State and Unknown Input Observers for Nonlinear Systems with Bounded Exogenous Inputs

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***Abstract*—A systematic design methodology for state observers for a large class of nonlinear systems with bounded exogenous inputs (disturbance inputs and sensor noise) is proposed. The nonlinearities under consideration are characterized by an incremental quadratic constraint parametrized by a set of multiplier matrices. Linear matrix inequalities are developed to construct observer gains which ensure that a performance output based on the state estimation error satisfies a prescribed degree of accuracy. Furthermore, conditions guaranteeing estimation of the unknown inputs in the absence of sensor noise to arbitrary degrees of accuracy are provided. The proposed scheme is illustrated with two numerical examples.**

I. Introduction

A problem of paramount importance in control is to design observers to estimate the state of nonlinear systems in the presence of disturbance inputs and measurement noise. Due to noisy measurements and external disturbances, it may not be possible to exactly reconstruct the plant state. In such environments, it is imperative to design observers which perform at pre-specified performance levels. For example, for some biomedical systems, it may be satisfactory to estimate a subset of the states of the system while attenuating the effect of modeling uncertainties and sensor noise [1]. Furthermore, in secure communication or cyber-physical systems, an observer that detects and reconstructs the measurement or state attack (noise) signal enables a counterattack protocol to be constructed to mitigate the attack signal [2], [3]. Hence, the development of observers which reconstruct states along with exogenous disturbances becomes a critical problem.

A variety of methods for constructing observers for nonlinear systems are available in the control literature. In [4], the problem of designing observers for continuous-time systems with Lipschitz nonlinearities is investigated using input-to-state stability properties (ISS). The error system is decomposed into cascaded systems, and linear matrix inequalities (LMIs) are proposed to solve the resulting observer design problem. A recent paper [5] extends this work to more general classes of nonlinear systems using ISS Lyapunov functions [6]. In [7], the authors solve a modified Riccati equation to design observers for Lipschitz nonlinear systems that are robust to uncertainties having large magnitudes. Alternative designs for Lipschitz nonlinear systems are proposed in [8] using the $\mathcal{H}_\infty$ formalism and [9] which uses the differential mean-value-theorem to formulate linear parameter varying descriptions of the error system. The above results were extended in [10] to take into account the presence of monotone nonlinearities. In [11], a prediction-correction formulation is proposed which yields observation error bounds within zonotopes using interval analysis. Other interval estimation techniques are reported in [12], [13] for nonlinear time-varying systems. In [14], a robust observer is designed to handle diverging parametric uncertainties generated by bounded disturbances using an extension of Barbalat's lemma and principles of persistent excitation. Parameter estimation drift has also been investigated in [15] using an $\mathcal{H}_\infty$ framework, and in [16] using a robust adaptive observer based on a normalized dead zone. Robust adaptive observers are also proposed in [17] based on input-to-state practical stability (ISpS) Lyapunov functions. A switched gain approach is proposed in [18] to reduce the effects of measurement noise. This problem was also investigated in [19] using an observer bank with adapting gains. In [20], the adaptive high gain approach was extended to a class of triangular systems. A high-gain observer for a broad class of nonlinear systems is proposed in [21] exploiting homogeneity and dynamic scaling. An implementation of high-gain observers using an extended Kalman filter is proposed in [22] to address issues arising from sensitivity to measurement noise. A discussion of high-gain observers with an application to the control of minimum-phase systems is presented in [23]. An extended state observer is proposed in [24] by transforming the error dynamics into a suitable form. A recent paper addresses the case when a diffeomorphism does not exist that transforms the nonlinear system into normal form [25]. For nonlinear systems with bounded Jacobian matrices, an observer design method is presented in [26] with applications to slip angle estimation. A high-gain observer based state-feedback controller is designed for nonlinear systems in [27]. Observers for nonlinear systems are constructed using differential geometry and contraction analysis in [28]–[30].

Unknown input reconstruction for linear systems has been studied using linear observers in [31]–[33], and sliding mode and higher-order sliding mode observers are proposed in [34]–[37] for unknown input reconstruction. Other unknown input observer architecture for globally Lipschitz nonlinear systems are proposed in [38]–[48]. Extensions to monotone nonlinearities and slope-restricted nonlinearities are discussed in [49]–[51].

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In this paper, we present a systematic framework for the design of observers for a wide class of nonlinear dynamical systems in the presence of bounded exogenous inputs. The nonlinearities under consideration are characterized by a set of symmetric matrices. The advantage of this characterization is that it provides a general framework for representing many common nonlinearities occurring in physical models while providing less conservative feasibility results, as demonstrated in [52]. Sufficient conditions employing these matrix characterizations are provided in the form of LMIs for the observer design. Guarantees of the observer performance and disturbance attenuation properties in the presence of exogenous inputs are discussed. Finally, sufficient conditions for the reconstruction of the unknown exogenous inputs are provided.

The paper is organized as follows. In Section II, we present the notation used in the remainder of the paper. We discuss the class of nonlinear systems considered in Section III, where, additionally, the problem is stated formally, and the proposed observer is presented. Section IV investigates the error dynamics of the observer and formulates a basic matrix inequality result for observer design with guaranteed performance. This basic inequality is utilized in constructing linear matrix inequalities for the computation of observer gains for different classes of nonlinearities in Section V. Conditions for unknown input estimation with linear and nonlinear error dynamics are provided separately in Section VI, and connections are made with results in the previous unknown input estimation literature for linear systems. The proposed methodology is tested on two simulation examples in Section VII. Contributions of this paper are highlighted and conclusions are offered in Section VIII.

II. Notation

We denote by $\mathbb{N}$ the set of natural numbers, $\mathbb{R}$ the set of real numbers and, for any $m, n \in \mathbb{N}$, $\mathbb{R}^n$ is the set of real n-vectors and $\mathbb{R}^{n\times m}$ is the set of real $n \times m$ matrices. For any matrix P, its transpose is denoted by $P^\top$, and its induced Euclidean norm (equivalently, maximum singular value) by $\|P\|$. For any vector $v \in \mathbb{R}^n$, we consider the 2-norm $\|v\| = (v^\top v)^{\frac{1}{2}}$. For a bounded function $v(\cdot) : \mathbb{R} \to \mathbb{R}^n$, we consider the norm $\|v(\cdot)\|_\infty = \sup_{t\in\mathbb{R}} \|v(t)\|$. A matrix M is symmetric if $M = M^\top$ and we use the star notation to avoid rewriting symmetric terms, that is,

$$\begin{bmatrix} M_a & M_b \\ \star & M_c \end{bmatrix} \equiv \begin{bmatrix} M_a & M_b \\ M_b^\top & M_c \end{bmatrix}.$$

We also use $\mathcal{D}f$ to denote the derivative of a differentiable function f.

III. Problem Statement and Proposed Solution

We consider a **nonlinear system** with disturbance input, measured output and measurement noise described by

$$\boxed{\begin{aligned} \dot{x} &= Ax + B_n f(t, y, q) + Bw + g(t, y) \quad &(1a)\\ q &= C_q x + D_{qn} f(t, y, q) + D_{qw} w \quad &(1b)\\ y &= Cx + Dw \quad &(1c) \end{aligned}}$$

where $t \in \mathbb{R}$ is the **time variable**, $x(t) \in \mathbb{R}^{n_x}$ is the **state**, $y(t) \in \mathbb{R}^{n_y}$ is the **measured output** and the vector $w(t) \in \mathbb{R}^{n_w}$ models the **disturbance input** and the **measurement noise** combined into one term; we refer to it as the **exogenous input**. This exogenous input is unknown but bounded.

The vector $f(t, y, q) \in \mathbb{R}^{n_f}$ models nonlinearities of known structure, but because this term depends on the state x (through q), it cannot be instantaneously determined from measurements. The vector $g(t, y) \in \mathbb{R}^{n_g}$ represents nonlinearities which can be calculated instantaneously from measurements. The matrices $A \in \mathbb{R}^{n_x\times n_x}, B \in \mathbb{R}^{n_x\times n_w}, B_n \in \mathbb{R}^{n_x\times n_f}$ $C \in \mathbb{R}^{n_y\times n_x}$ and $D \in \mathbb{R}^{n_y\times n_w}$ describe how the variables x, w and f enter the state and output equations of the system. The vector $q \in \mathbb{R}^{n_q}$ is a state-dependent argument of the nonlinearity f, and is characterized by the matrices $C_q \in \mathbb{R}^{n_q\times n_x}$, $D_{qn} \in \mathbb{R}^{n_q\times n_f}$ and $D_{qw} \in \mathbb{R}^{n_q\times n_w}$ as shown in (1b). The quantity q is not measured instantaneously and has to be estimated. The $D_{qw}w$ term enables us to model an exogenous input acting through the nonlinearity f.

Example 1. The implicit definition of q is useful in modeling some nonlinear systems. For example, consider the system:

$$\begin{aligned} \dot{x}_1 &= x_2 + w_1 \\ \dot{x}_2 &= 0.5 \sin(x_1 + \dot{x}_2 + w_3) \\ y &= x_1 + w_2. \end{aligned}$$

With $q := x_1 + \dot{x}_2 + w_3$ and $f(t, y, q) := \sin(q)$ the system can be described by

$$\begin{aligned} \dot{x}_1 &= x_2 + w_1 \\ \dot{x}_2 &= 0.5 f(t, y, q) \\ q &= x_1 + 0.5 f(t, y, q) + w_3 \\ y &= x_1 + w_2. \end{aligned}$$

Remark 1. Note that system description (1) is a compact representation of a system containing an input disturbance w_x and measurement noise w_y. That is, the system

$$\begin{aligned} \dot{x} &= Ax \ + B_n f(t, y, q) + \tilde{B} w_x + g(t, y) \\ q &= C_q x + D_{qn} f(t, y, q) + \tilde{D}_{qw} w_x \\ y &= Cx + \tilde{D} w_y \end{aligned}$$

can be written compactly in the format (1) by defining

$$w = \begin{bmatrix} w_x^\top & w_y^\top \end{bmatrix}^\top$$

and constructing B, D_{qw}, and D from $\tilde{B}$, $\tilde{D}_{qw}$, and $\tilde{D}$. We use this compact representation for clarity in presentation.

