y}_a - \boldsymbol{D}\zeta$, the dynamics of the state estimation error (13) become

$$\begin{aligned}\dot{\boldsymbol{e}} &= \boldsymbol{A}\boldsymbol{e} - \boldsymbol{L}_a\left(\boldsymbol{y}_h - \hat{\boldsymbol{y}}_a\right) + \boldsymbol{B}_2\boldsymbol{u}_2 - \boldsymbol{B}_2\boldsymbol{E}_a \\ &= (\boldsymbol{A} - \boldsymbol{L}_a\boldsymbol{C}_a)\,\boldsymbol{e} + \boldsymbol{L}_a\boldsymbol{D}\zeta + \boldsymbol{B}_2\boldsymbol{u}_2 - \boldsymbol{B}_2\boldsymbol{E}_a.\end{aligned} \tag{14}$$

Consider the Lyapunov function $V = \frac{1}{2}\boldsymbol{e}^\top\boldsymbol{P}_a\boldsymbol{e}$ for $t \ge t_0 + T(\epsilon)$. Evaluating the time derivative of V on the solutions of (14), we obtain

$$\begin{aligned}\dot{V} &= \boldsymbol{e}^\top\boldsymbol{P}\left(\boldsymbol{A} - \boldsymbol{L}_a\boldsymbol{C}_a\right)\boldsymbol{e} + \boldsymbol{e}^\top\boldsymbol{P}_a\boldsymbol{L}_a\boldsymbol{D}\zeta + \boldsymbol{e}^\top\boldsymbol{P}_a\boldsymbol{B}_2\boldsymbol{u}_2 - \boldsymbol{e}^\top\boldsymbol{P}_a\boldsymbol{B}_2\boldsymbol{E}_a \\ &= -\boldsymbol{e}^\top\boldsymbol{Q}_a\boldsymbol{e} + \boldsymbol{e}^\top\boldsymbol{P}_a\boldsymbol{L}_a\boldsymbol{D}\zeta + (\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e})^\top\boldsymbol{u}_2 - (\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e})^\top\boldsymbol{E}_a \\ &= -\boldsymbol{e}^\top\boldsymbol{Q}_a\boldsymbol{e} + \boldsymbol{e}^\top\boldsymbol{P}_a\boldsymbol{L}_a\boldsymbol{D}\zeta - (\boldsymbol{F}_a\boldsymbol{D}\zeta)^\top\boldsymbol{u}_2 \\ &\quad + (\boldsymbol{F}_a\boldsymbol{D}\zeta)^\top\boldsymbol{E}_a + (\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{F}_a\boldsymbol{D}\zeta)^\top(\boldsymbol{u}_2 - \boldsymbol{E}_a).\end{aligned}$$

If $\boldsymbol{F}_a(\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{D}\zeta) = \boldsymbol{0}$, then

$$(\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{F}_a\boldsymbol{D}\zeta)^\top(\boldsymbol{u}_2 - \boldsymbol{E}_a) = 0. \tag{15}$$

On the other hand, if $\boldsymbol{F}_a(\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{D}\zeta) \ne \boldsymbol{0}$, then

$$\begin{aligned}&(\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{F}_a\boldsymbol{D}\zeta)^\top(\boldsymbol{u}_2 - \boldsymbol{E}_a) \\ &\quad \le -(\eta - \rho)\|\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{F}_a\boldsymbol{D}\zeta\| \le 0.\end{aligned} \tag{16}$$

It follows from (15) and (16) that in both cases we have

$$\dot{V} \le -\boldsymbol{e}^\top\boldsymbol{Q}_a\boldsymbol{e} + \boldsymbol{e}^\top\boldsymbol{P}_a\boldsymbol{L}_a\boldsymbol{D}\zeta - (\boldsymbol{F}_a\boldsymbol{D}\zeta)^\top\boldsymbol{u}_2 + (\boldsymbol{F}_a\boldsymbol{D}\zeta)^\top\boldsymbol{E}_a. \tag{17}$$

Performing some manipulations gives

$$\begin{aligned}\dot{V} &\le -\lambda_{\min}(\boldsymbol{Q}_a)\|\boldsymbol{e}\|^2 + \|\boldsymbol{P}_a\boldsymbol{L}_a\|\|\boldsymbol{D}\|\|\zeta\|\|\boldsymbol{e}\| \\ &\quad + (\eta + \rho)\|\boldsymbol{F}_a\|\|\boldsymbol{D}\|\|\zeta\| \\ &= -2\mu_a V + \kappa_1\epsilon\sqrt{V} + \kappa_2\epsilon,\end{aligned} \tag{18}$$

where

$$\mu_a = \frac{\lambda_{\min}(\boldsymbol{Q}_a)}{\lambda_{\max}(\boldsymbol{P}_a)}, \qquad \kappa_1 = \frac{\sqrt{2}\beta\|\boldsymbol{P}_a\boldsymbol{L}_a\|}{\sqrt{\lambda_{\max}(\boldsymbol{P}_a)}}$$

and $\kappa_2 = (\eta + \rho)\beta\|\boldsymbol{F}_a\|$. It follows from (18) that

$$\dot{V} \le -\mu_a V - \left(\sqrt{V} - R_-\right)\left(\sqrt{V} - R_+\right), \tag{19}$$

where

$$R_- = \frac{\kappa_1\epsilon - \sqrt{\kappa_1^2\epsilon^2 + 4\mu_a\kappa_2\epsilon}}{2\mu_a} < 0$$

and

$$R_+ = \frac{\kappa_1\epsilon + \sqrt{\kappa_1^2\epsilon^2 + 4\mu_a\kappa_2\epsilon}}{2\mu_a} > 0.$$