The compact representation is illustrated further using the following example.

Example 2. Consider the system:

$$\begin{aligned} \dot{x}_1 &= x_1 + 2w_x \\ \dot{x}_2 &= 2x_1 + \exp(-x_1 + 3w_x) \\ y &= 3x_2 - 5w_y. \end{aligned}$$

Then, $w = \begin{bmatrix} w_x & w_y \end{bmatrix}^\top$ and

$$B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \qquad D_{qw} = \begin{bmatrix} 3 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & -5 \end{bmatrix}.$$

Remark 2. If the plant has a control input or other known inputs, this can be included in the term g.

The plant trajectories are defined as continuous functions $x(\cdot) : [t_0, t_1) \to \mathbb{R}^{n_x}$, with $0 < t_1 \leq \infty$ satisfying equations (1a)–(1b).

In this paper we characterize nonlinearities via their incremental multiplier matrices.

Definition 1(Incremental Multiplier Matrices). A symmetric matrix $M \in \mathbb{R}^{(n_q+n_f)\times(n_q+n_f)}$ is an **incremental multiplier matrix** (δMM) for f if it satisfies the following **incremental quadratic constraint** (δQC) for all $t \in \mathbb{R}$, $y \in \mathbb{R}^{n_y}$ and $q_1, q_2 \in \mathbb{R}^{n_q}$:

$$\begin{bmatrix}\Delta q\\ \Delta f\end{bmatrix}^\top M \begin{bmatrix}\Delta q\\ \Delta f\end{bmatrix} \geq 0, \tag{2}$$

where $\Delta q \triangleq q_1 - q_2$ and $\Delta f \triangleq f(t, y, q_1) - f(t, y, q_2)$.

Example 3. Consider the nonlinearity $f(t, y, q) = q|q|$, which is not globally Lipschitz. The nonlinearity f satisfies the inequality

$$(q_1|q_1| - q_2|q_2|)(q_1 - q_2) \geq 0,$$

for any $q_1, q_2 \in \mathbb{R}$. This can be rewritten as

$$\begin{bmatrix} q_1 - q_2\\ q_1|q_1| - q_2|q_2|\end{bmatrix}^\top \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \begin{bmatrix} q_1 - q_2\\ q_1|q_1| - q_2|q_2|\end{bmatrix} \geq 0\,.$$

Hence, an δMM for $f(q)$ is

$$M = \kappa \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

for any $\kappa > 0$.

Remark 3. Clearly, if a nonlinearity has a non-zero incremental multiplier matrix, it is not unique. Any positive scaling of an δMM is also an δMM.

To ensure that the implicit description of q results in a unique explicit description of q we need the following assumption on the nonlinearity f and D_{qn}.

Assumption 1. The nonlinear function f satisfies an incremental quadratic constraint. Furthermore, there exists a continuous function ψ such that for every $t \in \mathbb{R}$, $y \in \mathbb{R}^{n_y}$ and $\tilde{q} \in \mathbb{R}^{n_q}$, the equation

$$q = \tilde{q} + D_{qn} f(t, y, q)$$

has a unique solution given by $q = \psi(t, y, \tilde{q})$, that is,

$$\psi(t, y, \tilde{q}) = \tilde{q} + D_{qn} f(t, y, \psi(t, y, \tilde{q})). \tag{3}$$

Thus, in model (1), q is explicitly given by

$$q = \psi(t, y, C_q x + D_{qw} w). \tag{4}$$

The utility of characterizing nonlinearities using incremental multipliers is that we can generalize our observer design strategy for a broad class of nonlinear systems. Incremental multiplier matrices for many common nonlinearities are provided in [52], [53].

IV. Observer Design

In this section, we propose an **observer** architecture and provide conditions that guarantee observer performance in the presence of exogenous inputs.

A. Proposed observer and error dynamics

Our proposed observer is described by

$$\boxed{\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + B_n f(t, y, \hat{q}) + L_1(\hat{y} - y) + g(t, y) && \text{(5a)}\\
\hat{q} &= C_q\hat{x} + D_{qn} f(t, y, \hat{q}) + L_2(\hat{y} - y) && \text{(5b)}\\
\hat{y} &= C\hat{x} && \text{(5c)}
\end{aligned}}$$

where $\hat{x}(t)$ is an **estimate** of the state $x(t)$ of the plant and $\hat{x}(0) = \hat{x}_0$ is an initial estimate of the initial plant state $x_0 = x(0)$. Such an observer is simply a copy of the plant modified with two correction terms: a **Luenberger-type correction term** $L_1(\hat{y} - y)$ characterized by the gain matrix $L_1 \in \mathbb{R}^{n_x \times n_y}$ and an **injection term** $L_2(\hat{y} - y)$ acting on the nonlinearity, characterized by the gain matrix $L_2 \in \mathbb{R}^{n_q \times n_y}$.

Let $e \triangleq \hat{x} - x$ be the **state estimation error** and let

$$\Delta q \triangleq \hat{q} - q.$$

Then, the **observer error dynamics** are described by

$$\dot{e} = (A + L_1 C)e + B_n \Delta f - (B + L_1 D)w \tag{6a}$$
$$\Delta f = f(t, y, q + \Delta q) - f(t, y, q) \tag{6b}$$
$$\Delta q = (C_q + L_2 C)e + D_{qn}\Delta f - (D_{qw} + L_2 D)w. \tag{6c}$$

B. $\mathcal{L}_\infty$-stability with specified performance

Let

$$\boxed{z = He} \tag{7}$$

be a user-defined **performance output** associated with the state estimation error. Next, we define $\mathcal{L}_\infty$-stability with performance level γ.

Definition 2. *The nonlinear system* (6) *with performance output* (7) *is* ***globally uniformly*** $\mathcal{L}_\infty$***-stable with performance level*** γ *if the following conditions are satisfied.*

(P1) **Global uniform exponential stability.** *The zero-input system (obtained by setting $w \equiv 0$) is globally uniformly exponentially stable about the origin.*

(P2) **Global uniform boundedness of the error state.** *For every initial condition $e(t_0) = e_0$, and every bounded exogenous input $w(\cdot)$, there exists a bound $\beta(e_0, \|w(\cdot)\|_\infty)$ such that*

$$\|e(t)\| \leq \beta(e_0, \|w(\cdot)\|_\infty)$$

for all $t \geq t_0$.

(P3) **Output response for zero initial error state.** *For zero initial error, $e(t_0) = 0$, and every bounded exogenous input $w(\cdot)$, we have*

$$\|z(t)\| \leq \gamma\|w(\cdot)\|_\infty$$

for all $t \geq t_0$.

(P4) **Global ultimate output response.** *For every initial condition, $e(t_0) = e_0$, and every bounded exogenous input $w(\cdot)$, we have*

$$\limsup_{t\to\infty} \|z(t)\| \le \gamma \|w(\cdot)\|_\infty \tag{8}$$

Moreover, convergence is uniform with respect to t_0.

For additional background and definitions, we refer the interested reader to [54].

Our **objective** is to design an observer of the form (5) for the nonlinear system (1) with unknown exogenous inputs whilst ensuring that the observer error dynamics are $\mathcal{L}_\infty$-stable with a specified performance level for a given performance output, described in (7). To this end, the following result is useful.

Lemma 1. *Consider a system with exogenous input w and performance output z described by*

$$\begin{aligned} \dot{e} &= F(t,e,w) \qquad (9a)\\ z &= G(t,e) \qquad (9b) \end{aligned}$$

where $t \in \mathbb{R}$, $e(t) \in \mathbb{R}^{n_x}$, $w(t) \in \mathbb{R}^{n_w}$ and $z(t) \in \mathbb{R}^{n_z}$. Suppose there exists a differentiable function $V : \mathbb{R}^{n_x} \to \mathbb{R}$ and scalars $\alpha, \beta_1, \beta_2 > 0$ and $\mu_1, \mu_2 \ge 0$ such that

$$\beta_1 \|e\|^2 \le V(e) \le \beta_2 \|e\|^2 \tag{10}$$

and

$$\begin{aligned} \mathcal{D}V(e)\, F(t,e,w) &\le -2\alpha \left(V(e) - \mu_1 \|w\|^2\right) \qquad (11a)\\ \|G(t,e)\|^2 &\le \mu_2 V(e) \qquad (11b) \end{aligned}$$

for all $t \ge 0$, $e \in \mathbb{R}^{n_x}$ and $w \in \mathbb{R}^{n_w}$, where $\mathcal{D}V$ denotes the derivative of V. Then system (9) is globally uniformly $\mathcal{L}_\infty$-stable with performance level

$$\gamma = \sqrt{\mu_1 \mu_2}. \tag{12}$$

Proof: Consider any solution $e(\cdot) : [t_0, t_1) \to \mathbb{R}^{n_x}$ of (9a) with $e(t_0) = e_0$ and let $\tilde{V} := V(e(t))$. We begin by noting that

$$\dot{\tilde{V}} = \mathcal{D}V(e)\, F(t,e,w).$$

Condition (11a) implies that

$$\dot{\tilde{V}} \le -2\alpha(\tilde{V} - \mu_1 \|w\|^2). \tag{13}$$

Multiplying both sides of (13) by $e^{2\alpha t}$ and re-arranging yields

$$\frac{d}{dt}(e^{2\alpha t}\tilde{V}) \le 2\alpha\mu_1 \|w\|^2 e^{2\alpha t} \le 2\alpha\mu_1 \|w(\cdot)\|_\infty^2 e^{2\alpha t},$$

which, upon integrating over $[t_0, t]$ results in

$$\begin{aligned} e^{2\alpha t}\tilde{V}(t) - e^{2\alpha t_0}\tilde{V}(t_0) &\le \int_{t_0}^{t} 2\alpha\mu_1 \|w(\cdot)\|_\infty^2 e^{2\alpha\tau} d\tau\\ &= \mu_1 \|w(\cdot)\|_\infty^2 (e^{2\alpha t} - e^{2\alpha t_0}). \qquad (14) \end{aligned}$$

Now, we multiply both sides of (14) by $e^{-2\alpha t}$ and re-arrange to obtain

$$\begin{aligned} \tilde{V}(t) &\le e^{-2\alpha(t-t_0)}\tilde{V}(t_0) + \mu_1 \|w(\cdot)\|_\infty^2 (1 - e^{-2\alpha(t-t_0)})\\ &\le e^{-2\alpha(t-t_0)}\tilde{V}(t_0) + \mu_1 \|w(\cdot)\|_\infty^2 \end{aligned}$$

for $t \ge t_0$. Recalling that $\tilde{V}(t) = V(e(t))$, we finally obtain that

$$V(e(t)) \le e^{-2\alpha(t-t_0)} V(e_0) + \mu_1 \|w(\cdot)\|_\infty^2. \tag{15}$$

Using (10) we see that

$$\beta_1 \|e(t)\|^2 \le e^{-2\alpha(t-t_0)} \beta_2 \|e_0\|^2 + \mu_1 \|w(\cdot)\|_\infty^2.$$

Hence,

$$\|e(t)\| \le \sqrt{\beta_2/\beta_1}\; e^{-\alpha(t-t_0)} \|e_0\| + \sqrt{\mu_1/\beta_1}\; \|w(\cdot)\|_\infty. \tag{16}$$

This yields Properties (P1) and (P2) of Definition 2. Substituting (9b) and (11b) into (15) gives

$$\|z(t)\|^2 \le \mu_2 e^{-2\alpha(t-t_0)} V(e_0) + \mu_2\mu_1 \|w(\cdot)\|_\infty^2.$$

Therefore,

$$\|z(t)\| \le \sqrt{\mu_2 V(e_0)}\; e^{-\alpha(t-t_0)} + \gamma \|w(\cdot)\|_\infty, \tag{17}$$

where $\gamma = \sqrt{\mu_1\mu_2}$. This yields Properties (P3) and (P4) of Definition 2. ■

C. Sufficient conditions for observer design with guaranteed performance

We now use the above result to obtain sufficient conditions on the observer gain matrices so that the error system (6) has the desired performance.

Theorem 1. *Consider plant (1) and suppose that there are scalars $\alpha > 0$, $\mu \ge 0$, a symmetric matrix $P \succ 0$, matrices L_1, L_2 and an incremental multiplier matrix M for f such that the matrix inequalities*

$$\Phi + \Gamma^\top M \Gamma \preceq 0 \tag{18a}$$

$$\begin{bmatrix} P & \star \\ H & \mu I \end{bmatrix} \succeq 0 \tag{18b}$$

are satisfied where

$$\Phi = \begin{bmatrix} \Phi_{11} & PB_n & -P(B + L_1 D) \\ \star & 0 & 0 \\ \star & 0 & -2\alpha I \end{bmatrix} \tag{19}$$

with

$$\Phi_{11} = P(A + L_1 C) + (A + L_1 C)^\top P + 2\alpha P \tag{20}$$

and

$$\Gamma = \begin{bmatrix} C_q + L_2 C & D_{qn} & -D_{qw} - L_2 D \\ 0 & I & 0 \end{bmatrix}. \tag{21}$$

Then observer (5) results in error dynamics which are $\mathcal{L}_\infty$-stable with performance level

$$\gamma = \sqrt{\mu}$$

for the performance output $z = He$.

Proof: We will show that the error dynamics

$$\dot{e} = (A + L_1 C)e + B_n \Delta f - (B + L_1 D)w$$

with performance output He satisfy the hypotheses of Lemma 1 with $V(e) = e^\top P e$. This choice of V satisfies the Rayleigh inequality

$$\lambda_{\min}(P)\|e\|^2 \le V(e) \le \lambda_{\max}(P)\|e\|^2$$

for any $e \in \mathbb{R}^{n_x}$. Hence, (10) is satisfied with $\beta_1 = \lambda_{\min}(P)$ and $\beta_2 = \lambda_{\max}(P)$.

The time-derivative of $V(e)$ evaluated along a trajectory of the error dynamics is

$$\mathcal{D}V(e)\dot{e} = 2e^\top P\left[(A + L_1 C)e + B_n \Delta f - (B + L_1 D)w\right]. \tag{22}$$

With $\xi = \begin{bmatrix} e^\top & \Delta f^\top & w^\top \end{bmatrix}^\top$, it follows from (22) and (18a) that

$$\begin{aligned} &\mathcal{D}V(e)\dot{e} + 2\alpha V - 2\alpha\|w\|^2 + \xi^\top \Gamma^\top M \Gamma \xi \\ &= \xi^\top \left(\Phi + \Gamma^\top M \Gamma\right) \xi \le 0. \end{aligned} \tag{23}$$

Recalling the description of Δq in (6c), we see that

$$\Gamma\xi = \begin{bmatrix} \Delta q \\ \Delta f \end{bmatrix}$$

and, since M is an incremental multiplier matrix for f,

$$\xi^\top \Gamma^\top M \Gamma \xi = \begin{bmatrix} \Delta q \\ \Delta f \end{bmatrix}^\top M \begin{bmatrix} \Delta q \\ \Delta f \end{bmatrix} \ge 0.$$

It now follows from (23) that

$$\mathcal{D}V(e)\,\dot{e} \le -2\alpha(V - \|w\|^2),$$

that is, (11a) holds with $\mu_1 = 1$.

Since $\mu > 0$, taking a Schur complement in (18b) results in

$$P - \mu^{-1} H^\top H \succeq 0.$$

Pre-multiplying this inequality by by $e^\top$ and post-multiplying it by e, we get

$$e^\top H^\top H e - \mu e^\top P e \le 0,$$

which implies that

$$\|He\|^2 \le \mu V(e),$$

that is, (11b) holds with $G(t, e) = He$ and $\mu_2 = \mu$. Using Lemma 1, we obtain the desired result with $\gamma = \sqrt{\mu}$. ■

D. LMI conditions with fixed L_2

The matrix inequality (18b) is linear in the variables P and μ. However matrix inequality (18a) is not an LMI in the variables α, P, M, L_1, L_2 and M. One way to obtain LMI conditions is to fix α and L_2 and introduce a new variable $Y_1 \triangleq PL_1$. Then, inequality (18a) can be rewritten as in (24), which is an LMI in P, Y_1 and M, where Γ is defined in (21). This is summarized in the following corollary of Theorem 1.

Corollary 1. *Consider plant (1) and suppose that, for a given scalar $\alpha > 0$ and matrix L_2, there is a scalar $\mu \ge 0$, a symmetric matrix $P \succ 0$, a matrix Y_1 and an incremental multiplier matrix M for f such that the LMI conditions*

$$\Xi + \Gamma^\top M \Gamma \preceq 0 \tag{24}$$

and (18b) hold, where

$$\Xi = \begin{bmatrix} \Xi_{11} & PB_n & -PB - Y_1 D \\ \star & 0 & 0 \\ \star & 0 & -2\alpha I \end{bmatrix} \tag{25}$$

with

$$\Xi_{11} = PA + A^\top P + Y_1 C + C^\top Y_1^\top + 2\alpha P, \tag{26}$$

and Γ is defined in (21). Then the observer (5) with

$$L_1 = P^{-1} Y_1 \tag{27}$$

has error dynamics which are $\mathcal{L}_\infty$-stable with performance level $\gamma = \sqrt{\mu}$ for output He.

Proof: With L_1 given by (27), inequality (24) is equivalent to inequality (18a) of Theorem 1. ■

Remark 4. With α and L_2 fixed, conditions (24) and (18b) are LMIs in P, Y_1, M and μ. An issue that remains to be addressed is the selection of the positive scalar α. A line search algorithm can be used to ensure that the selection of α is optimal in some sense. A larger α ensures faster convergence. In Section V we consider the problem of obtaining LMI conditions when L_2 is not fixed.

E. Estimating the performance output to arbitrary accuracy

In this subsection we present conditions which ensure that the effect of the unknown input w on the performance output z can be made arbitrarily small by appropriate observer construction.

Lemma 2. *Consider plant (1) with $D, D_{qw} = 0$. Suppose that there is a scalar $\alpha > 0$, a symmetric matrix $\tilde{P} \succ 0$, matrices $\tilde{L}_1$, L_2, $\tilde{F}$ and an incremental multiplier matrix $\tilde{M}$ for f such that*

$$\begin{bmatrix} \tilde{\Phi}_{11} & \tilde{P}B_n \\ \star & 0 \end{bmatrix} + \Gamma_0^\top \tilde{M} \Gamma_0 \preceq 0 \tag{28a}$$

$$B^\top \tilde{P} - \tilde{F} C = 0 \tag{28b}$$

where

$$\tilde{\Phi}_{11} = \tilde{P}(A + \tilde{L}_1 C) + (A + \tilde{L}_1 C)^\top \tilde{P} + 2\alpha \tilde{P} \tag{29}$$

and

$$\Gamma_0 = \begin{bmatrix} C_q + L_2 C & D_{qn} \\ 0 & I \end{bmatrix}. \tag{30}$$

Then for any matrices $H_i \in \mathbb{R}^{n_{z_i} \times n_x}$ and scalars $\mu_i > 0$, $i = 1, \dots, N$, there is a symmetric matrix $P \succ 0$, matrices L_1 and L_2 and an incremental multiplier matrix M for f such that inequalities (18a) and

$$\begin{bmatrix} P & \star \\ H_i & \mu_i I \end{bmatrix} \succeq 0 \qquad \text{for} \quad i = 1, \dots, N \tag{31}$$

hold.