Hence, as long as $\sqrt{V} > R_+$, that is, $V > R_+^2$, we have $(\sqrt{V} - R_-)(\sqrt{V} - R_+) < 0$. Therefore, if $V(t_0+T(\epsilon)) > R_+^2$ and $V(t) > R_+^2$ for $t \ge t_0 + T_f(\epsilon)$, then $\dot{V} \le -\mu_a V$, which implies that $V(t) \le \exp(-\mu_a(t - t_0 - T(\epsilon)))V(t_0 + T(\epsilon))$. Thus, we can find a finite time $T_f(\epsilon)$ such that $V(t) \le R_+^2$ for $t \ge t_0 + T(\epsilon)$, where

$$T_f(\epsilon) = T(\epsilon) + \frac{1}{\mu_a}\ln\left(\frac{V(t_0 + T(\epsilon))}{R_+^2}\right).$$

On the other hand, if $V(t_0+T(\epsilon)) \le R_+^2$, then $V(t) \le R_+^2$ for $t \ge t_0+T(\epsilon)$. So we can choose $T_f(\epsilon) = T(\epsilon)$. Therefore, there exists a finite time $T_f(\epsilon)$ such that $V(t) \le R_+^2$ for $t \ge t_0+T_f(\epsilon)$. We also have that $\frac{1}{2}\lambda_{\min}(\boldsymbol{P}_a)\,\|\boldsymbol{e}(t)\|^2 \le V(t) \le R_+^2$. Using this fact coupled with the definition of R_+ we have that $\|\boldsymbol{e}(t)\| \le \kappa(\epsilon)$. The proof of the theorem is complete. □

Remark 1. It follows from Theorem 1 that the state estimation error enters the closed ball $\{\boldsymbol{e} : \|\boldsymbol{e}\| \le \kappa(\epsilon)\}$ after a finite time $T_f(\epsilon)$. It is easy to verify that the radius of the above closed ball can be adjusted by the design parameter ϵ, and because $\lim_{\epsilon\to 0^+}\kappa(\epsilon) = 0$, the state estimation error $\boldsymbol{e}$ converges to the origin as ϵ goes to zero.

Corollary 1. *The hyperplane,*

$$\{(\boldsymbol{e},\zeta) : \sigma = \boldsymbol{F}_a(\boldsymbol{C}_a\boldsymbol{e} + \boldsymbol{D}\zeta) = \boldsymbol{0}\},$$

is invariant in $(\boldsymbol{e},\zeta)$*-space and is reached in finite time for sufficiently large* η.

Proof. Let $\bar{\boldsymbol{A}}_{ci} = \boldsymbol{O}$ and $\bar{\boldsymbol{b}}_{i1} = \boldsymbol{0}$ if $\gamma_i = 1$ and let $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}_2) = [f_1(\boldsymbol{x},\boldsymbol{u}_2)\ \cdots\ f_p(\boldsymbol{x},\boldsymbol{u}_2)]^\top$. Thus, using (9) for $\gamma_i > 1$ and the fact that $\zeta_i = 0$ if $\gamma_i = 1$, we obtain

$$\epsilon\dot{\zeta} = \bar{\boldsymbol{A}}_c\zeta + \epsilon\bar{\boldsymbol{B}}_1\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}_2), \tag{20}$$

where

$$\bar{\boldsymbol{A}}_c = \mathrm{diag}[\bar{\boldsymbol{A}}_{c1}\ \cdots\ \bar{\boldsymbol{A}}_{cp}]$$

and

$$\bar{\boldsymbol{B}}_1 = \mathrm{diag}[\bar{\boldsymbol{b}}_{11}\ \cdots\ \bar{\boldsymbol{b}}_{p1}].$$

Because $\boldsymbol{x}$ and $\boldsymbol{u}_2$ are bounded, we know that $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}_2)$ is also bounded. For $t \ge t_0 + T_f(\epsilon)$, it follows from (14) and (20) that