Proof: Suppose that (28) holds for some scalar $\alpha > 0$ and matrices $\tilde{P} = \tilde{P}^\top \succ 0$, $\tilde{L}_1$, L_2, $\tilde{F}$ and an incremental multiplier matrix $\tilde{M}$ for f. Consider any matrix $H \in \mathbb{R}^{n_z \times n_x}$ and scalar $\mu > 0$. First, select $\nu > 0$ such that

$$\nu \tilde{P} \succeq \mu_i^{-1} H_i^\top H_i \qquad \text{for} \quad i = 1, \dots, N \tag{32}$$

and define

$$P = \nu \tilde{P} \succ 0. \tag{33}$$

Then inequalities (32) are equivalent to (31). Defining

$$M = \nu\tilde{M} \quad \text{and} \quad F = \nu\tilde{F}$$

we note that (28) holds with $\tilde{P}$, $\tilde{M}$ and $\tilde{F}$ replaced by P, M and F, that is,

$$\begin{bmatrix} P(A+\tilde{L}_1C)+(A+\tilde{L}_1C)^\top P+2\alpha P & PB_n \\ \star & 0 \end{bmatrix} + \Gamma_0^\top M\Gamma_0 \preceq 0 \tag{34a}$$

$$B^\top P - FC = 0. \tag{34b}$$

Note that M is a scaled version of $\tilde{M}$, and is, therefore, an incremental multiplier matrix for f. Choosing any ζ satisfying

$$\zeta \geq \|F\|^2/4\alpha \tag{35}$$

we have $F^\top F \leq 4\alpha\zeta I$; hence $\frac{1}{2\alpha}C^\top F^\top FC - 2\zeta C^\top C \preceq 0$. Using (34b), we obtain

$$\frac{1}{2\alpha}PBB^\top P - 2\zeta C^\top C \preceq 0. \tag{36}$$

From (34a) and (36), we have

$$\begin{bmatrix} \tilde{\Xi}_{11} + \frac{1}{2\alpha}PBB^\top P & PB_n \\ \star & 0 \end{bmatrix} + \Gamma_0^\top M\Gamma_0 \preceq 0 \tag{37}$$

where

$$\tilde{\Xi}_{11} = P(A+\tilde{L}_1C)+(A+\tilde{L}_1C)^\top P+2\alpha P-2\zeta C^\top C.$$

With

$$L_1 = \tilde{L}_1 - \zeta P^{-1}C^\top \tag{38}$$

inequality (37) results in

$$\begin{bmatrix} \Phi_{11} + \frac{1}{2\alpha}PBB^\top P & PB_n \\ \star & 0 \end{bmatrix} + \Gamma_0^\top M\Gamma_0 \preceq 0 \tag{39}$$

where Φ_{11} is given by (20). Since $\alpha > 0$, using a Schur complement result, we see that inequality (39) is equivalent to

$$\begin{bmatrix} \begin{pmatrix} \Phi_{11} & PB_n \\ \star & 0 \end{pmatrix} & \begin{pmatrix} -PB \\ 0 \end{pmatrix} \\ \star & -2\alpha I \end{bmatrix} + \begin{bmatrix} \Gamma_0^\top M\Gamma_0 & 0 \\ 0 & 0 \end{bmatrix} \preceq 0$$

Since $D, D_{qw} = 0$, this inequality is precisely inequality (18a). ■

Combining Theorem 1 and Lemma 2 yields the following result on the existence of observers to *arbitrarily attenuate the effect of the exogenous input* on any performance output.

Corollary 2. *Suppose that the hypotheses Theorem 2 are satisfied. Then for any matrices $H_i \in \mathbb{R}^{n_{z_i}\times n_x}$ and scalars $\gamma_i > 0$, $i = 1, \ldots, N$, there exists an observer of the form (5) such that, for each $i = 1, \ldots, n$, the error dynamics are $\mathcal{L}_\infty$-stable with performance level γ_i for output H_ie.*

V. LMI Conditions for Computation of L_1 and L_2

Recall that condition (2) for observer design is not an LMI in the variables P, L_1, L_2 and M. In this section, we consider particular classes of nonlinearities and by introducing new matrix variables Y_1 and Y_2 we obtain LMIs (for fixed α) which are equivalent to (2). This yields LMI conditions for simultaneously computing the gains L_1 and L_2. We consider nonlinearities whose multiplier matrices have the structure

$$M = T^T\tilde{M}T \tag{40}$$

where $\tilde{M}$ is a symmetric matrix that belongs to a set of matrices which have some special structure and

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \tag{41}$$

is a fixed matrix with $T_{11} \in \mathbb{R}^{n_q\times n_q}$, $T_{12} \in \mathbb{R}^{n_q\times n_f}$, $T_{21} \in \mathbb{R}^{n_f\times n_q}$, and $T_{22} \in \mathbb{R}^{n_f\times n_f}$.

We will need the following technical result for developing linear matrix inequalities in the remainder of this section.

Lemma 3. *Consider the inequality,*

$$\Xi + \tilde{\Gamma}^\top\tilde{M}\tilde{\Gamma} \preceq 0 \tag{42}$$

where Ξ is given in (25) and

$$\tilde{\Gamma} \triangleq \begin{bmatrix} T_{11}C_q + \Sigma L_2C & S_{12} & -T_{11}D_{qw} - \Sigma L_2D \\ T_{21}C_q & S_{22} & -T_{21}D_{qw} \end{bmatrix} \tag{43}$$

with

$$\Sigma \triangleq T_{11} - S_{12}S_{22}^{-1}T_{21}, \tag{44a}$$

$$\begin{bmatrix} S_{12} \\ \hline S_{22} \end{bmatrix} \triangleq \begin{bmatrix} T_{12} + T_{11}D_{qn} \\ \hline T_{22} + T_{21}D_{qn} \end{bmatrix}. \tag{44b}$$

Then, with

$$L_1 = P^{-1}Y_1 + B_nS_{22}^{-1}T_{21}L_2 \tag{45}$$

and M given by (40) and (41), inequalities (42) and (18a) are equivalent.

Proof: Introduce the invertible matrix

$$Q \triangleq \begin{bmatrix} I & 0 & 0 \\ L_{12}C & I & -L_{12}D \\ 0 & 0 & I \end{bmatrix}, \tag{46}$$

where

$$L_{12} \triangleq -S_{22}^{-1}T_{21}L_2. \tag{47}$$

With M satisfying (40), it should be clear that (18a) is equivalent to

$$Q^\top\Phi Q + Q^\top\Gamma^\top T^\top\tilde{M}T\Gamma Q \preceq 0\,. \tag{48}$$

One can readily show that

$$Q^\top\Phi Q = \begin{bmatrix} \tilde{\Phi}_{11} & PB_n & -PB - P(L_1 + B_nL_{12})D \\ \star & 0 & 0 \\ \star & 0 & -2\alpha I \end{bmatrix} \tag{49}$$

where

$$\tilde{\Phi}_{11} = PA + A^\top P + P(L_1 + B_nL_{12})C + C^\top(L_1 + B_nL_{12})^\top P + 2\alpha P.$$

With L_1 given by (45), we see that

$$P(L_1 + B_n L_{12}) = P(L_1 - B_n S_{22}^{-1} T_{21} L_2) = Y_1$$

and recalling (26) and (25) we obtain $\tilde{\Phi}_{11} = \Xi_{11}$ and

$$Q^\top \Phi Q = \Xi. \tag{50}$$

Using

$$T_{21} L_2 + S_{22} L_{12} = 0 \tag{51}$$

and

$$T_{11} L_2 + S_{12} L_{12} = (T_{11} - S_{12} S_{22}^{-1} T_{21}) L_2 = \Sigma L_2\,, \tag{52}$$

one can compute that

$$T \Gamma Q = \tilde{\Gamma}, \tag{53}$$

where $\tilde{\Gamma}$ is given by (43). It now follows from (53) and (50) that inequalities (42) and (18a) are equivalent. ∎

In our analysis of specific classes of multiplier matrices, we require the following condition on T.

Assumption 2. *T and $T_{22} + T_{21} D_{qn}$ are invertible.*

We will need the following technical lemma from [52].

Lemma 4. *If Assumption 2 holds then, the matrix Σ defined by (44) is invertible.*

A. Block Symmetric Anti-Triangular δMM

We now consider the situation in which the incremental multiplier matrices M for f have the form,

$$M = T^\top \begin{bmatrix} 0 & M_{12} \\ M_{12}^\top & M_{22} \end{bmatrix} T \tag{54}$$

where $M_{12} \in \mathbb{R}^{n_q \times n_f}$, $M_{22} \in \mathbb{R}^{n_f \times n_f}$ are variable matrices with (M_{12}, M_{22}) in some set and $T \in \mathbb{R}^{(n_q+n_f)\times(n_q+n_f)}$ is a fixed matrix. It is assumed that M_{12} has full column rank. Thus

$$\tilde{M} = \begin{bmatrix} 0 & M_{12} \\ M_{12}^\top & M_{22}. \end{bmatrix} \tag{55}$$

An example of a nonlinearity with incremental multiplier matrices of the above form follows.

Example 4. Consider a nonlinearity f, which, for some scalar $c \geq 0$, satisfies

$$\Delta q^\top \Delta f \geq c \Delta q^\top \Delta q \tag{56}$$

for all t, y, q_1 and q_2. Hence, any matrix of the form

$$M = \kappa \begin{bmatrix} -2cI & I \\ I & 0 \end{bmatrix}$$

with $\kappa > 0$ is an incremental multiplier matrix for f. These matrices can be written as in (54) with

$$T = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \qquad M_{12} = \kappa I, \qquad M_{22} = -2\kappa c I$$

and M_{12} has full column rank.

Remark 5. Incrementally passive nonlinearities are a class of nonlinearities that satisfy (56) with $c = 0$. For example, the nonlinearity $q|q|$ is characterized by an anti-triangular block symmetric matrix of the form (54), as demonstrated in Example 3.

Lemma 5. *Suppose Assumption 2 holds and M_{12} is full column rank. Consider the matrix inequality*

$$\Xi + \Gamma_1^\top \tilde{M} \Gamma_1 + \Gamma_1^\top \Gamma_2 + \Gamma_2^\top \Gamma_1 \preceq 0 \tag{57}$$

where Ξ is given by (25), $\tilde{M}$ is given by (55) and

$$\Gamma_1 = \begin{bmatrix} T_{11} C_q & S_{12} & -T_{11} D_{qw} \\ T_{21} C_q & S_{22} & -T_{21} D_{qw} \end{bmatrix} \tag{58}$$

$$\Gamma_2 = \begin{bmatrix} 0 & 0 & 0 \\ Y_2 C & 0 & -Y_2 D \end{bmatrix} \tag{59}$$

with S_{12}, S_{22} and Σ given by (44). Then, with L_1 given by (45),

$$L_2 = (M_{12}^\top \Sigma)^\dagger Y_2 \tag{60}$$

and M given by (54) and (41), inequalities (57) and (18a) are equivalent.[1]

Proof: Recalling Lemma 3, we can prove this result by showing that (57) and (42) are equivalent. Recalling (43) we see that

$$\tilde{\Gamma} = \Gamma_1 + \tilde{\Gamma}_2, \tag{61}$$

where

$$\tilde{\Gamma}_2 = \begin{bmatrix} \Sigma L_2 C & 0 & -\Sigma L_2 D \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that

$$\tilde{M} \tilde{\Gamma}_2 = \begin{bmatrix} 0 & 0 & 0 \\ M_{12}^\top \Sigma L_2 C & 0 & -M_{12}^\top \Sigma L_2 D \end{bmatrix}.$$

With L_2 given by (60), we see that $M_{12}^\top \Sigma L_2 = Y_2$; hence

$$\tilde{M} \tilde{\Gamma}_2 = \begin{bmatrix} 0 & 0 & 0 \\ Y_2 C & 0 & -Y_2 D \end{bmatrix} = \Gamma_2.$$

Also, $\tilde{\Gamma}_2^\top \tilde{M} \tilde{\Gamma}_2 = 0$. Hence,

$$\begin{aligned} \tilde{\Gamma}^\top \tilde{M} \tilde{\Gamma} &= (\Gamma_1 + \tilde{\Gamma}_2)^\top \tilde{M} (\Gamma_1 + \tilde{\Gamma}_2) \\ &= \Gamma_1^\top \tilde{M} \Gamma_1 + \Gamma_1^\top \tilde{M} \tilde{\Gamma}_2 + \tilde{\Gamma}_2^\top \tilde{M} \Gamma_1 + \tilde{\Gamma}_2^\top \tilde{M} \tilde{\Gamma}_2 \\ &= \Gamma_1^\top \tilde{M} \Gamma_1 + \Gamma_1^\top \Gamma_2 + \Gamma_2^\top \Gamma_1, \end{aligned}$$

which implies that (57) and (42) are equivalent. ∎

Remark 6. The right inverse $(M_{12}^\top \Sigma)^\dagger$ exists because M_{12} is full column rank and, by Lemma 4, the matrix Σ is nonsingular.

Remark 7. Note that, for a fixed α, inequality (57) is an LMI in the variables P, Y_1, Y_2, M_{12} and M_{22}. Hence for plants whose nonlinear term f has multiplier matrices of the type considered in this section one can obtain observers of the form (5) by solving LMIs (57) and (18b) for P, Y_1, Y_2, M_{12}, M_{22} and letting L_1 and L_2 be given by (45) and (60).

[1] Here $(M_{12}^\top \Sigma)^\dagger$ denotes a right-inverse of the matrix $M_{12}^\top \Sigma$.

B. Block Diagonalizable δMM

We now consider the situation in which the incremental multiplier matrices M for f have the form,

$$M = T^\top \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix} T, \tag{62}$$

where the matrix variables $M_{11} \in \mathbb{R}^{n_q \times n_q}$, $M_{22} \in \mathbb{R}^{n_f \times n_f}$ are symmetric with (M_{11}, M_{22}) in some set. Furthermore, suppose that

$$M_{11} \succ 0\,,$$

and $T \in \mathbb{R}^{(n_q+n_f)\times(n_q+n_f)}$ is a fixed matrix. Hence,

$$\tilde{M} = \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix}. \tag{63}$$

Thus we consider multiplier matrices which are block diagonalizable using a fixed congruence transformation T.

Example 5. Consider an **incrementally sector bounded** nonlinearity f, which, for some scalars a and b, satisfies

$$a \le \frac{\Delta f}{\Delta q} \le b \tag{64}$$

for all $t, y, q_1 \neq q_2$ where all quantities are real numbers, $\Delta q = q_2 - q_1$ and $\Delta f = f(t, y, q_2) - f(t, y, q_1)$. Satisfaction of (64) is equivalent to satisfaction of

$$(\Delta f - a\Delta q)(b\Delta q - \Delta f) \ge 0$$

for all t, y, q_1 and q_2. Thus any matrix of the form

$$M = \kappa \begin{bmatrix} -ab & (a+b)/2 \\ \star & -1 \end{bmatrix} \tag{65}$$

with $\kappa > 0$ is an incremental multiplier matrix for f. These matrices can be expressed as in (62) with

$$T = \begin{bmatrix} (a-b)/2 & 0 \\ -(a+b)/2 & 1 \end{bmatrix}, \quad M_{11} = \kappa, \quad M_{22} = -\kappa$$

and $M_{11} \succ 0$.

We are now ready to formulate LMI conditions to compute the observer gains when the incremental multiplier matrix is of the form (62).

Lemma 6. *Suppose Assumption 2 holds and $M_{11} \succ 0$. Consider the inequality,*

$$\begin{bmatrix} \Xi + \phi_2^T M_{22}\phi_2 & \star \\ \phi_1 & -M_{11} \end{bmatrix} \preceq 0, \tag{66}$$

where Ξ is given in (25) and

$$\phi_1 = \begin{bmatrix} M_{11}T_{11}C_q + Y_2C & M_{11}S_{12} & -M_{11}T_{21}D_q - Y_2D \end{bmatrix} \tag{67a}$$

$$\phi_2 = \begin{bmatrix} T_{21}C_q & S_{22} & -T_{21}D_{qw} \end{bmatrix} \tag{67b}$$

with S_{12}, S_{22} and Σ given by (44). Then, with L_1 given by (45),

$$L_2 = (M_{11}\Sigma)^{-1}Y_2 \tag{68}$$

and M given by (62) and (41), inequalities (66) and (18a) are equivalent.

Proof: Recalling Lemma 3, we can prove this result by showing that (66) and (42) are equivalent where $\tilde{M}$ is given by (63). Since $M_{11} \succ 0$, we can use Schur complements to obtain that (66) is equivalent to

$$\Xi + \phi_1^\top M_{11}^{-1}\phi_1 + \phi_2^\top M_{22}\phi_2 \preceq 0. \tag{69}$$

Recalling (43) we see that

$$\tilde{\Gamma}^\top \tilde{M}\tilde{\Gamma} = \tilde{\phi}_1^\top M_{11}^{-1}\tilde{\phi}_1 + \phi_2^\top M_{22}\phi_2, \tag{70}$$

where ϕ_1 is given by (67b) and

$$\tilde{\phi}_1 = M_{11}\begin{bmatrix} T_{11}C_q + \Sigma L_2 C & S_{12} & -T_{11}D_{qw} - \Sigma L_2 D \end{bmatrix}.$$

With L_2 given by (68), we see that $M_{11}\Sigma L_2 = Y_2$; hence

$$\begin{aligned}\tilde{\phi}_1 &= \begin{bmatrix} M_{11}T_{11}C_q + Y_2C & M_{11}S_{12} & -M_{11}T_{21}D_q - Y_2D \end{bmatrix} \\ &= \phi_1. \end{aligned} \tag{71}$$

It now follows from (70) and (71) that inequalities (66) and (42) are equivalent. ■

Remark 8. Note that, for a fixed α, inequality (66) is an LMI in the variables P, Y_1, Y_2, M_{11} and M_{22}. Hence for plants whose nonlinear term f has multiplier matrices of the type considered in this section, one can obtain observers of the form (5) by solving LMIs (66) and (18b) for P, Y_1, Y_2, M_{11}, M_{22} and letting L_1 and L_2 be given by (45) and (68).

C. A General case

This case combines the previous two cases. Consider the situation in which the incremental multiplier matrices M for f have the form,

$$M = T^\top \begin{bmatrix} E_1\mathfrak{M}_{11}E_1^\top & \mathfrak{M}_{12}E_2^\top \\ \star & \mathfrak{M}_{22} \end{bmatrix} T \tag{72}$$

where $\mathfrak{M}_{11} \in \mathbb{R}^{n_{q_1}\times n_{q_1}}$, $\mathfrak{M}_{12} \in \mathbb{R}^{n_q \times n_{f_1}}$ $\mathfrak{M}_{22} \in \mathbb{R}^{n_f\times n_f}$ are variable matrices with $(\mathfrak{M}_{11}, \mathfrak{M}_{12}, \mathfrak{M}_{22})$ in some set with $\mathfrak{M}_{11} \succ 0$, $\begin{bmatrix} E_1 & \mathfrak{M}_{12}\end{bmatrix}$, has full column rank, and $T \in \mathbb{R}^{(n_q+n_f)\times(n_q+n_f)}$ is a fixed matrix. Thus

$$\tilde{M} = \begin{bmatrix} E_1\mathfrak{M}_{11}E_1^T & \mathfrak{M}_{12}E_2^T \\ \star & \mathfrak{M}_{22}. \end{bmatrix} \tag{73}$$

Example 6. Consider a two-dimensional vector-valued nonlinearity, where the first component is an incrementally sector bounded nonlinearity as in Example 5. The second component consists of a nonlinearity satisfying the conditions of Example 4, that is,

$$f(t, y, q) = \begin{bmatrix} f_1(t, y, q_1) \\ f_2(t, y, q_2) \end{bmatrix},$$

where

$$a \le \Delta f_1/\Delta q_1 \le b$$
$$\Delta q_2 \Delta f_2 \ge c\Delta q_2^2$$

for all nonzero Δq_1 where all quantities are real numbers and a, b, c are known constants. Using the results in Examples 4 and 5, any matrix of the form

$$M = \kappa \begin{bmatrix} -\kappa_1 ab & 0 & \kappa_1(a+b)/2 & 0 \\ \star & -2\kappa_2 c & 0 & \kappa_2 \\ \star & \star & -\kappa_1 & 0 \\ \star & \star & \star & 0 \end{bmatrix} \tag{74}$$

with $\kappa_1, \kappa_2 > 0$ is an incremental multiplier matrix for f. The family of matrices of the form (74) parameterized by $\kappa > 0$ can be expressed as in (40), with

$$T = \begin{bmatrix} (a-b)/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -(a+b)/2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and

$$\tilde{M} = \begin{bmatrix} \kappa_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa_2 \\ 0 & 0 & -\kappa_1 & 0 \\ 0 & \kappa_2 & 0 & -2\kappa_2 c \end{bmatrix}.$$

Thus for these matrices, $\tilde{M}$ can be expressed as in (72) with

$$\mathfrak{M}_{11} = \kappa_1 , \quad \mathfrak{M}_{12} = \begin{bmatrix} 0 \\ \kappa_2 \end{bmatrix} , \quad E_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \quad E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with $\mathfrak{M}_{11} \succ 0$ and $\begin{bmatrix} E_1 & \mathfrak{M}_{12} \end{bmatrix}$ being full column rank. □

We now present the following proposition.