$$\begin{aligned}
\sigma^\top\dot{\sigma} &= \sigma^\top(\boldsymbol{F}_a\boldsymbol{C}_a(\boldsymbol{A}-\boldsymbol{L}_a\boldsymbol{C}_a)\boldsymbol{e} + \boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{L}_a\boldsymbol{D}\zeta)\\
&\quad + \sigma^\top(\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{B}_2\boldsymbol{u}_2 - \boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{B}_2\boldsymbol{E}_a)\\
&\quad + \sigma^\top\left(\frac{1}{\epsilon}\boldsymbol{F}_a\boldsymbol{D}\bar{\boldsymbol{A}}_c\zeta + \boldsymbol{F}_a\boldsymbol{D}\bar{\boldsymbol{B}}_1\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}_2)\right)\\
&\le \|\boldsymbol{F}_a\boldsymbol{C}_a(\boldsymbol{A}-\boldsymbol{L}_a\boldsymbol{C}_a)\|\|\boldsymbol{e}\|\|\sigma\| + \|\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{L}_a\|\|\boldsymbol{D}\|\|\zeta\|\|\sigma\|\\
&\quad + \sigma^\top(\boldsymbol{B}_2^\top\boldsymbol{P}_a\boldsymbol{B}_2)\boldsymbol{u}_2 - \eta\sigma^\top(\boldsymbol{B}_2^\top\boldsymbol{P}_a\boldsymbol{B}_2)\frac{\sigma}{\|\sigma\|}\\
&\quad + \frac{1}{\epsilon}\|\boldsymbol{F}_a\|\|\bar{\boldsymbol{A}}_c\|\|\boldsymbol{D}\|\|\zeta\|\|\sigma\| + \|\boldsymbol{F}_a\|\|\bar{\boldsymbol{B}}_1\|\|\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}_2)\|\|\sigma\|\\
&\le \kappa(\epsilon)\|\boldsymbol{F}_a\boldsymbol{C}_a(\boldsymbol{A}-\boldsymbol{L}_a\boldsymbol{C}_a)\|\|\sigma\| + \beta\epsilon\|\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{L}_a\|\|\sigma\|\\
&\quad + \beta_1\|\boldsymbol{F}_a\|\|\bar{\boldsymbol{B}}_1\|\|\sigma\| - \eta\lambda_{\min}(\boldsymbol{B}_2^\top\boldsymbol{P}_a\boldsymbol{B}_2)\|\sigma\|\\
&\quad + \beta\|\boldsymbol{F}_a\|\|\bar{\boldsymbol{A}}_c\|\|\sigma\| + \lambda_{\max}(\boldsymbol{B}_2^\top\boldsymbol{P}_a\boldsymbol{B}_2)\|\boldsymbol{u}_2\|\|\sigma\|\\
&= -(\eta-\bar{\eta})\,\lambda_{\min}(\boldsymbol{B}_2^\top\boldsymbol{P}_a\boldsymbol{B}_2)\|\sigma\|,
\end{aligned} \tag{21}$$

with

$$\bar{\eta} = \frac{\kappa_3+\kappa_4+\kappa_5+\kappa_6+\kappa_7}{\lambda_{\min}(\boldsymbol{B}_2^\top\boldsymbol{P}_a\boldsymbol{B}_2)}$$

where

$$\kappa_3 = \kappa(\epsilon)\|\boldsymbol{F}_a\boldsymbol{C}_a(\boldsymbol{A}-\boldsymbol{L}_a\boldsymbol{C}_a)\|,$$
$$\kappa_4 = \beta\epsilon\|\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{L}_a\|,\qquad \kappa_5 = \rho\lambda_{\max}(\boldsymbol{B}_2^\top\boldsymbol{P}_a\boldsymbol{B}_2),$$
$$\kappa_6 = \beta\|\boldsymbol{F}_a\|\|\bar{\boldsymbol{A}}_c\|,\qquad \kappa_7 = \|\boldsymbol{F}_a\|\|\bar{\boldsymbol{B}}_1\|\|\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}_2)\|.$$

It follows from (21) that, if we choose η such that $\eta \ge \bar{\eta}+\varepsilon$, where ε is a small positive constant, then $\sigma^\top\dot{\sigma} \le -\varepsilon\|\sigma\|$, which implies that the above hyperplane is invariant. Then we can show that the sliding surface is reached in finite time using the same arguments as in Edwards and Spurgeon (1998, p. 53).

5. Unknown input reconstruction

Our objective in this subsection is to show that we can use the proposed architecture to estimate the unknown input $\boldsymbol{u}_2$. We will show that, for any $\delta > 0$, there exist a design parameter $\epsilon > 0$ and $T > 0$ such that

$$\|\boldsymbol{E}_a(\boldsymbol{y}_h^s,\hat{\boldsymbol{y}}_a,\eta) - \boldsymbol{u}_2\| < \delta \tag{22}$$

for $t > T$. Roughly speaking, the above means that, for $t > T$,

$$\boldsymbol{E}_a(\boldsymbol{y}_h^s,\hat{\boldsymbol{y}}_a,\eta) \approx \boldsymbol{u}_2. \tag{23}$$

It follows from the corollary that the manifold $\{(\boldsymbol{e},\zeta) : \sigma = \boldsymbol{F}_a(\boldsymbol{C}_a\boldsymbol{e}+\boldsymbol{D}\zeta) = \boldsymbol{0}\}$ is invariant and is reached after a finite time, and therefore

$$\begin{aligned}
\dot{\sigma} &= \boldsymbol{F}_a\boldsymbol{C}_a(\boldsymbol{A}-\boldsymbol{L}_a\boldsymbol{C}_a)\boldsymbol{e} + \boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{L}_a\boldsymbol{D}\zeta + \boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{B}_2\boldsymbol{u}_2\\
&\quad - \boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{B}_2\boldsymbol{E}_a(\boldsymbol{y}_h^s,\hat{\boldsymbol{y}}_a,\eta) + \boldsymbol{F}_a\boldsymbol{D}\dot{\zeta} = \boldsymbol{0}.
\end{aligned} \tag{24}$$

Substituting (12) into (24) and performing simple manipulations, we obtain

$$\begin{aligned}
\boldsymbol{E}_a(\boldsymbol{y}_h^s,\hat{\boldsymbol{y}}_a,\eta) - \boldsymbol{u}_2 &= \left(\boldsymbol{B}_2^\top\boldsymbol{P}_a\boldsymbol{B}_2\right)^{-1}\left(\boldsymbol{F}_a\boldsymbol{C}_a\boldsymbol{L}_a\boldsymbol{D}\zeta + \boldsymbol{F}_a\boldsymbol{D}\dot{\zeta}\right)\\
&\quad + \left(\boldsymbol{B}_2^\top\boldsymbol{P}_a\boldsymbol{B}_2\right)^{-1}\boldsymbol{F}_a\boldsymbol{C}_a(\boldsymbol{A}-\boldsymbol{L}_a\boldsymbol{C}_a)\boldsymbol{e}.
\end{aligned} \tag{25}$$

We then show that for sufficiently small ϵ, $\|\boldsymbol{e}(t)\|$, $\|\zeta(t)\|$ and $\|\dot{\zeta}\|$ become negligible after a finite time, which results in (22). Note that $\boldsymbol{E}_a$ is a discontinuous function. Two practical approaches can be used to obtain an estimate of $\boldsymbol{u}_2$. The first approach is to pass the output of the injection term $\boldsymbol{E}_a$ through a low-pass filter. The second approach is to use a boundary layer to smooth out the discontinuous $\boldsymbol{E}_a$. More details can be found in Edwards and Spurgeon (1998).

By Lemma 1, we have $\|\zeta(t)\| \le \beta\epsilon$ for $t \ge t_0 + T(\epsilon)$. By Theorem 1, we have $\|\boldsymbol{e}(t)\| \le \kappa(\epsilon)$ for $t \ge t_0 + T_f(\epsilon)$, where $\lim_{\epsilon\to 0^+}\kappa(\epsilon) = 0$. Thus, in order to get (22), it remains to show that $\|\dot{\zeta}(t)\|$ becomes negligible after a finite time if ϵ is sufficiently small. In what follows, we perform preliminary manipulations before formally proceeding with the proof. Recall that $\zeta = [\zeta_1^\top\ \cdots\ \zeta_p^\top]^\top$ and $\zeta_i = \dot{\zeta}_i = 0$ if $\gamma_i = 1$. Thus, we only need to prove that, if $\gamma_i > 1$, then $\|\dot{\zeta}_i(t)\|$ becomes negligible after a finite time for sufficiently small ϵ. We first rewrite (9) as

$$\begin{aligned}
\dot{\zeta}_i(t) &= \frac{1}{\epsilon}\bar{\boldsymbol{A}}_{ci}\zeta_i(t) + \bar{\boldsymbol{b}}_{i1}f_i(\boldsymbol{x}(t),\boldsymbol{u}_2(t))\\
&= \frac{1}{\epsilon}\bar{\boldsymbol{A}}_{ci}\zeta_i(t) + \boldsymbol{v}_i(t),
\end{aligned} \tag{26}$$

where $\boldsymbol{v}_i(t) = \bar{\boldsymbol{b}}_{i1}f_i(\boldsymbol{x}(t),\boldsymbol{u}_2(t))$. Because $\boldsymbol{x}(t)$ and $\boldsymbol{u}_2(t)$ are bounded, it follows from the definition of $f_i(\boldsymbol{x}(t),\boldsymbol{u}_2(t))$ that it is also bounded. Thus, $\boldsymbol{v}_i(t)$ is a bounded measurable function. It is well known that the solution to (26) has the form

$$\begin{aligned}
\zeta_i(t) &= \exp\left(\frac{1}{\epsilon}\bar{\boldsymbol{A}}_{ci}(t-t_0)\right)\zeta_i(t_0)\\
&\quad + \int_{t_0}^{t}\exp\left(\frac{1}{\epsilon}\bar{\boldsymbol{A}}_{ci}(t-s)\right)\boldsymbol{v}_i(s)\mathrm{d}s.
\end{aligned} \tag{27}$$

Performing a change of variables in the integral of (27) by $z = (t-s)/\epsilon$, we obtain

$$\begin{aligned}
\zeta_i(t) &= \exp\left(\frac{1}{\epsilon}\bar{\boldsymbol{A}}_{ci}(t-t_0)\right)\zeta_i(t_0)\\
&\quad + \epsilon\int_{0}^{(t-t_0)/\epsilon}\exp\left(\bar{\boldsymbol{A}}_{ci}z\right)\boldsymbol{v}_i(t-\epsilon z)\mathrm{d}z.
\end{aligned} \tag{28}$$

To proceed, two notions regarding the function $\boldsymbol{v}_i(t)$ are defined in the following.