Proposition 1. *Suppose Assumption 2 holds, $\mathfrak{M}_{11} \succ 0$ and $\begin{bmatrix} E_1 & \mathfrak{M}_{12} \end{bmatrix}$ is full column rank. Consider the inequality,*

$$\begin{bmatrix} \Xi + \Gamma_1^\top \tilde{M} \Gamma_1 + \Gamma_1^\top \Gamma_2 + \Gamma_2^\top \Gamma_1 & \star \\ \phi_1 & -\mathfrak{M}_{11} \end{bmatrix} \preceq 0 \tag{75}$$

where Ξ is given by (25), $\tilde{M}$ is given by (73) and

$$\Gamma_1 = \begin{bmatrix} T_{11} C_q & S_{12} & -T_{11} D_{qw} \\ T_{21} C_q & S_{22} & -T_{21} D_{qw} \end{bmatrix}$$
$$\Gamma_2 = \begin{bmatrix} E_1 Y_{21} C & 0 & -E_1 Y_{21} D \\ E_2 Y_{22} C & 0 & -E_2 Y_{22} D \end{bmatrix}$$
$$\phi_1 = \begin{bmatrix} Y_{21} C & 0 & -Y_{21} D \end{bmatrix}$$

with S_{12}, S_{22} and Σ given by (44). Then, with L_1 given by (45),

$$L_2 = \Sigma^{-1} \begin{bmatrix} E_1^\top \\ \mathfrak{M}_{12}^\top \end{bmatrix}^\dagger \begin{bmatrix} \mathfrak{M}_{11}^{-1} Y_{21} \\ Y_{22} \end{bmatrix}$$

and M given by (72) and (41), inequalities (75) and (18a) are equivalent.

Proof: This can be proven using the techniques employed in the proofs of Lemmas 5 and 6. ■

Remark 9. Note that, for a fixed α, inequality (75) is an LMI in the variables P, Y_1, Y_{21}, Y_{22}, $\mathfrak{M}_{11}$, $\mathfrak{M}_{12}$, and $\mathfrak{M}_{22}$.

VI. Exogenous Input Estimation

In this section, we consider the problem of estimating components of the exogenous input w in addition to the plant state. Using the observers presented in the previous sections, we demonstrate how one can obtain an estimate of components of the exogenous input, given that w and its derivative $\dot{w}$ are bounded. Specifically, we are concerned with the estimation of

$$v \triangleq \mathcal{H} w,$$

. We require the following assumption.

Assumption 3. *There exists a matrix Θ such that $\Theta B = \mathcal{H}$.*

Remark 10. If all components of w are to be estimated, then $\mathcal{H} = I$ and Θ will need to be a left-inverse of B, necessitating B to have full column rank.

Herein, we will show that

$$\boxed{\hat{v} \triangleq \Theta L_1 (\hat{y} - y)} \tag{76}$$

is an estimate of v.

For simplicity, we consider nonlinear functions with q as their only argument. This yields a plant of the form

$$\dot{x} = Ax + B_n f(q) + Bw + g(t,y) \tag{77a}$$
$$q = C_q x + D_{qn} f(q) + D_{qw} w \tag{77b}$$
$$y = Cx + Dw. \tag{77c}$$

For such plants the proposed observers are described by

$$\dot{\hat{x}} = A\hat{x} + B_n f(\hat{q}) + L_1(\hat{y} - y) + g(t,y) \tag{78a}$$
$$\hat{q} = C_q \hat{x} + D_{qn} f(\hat{q}) + L_2 (\hat{y} - y) \tag{78b}$$
$$\hat{y} = C\hat{x}. \tag{78c}$$

We make the following additional assumptions for the class of systems considered in this subsection.

Assumption 4. *The function f is differentiable and there is a scalar κ_1 such that $\|\mathcal{D}f(q)\| \leq \kappa_1$ for all $q \in \mathbb{R}^{n_q}$ and $\kappa_1 \|D_{qn}\| < 1$.*

Assumption 5. *The derivative $\dot{x}$ of the state of plant (77) is bounded.*

Remark 11. Assumption 4 implies that the nonlinearity f is globally κ_1-Lipschitz, that is, $\|f(\hat{q}) - f(q)\| \leq \kappa_1 \|\hat{q} - q\|$ for all $\hat{q}, q \in \mathbb{R}^{n_q}$. This assumption also guarantees that there exists a scalar κ_2 such that $\|\mathcal{D}f(\hat{q}) - \mathcal{D}f(q)\| \leq \kappa_2$ for all $\hat{q}, q \in \mathbb{R}^{n_q}$.

A. Estimating the exogenous input $v = \mathcal{H}w$

We now state and prove the following theorem that provides sufficiency conditions for the estimation of the exogenous input v to a specifed degree of accuracy.

Theorem 2. *Consider plant (77) satisfying Assumptions 3, 4 and 5. Suppose there exist scalars $\alpha > 0$, $\mu_1, \mu_2, \mu_3 \geq 0$, a symmetric matrix $P \succ 0$, matrices L_1 and L_2 and an incremental multiplier matrix M for f, such that*

$$\hat{\Phi} + \hat{\Gamma}^\top M \hat{\Gamma} \preceq 0 \tag{79a}$$
$$\begin{bmatrix} P & \star \\ \Theta A & \mu_1 I \end{bmatrix} \succeq 0 \tag{79b}$$
$$\begin{bmatrix} P & \star \\ C_q + L_2 C & \mu_2 I \end{bmatrix} \succeq 0 \tag{79c}$$
$$\begin{bmatrix} P & \star \\ \Theta & \mu_3 I \end{bmatrix} \succeq 0 \tag{79d}$$

where

$$\hat{\Phi} = \begin{bmatrix} \Phi_{11} & PB_n & -P(B + L_1 D) & PB_n \\ \star & 0 & 0 & 0 \\ \star & 0 & -2\alpha I & 0 \\ \star & 0 & 0 & -2\alpha I \end{bmatrix}$$

Φ_{11} is given by (20) and

$$\hat{\Gamma} = \begin{bmatrix} C_q + L_2 C & D_{qn} & -D_{qw} - L_2 D & 0 \\ 0 & I & 0 & 0 \end{bmatrix}.$$

Then observer (78) with $\hat{v}$ given by (76) yields the exogenous input estimation error bound:

$$\limsup_{t\to\infty} \|\hat{v}-v\| \leq \gamma_1\|w(\cdot)\|_\infty + \gamma_2\|\dot{w}(\cdot)\|_\infty + \gamma_3\|\dot{x}(\cdot)\|_\infty, \tag{80}$$

where

$$\gamma_1 = \sqrt{\mu_1} + \tilde{\kappa}_1\|\Theta B_n\|(\sqrt{\mu_2} + \|D_{qw} + L_2 D\|) \tag{81a}$$
$$\gamma_2 = \sqrt{\mu_3}(1 + \tilde{\kappa}_2\|D_{qw}\|) \tag{81b}$$
$$\gamma_3 = \sqrt{\mu_3}\tilde{\kappa}_2\|C_q\| \tag{81c}$$
$$\tilde{\kappa}_1 = \kappa_1(1-\kappa_1\|D_{qn}\|)^{-1} \tag{81d}$$
$$\tilde{\kappa}_2 = \kappa_2(1-\kappa_1\|D_{qn}\|)^{-1}. \tag{81e}$$

Proof: Let
$$\Delta f = f(\hat{q}) - f(q).$$

Recalling the plant description (77) and the observer description (78), we see that the error dynamics can be described by
$$\dot{e} = Ae + B_n\Delta f + L_1(\hat{y} - y) - Bw. \tag{82}$$

Multiplying this equation by Θ introduced in Assumption 3 and recalling definition (76) of $\hat{v}$ results in
$$\Theta\dot{e} = \Theta A e + \Theta B_n \Delta f + \hat{v} - \mathcal{H}w.$$

Hence,
$$\hat{v} - v = \Theta\dot{e} - \Theta A e - \Theta B_n\Delta f.$$

This implies that
$$\|\hat{v} - v\| \leq \|\Theta A e\| + \|\Theta B_n\|\|\Delta f\| + \|\Theta\dot{e}\|. \tag{83}$$

As a consequence of Assumption 4,
$$\|f(\hat{q})-f(q)\| \leq \kappa_1\|\hat{q} - q\|.$$

Since
$$\begin{aligned}\|\hat{q} - q\| &= \|(C_q+L_2C)e + D_{qn}\Delta f - (D_{qw}+L_2D)w\|\\ &\leq \|(C_q+L_2C)e\| + \|D_{qn}\|\|\Delta f\|\\ &\quad+ \|D_{qw}+L_2D\|\|w\|,\end{aligned}$$

we see that
$$\|\Delta f\| \leq \tilde{\kappa}_1(\|(C_q+L_2C)e\| + \|(D_{qw}+L_2D)\|\|w\|),$$

where $\tilde{\kappa}_1$ is given by (81d). Therefore,
$$\begin{aligned}\|\hat{v} - v\| &\leq \tilde{\kappa}_1\|\Theta B_n\|\|(D_{qw}+L_2D)\|\|w\|\\ &\quad+ \tilde{\kappa}_1\|\Theta B_n\|\|(C_q + L_2C)e\| + \|\Theta A e\| + \|\Theta\dot{e}\|.\end{aligned} \tag{84}$$

We now use Theorem 1 to obtain ultimate bounds on $\|\Theta A e\|$, $\|(C_q + L_2 C)e\|$ and $\|\Theta\dot{e}\|$.