Definition 1. A function $\boldsymbol{v}_i(t)$ is left-continuous if, for all t, we have $\lim_{\epsilon\to 0^+} \boldsymbol{v}_i(t-\epsilon) = \boldsymbol{v}_i(t)$.

Definition 2. A function $\boldsymbol{v}_i(t)$ defined on $S \subset \mathbb{R}$ is weakly uniformly continuous if, for every $\nu > 0$, there exists a $\delta > 0$ such that, for each interval $\Omega \subset S$ with length less than δ, $\|\boldsymbol{v}_i(s) - \boldsymbol{v}_i(t)\| < \nu$ for $s, t \in \Omega$.

Remark 2. A function with simple jump discontinuities, for example, a square wave, can always be made left-continuous by changing the function values at the points of discontinuity. Note that a uniformly continuous function is weakly uniformly continuous. However, a weakly uniformly continuous function is not necessarily uniformly continuous. The square wave on the complement of its switching points is an example of a function that is weakly uniformly continuous but not uniformly continuous. If S is connected, then the two notions of uniform continuity are equivalent.

In the following, we use S_1+S_2, where $S_1, S_2 \subset \mathbb{R}$, to denote the set $\{s_1+s_2 : s_1 \in S_1, s_2 \in S_2\}$. If S_1 or S_2 is empty, then S_1+S_2 is defined to be empty.

Theorem 2. *Consider the dynamics (26), where $\bar{\boldsymbol{A}}_{ci}$ is Hurwitz and $\boldsymbol{v}_i(t)$ is bounded. Let J denote the set of points at which $\boldsymbol{v}_i(t)$ is discontinuous and let $\tau > t_0 > 0$. If $\boldsymbol{v}_i(t)$ is left-continuous, then*

$$\lim_{\epsilon\to 0^+} \dot{\zeta}_i(t) = \boldsymbol{0}$$

for each $t > t_0 \geq 0$. Moreover, if $\boldsymbol{v}_i(t)$ is also weakly uniformly continuous on $[\tau,\infty)\setminus J$, then the convergence of $\dot{\zeta}_i(t)$ to $\boldsymbol{0}$ as $\epsilon \to 0^+$ is uniform on $[\tau,\infty)\setminus(J+(0,\xi))$ for each $\xi > 0$. In particular, if $\boldsymbol{v}_i(t)$ is uniformly continuous, then the convergence is uniform on $[\tau,\infty)$.

Proof. It follows from (28) that

$$\frac{\zeta_i(t)}{\epsilon} = \exp\left(\frac{1}{\epsilon}\bar{\boldsymbol{A}}_{ci}(t-t_0)\right)\frac{\zeta_i(t_0)}{\epsilon} + \int_0^{(t-t_0)/\epsilon} \exp\left(\bar{\boldsymbol{A}}_{ci}z\right)\boldsymbol{v}_i(t-\epsilon z)\mathrm{d}z. \tag{29}$$

Note that the matrix $\bar{\boldsymbol{A}}_{ci}$ is Hurwitz. Let $-\lambda$, $\lambda > 0$, denote the maximum of the real parts of its eigenvalues. The first term on the right-hand side of (29) satisfies

$$\left\|\exp\left(\frac{1}{\epsilon}\bar{\boldsymbol{A}}_{ci}(t-t_0)\right)\frac{\zeta_i(t_0)}{\epsilon}\right\| \leq \frac{M_1}{\epsilon}\|\zeta_i(t_0)\| \exp\left(-\frac{1}{\epsilon}\lambda(t-t_0)\right). \tag{30}$$

Because the initial conditions for the high-gain observers are always bounded, there exists $M_2 > 0$ such that $\|\zeta_i(t_0)\| \leq M_2/\epsilon^{\gamma_i - 1}$. Substituting the above into (30), we obtain

$$\left\|\exp\left(\frac{1}{\epsilon}\bar{\boldsymbol{A}}_{ci}(t-t_0)\right)\frac{\zeta_i(t_0)}{\epsilon}\right\| \leq \frac{M_1M_2}{\epsilon^{\gamma_i}}\exp\left(-\frac{1}{\epsilon}\lambda(t-t_0)\right). \tag{31}$$

By calculus, for each $t > t_0 \geq 0$, we have

$$\lim_{\epsilon\to 0^+}\frac{1}{\epsilon^{\gamma_i}}\exp\left(-\frac{\lambda(t-t_0)}{\epsilon}\right) = 0,$$

and thus,

$$\lim_{\epsilon\to 0^+}\exp\left(\frac{1}{\epsilon}\bar{\boldsymbol{A}}_{ci}(t-t_0)\right)\frac{\zeta_i(t_0)}{\epsilon} = \boldsymbol{0}. \tag{32}$$

We next consider the second term on the right-hand side of (29). Let

$$\boldsymbol{g}_i(z) = \exp\left(\bar{\boldsymbol{A}}_{ci}z\right)\boldsymbol{v}_i(t-\epsilon z) I_{[0,(t-t_0)/\epsilon)}(z),$$

where $I_{[0,(t-t_0)/\epsilon)}(z)$ denotes the indicator function for the interval $[0,(t-t_0)/\epsilon)$. Observe that, for each $z \geq 0$, if $\boldsymbol{v}_i(t)$ is left-continuous, then

$$\lim_{\epsilon\to 0^+}\boldsymbol{g}_i(z) = \exp\left(\bar{\boldsymbol{A}}_{ci}z\right)\boldsymbol{v}_i(t).$$

Because $\boldsymbol{v}_i(t)$ is bounded, there exists $M_3 > 0$ such that $\|\boldsymbol{v}_i(t)\| \leq M_3$ for all $t > t_0$. Thus, for each $z \geq 0$ and $\epsilon \in (0, 1)$, $\|\boldsymbol{g}_i(z)\| \leq M_1M_3\exp(-\lambda z)$. Because $\lambda > 0$, the function $\exp(-\lambda z)$ is integrable on $[0,\infty)$. Then we can thus apply Lebesgue's dominated convergence theorem Bartle (1966, page 45) (to each component) such that, for each $t > t_0 \geq 0$,

$$\begin{aligned}
\lim_{\epsilon\to 0^+}\int_0^{(t-t_0)/\epsilon} & \exp\left(\bar{\boldsymbol{A}}_{ci}z\right)\boldsymbol{v}_i(t-\epsilon z)\mathrm{d}z \\
&= \lim_{\epsilon\to 0^+}\int_0^{\infty}\boldsymbol{g}_i(z)\mathrm{d}z \\
&= \left(\int_0^{\infty}\exp\left(\bar{\boldsymbol{A}}_{ci}z\right)\mathrm{d}z\right)\boldsymbol{v}_i(t) \\
&= \bar{\boldsymbol{A}}_{ci}^{-1}\left(\lim_{z\to\infty}\exp\left(\bar{\boldsymbol{A}}_{ci}z\right) - \boldsymbol{I}_{\gamma_i}\right)\boldsymbol{v}_i(t) \\
&= -\bar{\boldsymbol{A}}_{ci}^{-1}\boldsymbol{v}_i(t).
\end{aligned} \tag{33}$$

We have $\lim_{z\to\infty}\exp\left(\bar{\boldsymbol{A}}_{ci}z\right) = \boldsymbol{0}$ because $\bar{\boldsymbol{A}}_{ci}$ is Hurwitz. Therefore, it follows from (29), (32) and (33) that

$$\lim_{\epsilon\to 0^+}\frac{\zeta_i(t)}{\epsilon} = -\bar{\boldsymbol{A}}_{ci}^{-1}\boldsymbol{v}_i(t). \tag{34}$$

Combining (26) and (34), we conclude that

$$\lim_{\epsilon\to 0^+}\dot{\zeta}_i(t) = \bar{\boldsymbol{A}}_{ci}\lim_{\epsilon\to 0^+}\frac{\zeta_i(t)}{\epsilon} + \boldsymbol{v}_i(t) = \boldsymbol{0}$$

for each $t > t_0 \geq 0$. Now we consider the case when $\boldsymbol{v}_i(t)$ is also weakly uniformly continues on $[\tau,\infty)\setminus J$. Let $\nu > 0$ and $\tau > t_0 > 0$. We can use (29) to estimate the difference between $\zeta_i(t)/\epsilon$ and $-\bar{\boldsymbol{A}}_{ci}^{-1}\boldsymbol{v}_i(t)$. For $t \geq \tau$, it follows from (31) that there exists a constant $\mu_1 \in (0, 1)$ such that

$$\left\|\exp\left(\frac{1}{\epsilon}\bar{\boldsymbol{A}}_{ci}(t-t_0)\right)\frac{\zeta_i(t_0)}{\epsilon}\right\| \leq \frac{\nu}{2}$$

for $\epsilon \in (0, \mu_1)$. We next analyze the difference between the second term on the right-hand side of (29) and $-\bar{\boldsymbol{A}}_{ci}^{-1}\boldsymbol{v}_i(t)$. Let $t \geq \tau$. Then we have

$$\begin{aligned}
&\left\|\int_0^{(t-t_0)/\epsilon}\exp\left(\bar{\boldsymbol{A}}_{ci}z\right)\boldsymbol{v}_i(t-\epsilon z)\mathrm{d}z + \bar{\boldsymbol{A}}_{ci}^{-1}\boldsymbol{v}_i(t)\right\| \\
&\quad\leq M_1\int_0^{(t-t_0)/\epsilon}\exp(-\lambda z)\|\boldsymbol{v}_i(t-\epsilon z) - \boldsymbol{v}_i(t)\|\mathrm{d}z \\
&\quad+ \left\|\int_0^{(t-t_0)/\epsilon}\exp\left(\bar{\boldsymbol{A}}_{ci}z\right)\boldsymbol{v}_i(t)\mathrm{d}z + \bar{\boldsymbol{A}}_{ci}^{-1}\boldsymbol{v}_i(t)\right\|.
\end{aligned}$$