We first note that inequality (79a) implies that inequality (18a) of Theorem 1 holds. It now follows from Theorem 1 that satisfaction of (79a) and (79b) implies that the error system (82) with performance output $\Theta A e$ is $\mathcal{L}_\infty$-stable with performance level $\sqrt{\mu_1}$. Hence the ultimate bound on $\Theta A e$ satisfies
$$\limsup_{t\to\infty} \|\Theta A e(t)\| \leq \sqrt{\mu_1}\|w(\cdot)\|_\infty. \tag{85}$$

In a similar fashion, satisfaction of (79a) and (79c) implies that
$$\limsup_{t\to\infty} \|(C_q + L_2C)e(t)\| \leq \sqrt{\mu_2}\|w(\cdot)\|_\infty. \tag{86}$$

To obtain an ultimate an ultimate bound $\|\Theta\dot{e}\|$, we note that for the plant (77) and the corresponding observer (78), we get the error dynamics:
$$\dot{e} = (A + L_1C)\dot{e} + B_n\Delta f - (B + L_1D)w \tag{87}$$
$$\hat{q} - q = (C_q + L_2C)e + D_{qn}\Delta f - (D_{qw} + L_2D)w. \tag{88}$$

Next, we take the time-derivative of (87) to obtain
$$\ddot{e} = (A + L_1C)\dot{e} + B_n\frac{d\Delta f}{dt} - (B + L_1D)\dot{w}.$$

Note that
$$\begin{aligned}\frac{d\Delta f}{dt} &= \frac{d}{dt}(f(\hat{q}) - f(q))\\ &= \mathcal{D}f(\hat{q})\dot{\hat{q}} - \mathcal{D}f(q)\dot{q},\\ &= \mathcal{D}f(\hat{q})(\dot{\hat{q}} - \dot{q}) + (\mathcal{D}f(\hat{q}) - \mathcal{D}f(q))\,\dot{q}\end{aligned}$$

and
$$\begin{aligned}\dot{\hat{q}} - \dot{q} &= (C_q + L_2C)\dot{e} + D_{qn}\frac{d\Delta f}{dt} - (D_{qw} + D)\dot{w}\\ \dot{q} &= C_q\dot{x} + D_{qn}\mathcal{D}f(q)\dot{q} + D_{qw}\dot{w}.\end{aligned}$$

Hence,
$$\ddot{e} = (A + L_1C)\dot{e} + B_n\tilde{f}(t,\tilde{q}) + B_{\tilde{w}}\tilde{w} \tag{89a}$$
$$\tilde{f}(t,\tilde{q}) = \mathcal{D}f(\hat{q}(t))\tilde{q} \tag{89b}$$
$$\tilde{q} = (C_q + L_2C)\dot{e} + D_{qn}\tilde{f}(t,\tilde{q}) - (D_{qw} + L_2D)\dot{w}, \tag{89c}$$

where $\tilde{w} = \begin{bmatrix}\dot{w} & \tilde{w}_2\end{bmatrix}$ with
$$\tilde{w}_2 = (\mathcal{D}f(\hat{q}) - \mathcal{D}f(q))\,\dot{q} \tag{90}$$

and
$$B_{\tilde{w}} = \begin{bmatrix}-B - L_1D & B_n\end{bmatrix}.$$

Since M is an incremental multiplier for f, [53, Lemma 4.4] tells us that
$$\begin{bmatrix}\tilde{q}\\ \mathcal{D}f(\hat{q})\tilde{q}\end{bmatrix}^\top M \begin{bmatrix}\tilde{q}\\ \mathcal{D}f(\hat{q})\tilde{q}\end{bmatrix} \geq 0$$

for all $\hat{q}, \tilde{q} \in \mathbb{R}^{n_q}$. Hence M is an incremental multiplier matrix for $\tilde{f}$.

Considering (89) as a system with state $\dot{e}$, exogenous input $\tilde{w}$ and performance output $\Theta\dot{e}$, it follows from Theorem 1 that satisfaction of (79a) and (79d) implies $\mathcal{L}_\infty$-stability with performance level $\sqrt{\mu_3}$. Hence the ultimate bound on $\Theta\dot{e}(t)$ satisfies
$$\limsup_{t\to\infty} \|\Theta\,\dot{e}(t)\| \leq \sqrt{\mu_3}\|\tilde{w}(\cdot)\|_\infty. \tag{91}$$

To obtain a bound on $\tilde{w}$, we first use (89a) and Assumption 4 to obtain
$$\begin{aligned}\|\dot{q}\| &\leq \|C_q\|\|\dot{x}\| + \|D_{qn}\|\|\mathcal{D}f(q)\|\|\dot{q}\| + \|D_{qw}\|\|\dot{w}\|\\ &\leq \|C_q\|\||\dot{x}\| + \kappa_1\|D_{qn}\|\|\dot{q}\| + \|D_{qw}\|\|\dot{w}\|.\end{aligned}$$

Thus,
$$\|\dot{q}\| \leq (1 - \kappa_1\|D_{qn}\|)^{-1}(\|C_q\|\|\dot{x}\| + \|D_{qw}\|\dot{w}\|).$$

Recalling (90) and Remark 11,

$$\begin{aligned}\|\tilde{w}_2\| &\le \|\mathcal{D}f(\hat{q}) - \mathcal{D}f(q)\|\|\dot{q}\| \\ &\le \tilde{\kappa}_2(\|C_q\|\|\dot{x}\| + \|D_{qw}\|\|\dot{w}\|),\end{aligned}$$

where $\tilde{\kappa}_2$ is given by (81e). Therefore,

$$\begin{aligned}\|\tilde{w}\| &\le \|\dot{w}\| + \|\tilde{w}_2\| \\ &\le (1 + \tilde{\kappa}_2\|D_{qw}\|)\|\dot{w}\| + \tilde{\kappa}_2\|C_q\|\|\dot{x}\| \,. \end{aligned} \tag{92}$$

Using (84) and taking limit superiors yields

$$\begin{aligned}\limsup_{t\to\infty}\|\hat{v} - v\| \le& \limsup_{t\to\infty}\|\Theta A e(t)\| \\ &+ \tilde{\kappa}_1\|\Theta B_n\| \limsup_{t\to\infty}\|(C_q + L_2 C)e(t)\| \\ &+ \limsup_{t\to\infty}\|\Theta\dot{e}(t)\| \\ &+ \tilde{\kappa}_1\|\Theta B_n\|\|(D_{qw} + L_2 D)\|\|w(\cdot)\|_\infty . \end{aligned} \tag{93}$$

Recalling (85), (86), (91) and (92) yields the bound in (80). ∎

B. Estimating $v = \mathcal{H}w$ to an arbitrary degree of accuracy

The following result is a simple consequence of Theorem 2.

Corollary 3. *Consider plant (77) satisfying Assumptions 3–5. Suppose that, for every $\mu > 0$ there is a scalar $\alpha > 0$, a symmetric matrix $P \succ 0$, matrices L_1 and L_2 and an incremental multiplier matrix M for f that satisfy (79) with $\mu_1 = \mu_2 = \mu_3 = \mu$ and $D_{qw} + L_2 D = 0$. Let w be a bounded exogenous input with bounded derivative. Then, for any $\varepsilon > 0$, there exists an observer of the form (78) that satisfies*

$$\limsup_{t\to\infty}\|\hat{v}(t) - v(t)\| \le \varepsilon. \tag{94}$$

where $\hat{v}$ is given by (76).

Proof: For a given $\varepsilon > 0$, choose $\mu > 0$ to satisfy

$$\mu \le \left(\frac{\varepsilon}{\tilde{\gamma}_1\|w(\cdot)\|_\infty + \tilde{\gamma}_2\|\dot{w}(\cdot)\|_\infty + \tilde{\gamma}_3\|\dot{x}(\cdot)\|_\infty}\right)^2,$$

where $\tilde{\gamma}_1 = 1 + \tilde{\kappa}_1\|\Theta B_n\|$, $\tilde{\gamma}_2 = 1 + \tilde{\kappa}_2\|D_{qw}\|$, and $\tilde{\gamma}_2 = \tilde{\kappa}_2\|C_q\|$. With $D_{qw} + L_2 D = 0$, it follows from Theorem 2 and our choice of μ above, that observer (78) satisfies the bound (94). ∎

The next result follows from Corollary 3 and Lemma 2.

Corollary 4. *Consider the plant (77) with $D = 0$, $D_{qw} = 0$ and satisfying Assumptions 3–5. Suppose there is a scalar $\alpha > 0$, a symmetric matrix $\tilde{P} \succ 0$, matrices $\tilde{L}_1$, L_2 and $\tilde{F}$ and an incremental multiplier matrix $\tilde{M}$ for f that satisfy (28a) and*

$$\begin{bmatrix} B & -B_n \end{bmatrix}^\top \tilde{P} - \tilde{F}C = 0. \tag{95}$$

Let w be a bounded exogenous input with bounded derivative. Then, for any $\varepsilon > 0$, there exists an observer of the form (78) so that (94) holds where $\hat{v}$ is given by (76).

Remark 12. Inequality (79a) is not an LMI in the variables α, P, L_1, L_2 and M. However, for a fixed α, one can obtain an equivalent LMI using the approaches taken in sections IV-D and V.

We will illustrate in the following theorem that Corollary 4 is a generalized result of well established conditions for constructing unknown input observers for linear systems satisfying the so-called 'matching condition'—see for example: [41], [55]–[57].