We analyze the terms in the above sum separately. Let $S = [\tau,\infty)\setminus(J+(0,\xi))$. The set S is closed and it may be empty. For each $t \in S$, the distance from t to the nearest point of discontinuity less than t is at least ξ since all points with distance less than ξ to the right of a point of discontinuity are removed. Note that it is possible for S to contain a point of discontinuity. Let $t \in S$. By assumption, $\boldsymbol{v}_i(t)$ is weakly uniformly continuous on $[\tau,\infty)\setminus J$. Because $(t-\xi, t) \subset [\tau,\infty)\setminus J$, we can choose δ, independent of t, such that

$0 < \delta < \xi$ and $\|\boldsymbol{v}_i(s) - \boldsymbol{v}_i(w)\| \leq \lambda\nu/(8M_1)$ for $s, w \in (t-\delta, t)$. Because $\boldsymbol{v}_i(t)$ is left-continuous, we conclude, by letting $w \to t^-$, that $\|v(s) - v(t)\| \leq \lambda\nu/(8M_1)$ for $s \in (t-\delta, t]$. Then we have

$$
\begin{aligned}
& M_1 \int_0^{(t-t_0)/\epsilon} \exp(-\lambda z) \|\boldsymbol{v}_i(t-\epsilon z) - \boldsymbol{v}_i(t)\| \mathrm{d}z \\
&= M_1 \left(\int_0^{\delta/\epsilon} \exp(-\lambda z) \|\boldsymbol{v}_i(t-\epsilon z) - \boldsymbol{v}_i(t)\| \mathrm{d}z \right. \\
&\quad \left. + \int_{\delta/\epsilon}^{(t-t_0)/\epsilon} \exp(-\lambda z) \|\boldsymbol{v}_i(t-\epsilon z) - \boldsymbol{v}_i(t)\| \mathrm{d}z \right) \\
&< \frac{\lambda\nu}{8} \int_0^{\infty} \exp(-\lambda z)\, \mathrm{d}z + 2M_1M_3 \int_{\delta/\epsilon}^{\infty} \exp(-\lambda z)\, \mathrm{d}z \\
&= \frac{\nu}{8} + \frac{2M_1M_3}{\lambda} \exp\left(-\frac{\lambda\delta}{\epsilon}\right).
\end{aligned}
$$

Choose $\mu_2 \in (0,1)$ such that $2M_1M_3 \exp(-\lambda\delta/\epsilon)/\lambda \leq \nu/8$ for $\epsilon \in (0, \mu_2)$. It follows that, for $t \in S$ and $\epsilon \in (0, \mu_2)$, $M_1 \int_0^{(t-t_0)/\epsilon} \exp(-\lambda z) \|\boldsymbol{v}_i(t-\epsilon z) - \boldsymbol{v}_i(t)\| \mathrm{d}z \leq \nu/4$. On the other hand, we have

$$
\begin{aligned}
& \left\| \int_0^{(t-t_0)/\epsilon} \exp\left(\bar{\boldsymbol{A}}_{ci} z\right) \boldsymbol{v}_i(t) \mathrm{d}z + \bar{\boldsymbol{A}}_{ci}^{-1} \boldsymbol{v}_i(t) \right\| \\
&= \left\| \bar{\boldsymbol{A}}_{ci}^{-1} \exp\left(\frac{1}{\epsilon} \bar{\boldsymbol{A}}_{ci}(t-t_0)\right) \boldsymbol{v}_i(t) \right\| \\
&\leq M_1M_3 \left\|\bar{\boldsymbol{A}}_{ci}^{-1}\right\| \exp\left(-\frac{\lambda(t-t_0)}{\epsilon}\right) \\
&\leq M_1M_3 \left\|\bar{\boldsymbol{A}}_{ci}^{-1}\right\| \exp\left(-\frac{\lambda(\tau-t_0)}{\epsilon}\right),
\end{aligned}
$$

because $\lambda > 0$, $\epsilon > 0$ and $t \geq \tau$. Choose $\mu_3 \in (0,1)$ such that $M_1M_3\|\bar{\boldsymbol{A}}_{ci}^{-1}\| \exp(-\lambda(\tau-t_0)/\epsilon) < \nu/4$ for $\epsilon \in (0, \mu_3)$. Let $\mu = \min\{\mu_1, \mu_2, \mu_3\}$. Then, combining the above inequalities, we conclude that, for $t \in S$ and $\epsilon \in (0, \mu)$, $\|\zeta_i(t)/\epsilon + \bar{\boldsymbol{A}}_{ci}^{-1}\boldsymbol{v}_i(t)\| \leq \nu$, which implies that $\zeta_i(t)/\epsilon$ converges uniformly to $-\bar{\boldsymbol{A}}_{ci}^{-1}\boldsymbol{v}_i(t)$ on S. The uniform convergence of $\dot{\zeta}_i(t)$ to $\boldsymbol{0}$ on $[\tau, \infty) \setminus (J + (0, \xi))$ follows immediately. If $\boldsymbol{v}_i(t)$ is uniformly continuous, then J is empty, which implies that $J + (0, \xi)$ is empty. Thus, it follows that the convergence of $\dot{\zeta}_i(t)$ to $\boldsymbol{0}$ is uniform on $[\tau, \infty)$, and the proof of the theorem is complete. $\square$