Theorem 3. *Consider plant (77) with $f = 0$, $D = 0$ satisfying Assumption 3. Suppose that there is a symmetric matrix $\tilde{P} \succ 0$, and matrices $\tilde{Y}$ and $\tilde{F}$ such that*

$$\begin{aligned}\tilde{P}A + A^\top\tilde{P} + \tilde{Y}C + C^\top\tilde{Y}^\top \prec 0 \\ B^\top\tilde{P} - \tilde{F}C = 0\end{aligned}$$

Let w be a bounded exogenous input with bounded derivative. Then, for any $\varepsilon > 0$, there is an observer of the form (78) with $L_2 = 0$ so that (94) holds where $\hat{v}$ is given by (76).

VII. Examples

In this section, we illustrate the performance of the proposed observers on two nonlinear systems with additive bounded disturbances. All LMIs were solved using the `CVX` [58] package in `MATLAB`.

A. Example 1: Robotic manipulator with Unknown Load

For this example, we use a single-link robotic manipulator described by

$$\dot{x}_1 = x_2, \tag{97a}$$

$$\dot{x}_2 = \frac{k}{J_m}(x_3 - x_1) - \frac{b_V}{J_m}x_2 + \frac{K_\tau}{J_m}u, \tag{97b}$$

$$\dot{x}_3 = x_4, \tag{97c}$$

$$\dot{x}_4 = -\frac{k}{J_l}(x_3 - x_1) - \frac{mgb}{J_l}\sin x_3 + \frac{Fb}{J_l}, \tag{97d}$$

$$y_1 = x_1 + w_s, \tag{97e}$$

This is a modification of the model in [59]. Specifically, we are adding the noise input w_1 to the measurement channels. The parameter descriptions and their nominal values are given in Table I.

TABLE I
Model Parameters for Single-Link Flexible Robot

Parameter	Description	Nominal Value
J_m	Inertia of the motor	0.0037 kg·m^2
J_l	Inertia of the link	0.0093 kg·m^2
m	Mass of the link	0.21 kg
b	Center of mass of the link	0.15 m
k	Elastic constant	0.18 N·m/rad
b_V	Viscous friction coefficient	0.0083 N·m/V
K_τ	Amplifier gain	0.08 N·m/V
g	Acceleration due to gravity	9.81 m/s^2

Let the exogenous input be $w = \begin{bmatrix} F & w_s \end{bmatrix}^\top$. Then the robot model (97) can be represented in the form (1), where the system matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_m} & -\frac{b_V}{J_m} & \frac{k}{J_m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_l} & 0 & -\frac{k}{J_l} & 0 \end{bmatrix}, \; B_u = \begin{bmatrix} 0 \\ \frac{K_\tau}{J_m} \\ 0 \\ 0 \end{bmatrix},$$

$$B_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{-mgb}{J_l} \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{b}{J_l} & 0 \end{bmatrix},$$

and the measurement output matrices are

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

We restrict the unknown load $|F| \leq 0.5$ N and the noise in the measurement channels $|w_s| \leq 0.1$. The performance output is chosen to be

$$z = He = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} e = e_3.$$

The nonlinearity under consideration is $f = \sin x_3$, which can be represented as a convex combination of the matrices,

$$\theta_1 = \begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix}^\top, \; \theta_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^\top.$$

An incremental multiplier matrix for this nonlinearity is

$$M = \operatorname{diag}\left(\begin{bmatrix} 30.66 & -11.96 & -11.96 & -11.96 & -0.01 \end{bmatrix}\right)$$

obtained using the method described in [52, Section 5.2.1]. Note that M is diagonal, hence we can write $\tilde{M} := M$ and $T := I$ in (63). Now, $q = x_3$, so $C_q = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$ and $D_{qn} = 0$. Choosing $\alpha = 1$, we solve (66) to obtain

$$P = \begin{bmatrix} 0.74 & 0.1 & -0.24 & -0.01 \\ 0.1 & 0.4 & -0.25 & 0.04 \\ -0.24 & -0.25 & 0.24 & 0.01 \\ -0.01 & 0.04 & 0.01 & 0.16 \end{bmatrix},$$

and observer gains

$$L_1 = \begin{bmatrix} -45.97 & -944.73 & -295.6 & 401.81 \end{bmatrix}^\top \text{ and } L_2 = 10$$

with $\gamma = 1.8230$. Simulations were performed with initial conditions $x(0) = \begin{bmatrix} 3 & 3 & -3 & -20 \end{bmatrix}^\top$ and $\hat{x}(0) \equiv 0$. The control input was kept constant at $u = 0.2 \sin 4t$ and the exogenous input

$$w = \begin{bmatrix} 0.5 & 0.1 \sin(2t) \end{bmatrix}^\top.$$

Therefore, $\rho_w = 0.5099$ The simulation results are shown in Fig. 1. The performance output z is eventually bounded within the expected performance bounds $\pm \gamma \rho_w = 0.9298$. We also note (by inspection) that the performance bounds are not very conservative, as demonstrated by the z (top) plot in Fig. 1.

B. Example 2: Unknown Input Reconstruction

In this example, we select an active magnetic bearing system that was investigated previously in [49], [52]. A motivation for choosing this example is that no observer of the form (5) exists for the system when $L_2 = 0$. This was illustrated previously in [52]. The model has the form

$$\dot{x} = \begin{bmatrix} x_2 \\ x_3 + x_3|x_3| \\ w \end{bmatrix}, \quad y = x_1. \tag{98}$$

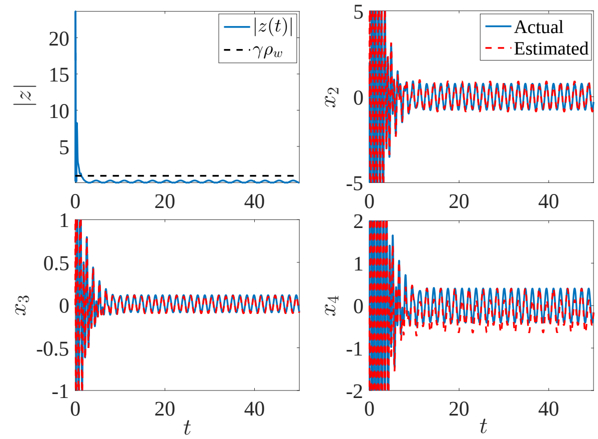

Fig. 1. (*Top*) Ultimate performance output bound. The dashed black line denotes $\gamma\rho_w = 0.9296$. (*Middle, Bottom*) Estimating states of the single-link flexible robot with uncertain load. The actual (blue) and estimated (red) state trajectories are shown.

To illustrate asymptotic estimation of the unknown input signal w, we rewrite the model (98) in the form (1) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad B_n = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C_q = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \; D_{qn} = D = 0,$$

$g = 0$, and $f(q) = q|q|$. Since the nonlinearity f is incrementally passive (see Remark 5), incremental multiplier matrices are given by

$$M = \kappa \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

for any $\kappa > 0$. We choose $z = x_3$, and fix our exponential decay rate $\alpha = 0.5$. Solving (79), we get

$$L_1 = \begin{bmatrix} -13974.8 & -606.6 & -2.3 \times 10^8 \end{bmatrix}^\top, \; L_2 = 9560.2,$$

$\kappa = 0.024$, and $\gamma = 0.061$. As the magnitude of γ is small, we expect to reconstruct the unknown input signal w. The unknown input is a random signal generated in Simulink. We test our proposed observer on the system (98) with the initial conditions $x(0) = \begin{bmatrix} 0.961 & 0.124 & 1.437 \end{bmatrix}^\top$ and $\hat{x}(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$. The response of the proposed observer is shown in Figure 2. We note that the observation error becomes arbitrarily small and the unknown input is estimated to satisfactory accuracy.

VIII. Conclusions

In this paper, we present a method for constructing observers for a class of nonlinear systems with unknown but bounded exogenous inputs (disturbance inputs and measurement noise.) Our contributions include: (i) a convex programming framework for designing observers for nonlinear systems with exogenous inputs; (ii) providing performance guarantees and explicit bounds on the unknown input reconstruction error; (iii) providing conditions for unknown input estimation in nonlinear systems with arbitrary accuracy; and, (iv) for linear

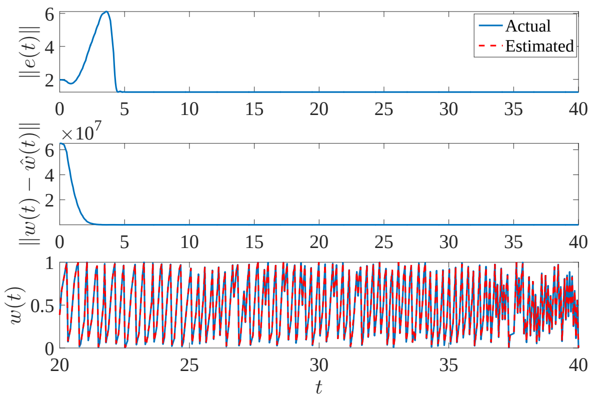

Fig. 2. (*Top*) State estimation error of the nonlinear system in (98). (*Middle*) Unknown input estimation error. The convergence of the norm of $e_w(t) \triangleq w(t) - \hat{w}(t)$ is illustrated. (*Bottom*) Unknown input estimate. The simulated randomly generated unknown input $w(t)$ is shown using the continuous blue line and the reconstruction $\hat{w}(t)$ is depicted with the red dashed line. Note that we plot $w, \hat{w}$ for $t \in [20, 40]$ because the initial estimation error is high.

error dynamics, demonstrating that our proposed LMIs are a generalization of existing conditions for unknown input observers.

Although our method handles a wide variety of nonlinearities, we have used convex relaxations to compute the $\mathcal{L}_\infty$-gain. This convexification introduces conservatism. An open problem is to reduce the implicit conservativeness in the proposed scheme.

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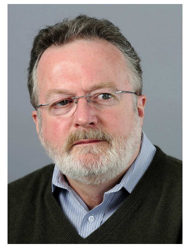

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