6. Numerical example

In this section, we illustrate the effectiveness of our proposed high-gain approximate differentiator based sliding-mode observer with a numerical example. We consider a linear time-invariant system modeled by

$$
\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -5 & -10 & -10 & -5 \end{bmatrix}
$$

and

$$
\boldsymbol{B}_1 = \boldsymbol{B}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{C}^{\top} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.
$$

The initial condition is $\boldsymbol{x}(0) = [0.5\ 0.5\ 0.5\ \ -0.5\ \ -0.5]^{\top}$, and the known input $\boldsymbol{u}_1$ is set to be a zero vector. The unknown input $\boldsymbol{u}_2$ consists of a square wave with amplitude 1 and frequency 1 Hz,

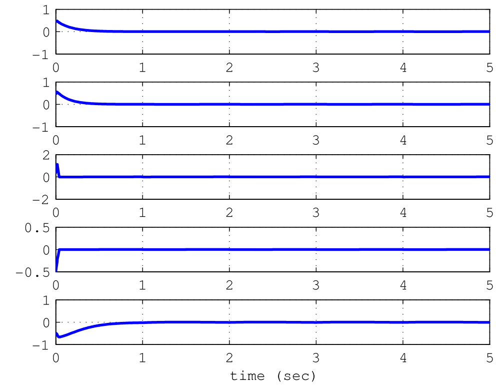

Fig. 1. State estimation errors: e_1 through e_5.

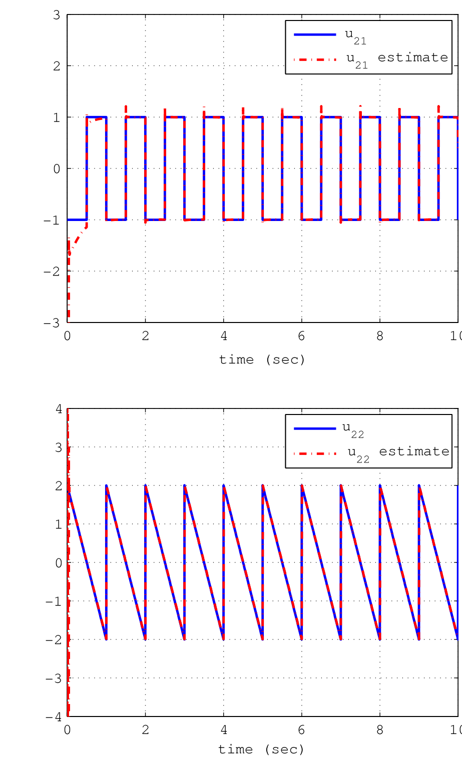

Fig. 2. Unknown input reconstruction.

and a sawtooth signal with amplitude 2 and frequency 1 Hz. It is easy to check that for this system $\operatorname{rank}(\boldsymbol{C}\boldsymbol{B}_2) \neq \operatorname{rank}\boldsymbol{B}_2$ because $\boldsymbol{c}_1\boldsymbol{B}_2 = \boldsymbol{0}$. Thus, we choose $\gamma_1 = r_1 = 3$ such that

$$
\boldsymbol{C}_a = \begin{bmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_1\boldsymbol{A} \\ \boldsymbol{c}_1\boldsymbol{A}^2 \\ \boldsymbol{c}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}
$$

is of full rank with $\operatorname{rank}(\boldsymbol{C}_a\boldsymbol{B}_2) = \operatorname{rank}\boldsymbol{B}_2$. We employ a high-gain observer to estimate the auxiliary outputs $y_{12} = \boldsymbol{c}_1\boldsymbol{A}\boldsymbol{x}$ and $y_{13} = \boldsymbol{c}_1\boldsymbol{A}^2\boldsymbol{x}$. The design parameters of the high-gain observer are selected to be $\alpha_{11} = 3$, $\alpha_{12} = 3$, $\alpha_{13} = 1$ and $\epsilon = 0.001$. Now we use the estimates of the auxiliary outputs to construct the sliding-mode observer described by (11). We choose $\kappa = 2.0659$ and $\eta = 50$ to obtain

$$
\boldsymbol{L}_a = \begin{bmatrix} 6 & 1 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2.0659 & 0 \\ 0 & 0 & 0 & 2.0659 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

and

$$\boldsymbol{F}_a = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

We set the initial states of the sliding-mode observer to be zero, that is, $\hat{\boldsymbol{x}}(0) = \boldsymbol{0}$, and select $S_{11} = S_{12} = S_{13} = 1.5$. The state estimation errors are shown in Fig. 1. The unknown input reconstruction is illustrated in Fig. 2.

7. Conclusions

In this paper, a sliding-mode observer has been designed for linear systems with unknown inputs, where the observer matching condition is not satisfied. High-gain approximate differentiators were employed to estimate auxiliary outputs that are then used in the sliding-mode observer design. The proposed high-gain approximate differentiator based sliding-mode observer has simple architecture, which lends itself to easy implementation and analysis.

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