Delayed Unknown Input Observers For Discrete-Time Linear Systems With Guaranteed Performance

Ankush Chakrabarty\textsuperscript{a,}*, Raid Ayoub\textsuperscript{b}, Stanislaw H. Zak\textsuperscript{a}, Shreyas Sundaram\textsuperscript{a}

\textsuperscript{a}School of Electrical and Computer Engineering at Purdue University, West Lafayette, IN
\textsuperscript{b}Strategic CAD Labs, Intel Corporation

Abstract

In this paper, a state and unknown input observer is proposed for discrete-time linear systems with bounded unknown inputs and measurement disturbances. The design procedure is formulated using a set of linear matrix inequalities, and leverages delayed (or fixed-lag) estimates. The observer error states and/or user-defined performance outputs are guaranteed to operate at pre-specified performance bounds. Furthermore, by employing sufficiently large delays, the observer is guaranteed to provide exact asymptotic state and input estimates for minimum-phase systems.

Keywords: unknown input observer; attack detection; linear matrix inequalities; delayed estimation; $\ell_\infty$ observer

1. Introduction

The problem of estimating the states of a dynamical system in the presence of unknown disturbance inputs arises in a variety of applications. Examples include security for cyber-physical systems where the attack vector is modeled as an unknown input \cite{1, 2}, networked or decentralized control where control input information at different nodes is unavailable and must be estimated \cite{3}, and fault detection and estimation in large-scale systems \cite{4}. Estimating system states in the presence of the disturbance inputs is usually done through robust state estimation. Such formalisms include set-valued observers \cite{5, 6, 7} or $\mathcal{H}_2$/$\mathcal{H}_\infty$ filtering \cite{8, 9, 10}. For disturbances whose stochastic descriptions are known, robust state estimators such as Kalman filters \cite{11, 12} and other minimum variance filters \cite{13, 14, 15} have been widely investigated.

For the case of completely arbitrary unknown inputs, the current literature contains a variety of unknown input observer (UIO) architectures. A discussion of strong observability and conditions for unknown input reconstruction can be found in \cite{16} and \cite{17}. Necessary and sufficient conditions for the existence of discrete time UIOs are proposed in \cite{18, 19}. A relaxation of these stringent conditions is discussed in \cite{20} by allowing delays in estimation. Recent discrete-time UIO methodologies have continued to explore the systematic use of time delays. For example, geometric conditions for observer design are discussed in \cite{21}.

In \cite{22}, a delayed observer with fictitious outputs is considered that enables state and unknown input reconstruction by exploiting left-invertibility properties of the system. In \cite{23}, the dimensionality issue is improved by constructing delayed observers for a reduced-order system. An adaptation gain term is used in \cite{24} to generate the unknown input using delayed observers. Other formulations of unknown input observers are found in \cite{25, 26, 27}.

One of the key insights established by the existing literature is that perfect asymptotic state and input estimation is possible in the presence of arbitrary unknown inputs \textit{if and only if} the system satisfies a so-called ‘minimum phase’ condition. Furthermore, under this condition, real-time estimation may not be possible unless the system satisfies certain “matching” conditions, which is the reason for introducing delays in estimation. These conditions pose certain challenges. First, one may be interested in estimating the states or inputs of non-minimum phase systems. Second, even if the system is minimum phase, the delay required to completely decouple the effects of the unknown inputs may be larger than desired. Thus, there is a need to construct observers that generate accurate estimates of plant states and unknown exogenous inputs with specified maximum bounds on estimation delays.

In this paper we address the above issues by formulating an observer that provides a guaranteed level of attenuation for \textit{bounded} unknown inputs with any specified maximum estimation delay. Specifically, we provide sufficient conditions in the form of linear matrix inequalities (LMIs) for the construction of the observer gains and compute peak-gain performance bounds on a pre-specified performance output of the observer. Additionally, we propose sufficient conditions which ensure that the unknown inputs can be reconstructed to a specified level of accuracy. Our observer generalizes existing approaches in that...
it achieves perfect attenuation of the unknown inputs if
the system is minimum-phase and the specified delay is
sufficiently large; however, when these conditions are not
satisfied, the performance of our observer degrades grace-
fully with the magnitude of the unknown inputs as long
as the system is detectable (which is a necessary condition
for the construction of any estimator).

2. Notation

We denote by \( \mathbb{R} \) the set of real numbers, \( \mathbb{N} \) the set
of natural numbers, and \( \mathbb{R}^{m \times n} \) the set of \( n \times m \)
matrices for any \( m, n \in \mathbb{N} \). For any vector \( v \in \mathbb{R}^{n} \),
we denote \( \|v\| = \sqrt{v^\top v} \). For a sequence of vectors \( \{v_k\}_{k=k_0} \),
we denote \( \|v\|_\infty \triangleq \sup_{k \geq k_0} \|v_k\| \); consequently, we say
a sequence \( \{v_k\} \) is \( \ell_\infty \) if \( \|v\|_\infty < \infty \). For any matrix
\( P \in \mathbb{R}^{n \times n} \), we denote \( P^\top \) as its transpose, and \( \|P\| \)
as the maximum singular value of \( P \). For a symmetric matrix
\( M = M^\top \), we use the star notation to avoid rewriting
symmetric terms, that is,
\[
\begin{bmatrix}
M_a & \star \\
M_b & M_c
\end{bmatrix} \equiv \begin{bmatrix}
M_a & M_b \\
M_b^\top & M_c
\end{bmatrix}.
\]

3. Problem Statement and Proposed Solution

3.1. Problem Statement

We consider a class of discrete-time linear systems mod-
eld by
\[
\begin{align}
\dot{x}_{k+1} &= Ax_k + Bu_k, \\
y_k &= Cx_k + Dw_k,
\end{align}
\]
(1a)
(1b)
where \( x_k \in \mathbb{R}^{n_x} \) denotes the state vector at the \( k \)-th
sampled time, \( w_k \in \mathbb{R}^{n_w} \) denotes the vector of exogenous
unknown inputs (e.g., disturbance inputs in the state and
output vector fields, measurement noise, attack vectors,
etc.). The measured output is denoted by \( y_k \in \mathbb{R}^{n_y} \). The
matrices \( A, B, C, D \) are of appropriate dimensions. The
initial sample time is \( k = 0 \). We make the following assump-
tions on the class of systems considered in this paper.

**Assumption 1.** The unknown inputs are bounded, that is,
the disturbance input sequence \( \{w_k\} \in \ell_\infty \).

Note that the bounds mentioned in Assumption 1 are
not necessarily known by the designer.

**Assumption 2.** The matrix \( G \triangleq \begin{bmatrix} B \\ D \end{bmatrix} \) has full
column rank.

**Remark 1.** Assumption 2 is mild as the linearly depend-
cent columns of \( G \) can be removed without affecting the
column space through which the exogenous inputs act.

Our objective is to construct a robust observer that
reconstructs the states \( x_k \) of the plant while attenuating
the effect of the unknown exogenous input \( w_k \). As dis-
cussed in Section 1, we will be considering observers that
allow a pre-specified delay in estimation. Before we in-
troduce the specific observer structure, it will be useful to
introduce some notation. For any \( \delta \in \mathbb{N} \), define
\[
Y_{k:k+\delta} \triangleq \begin{bmatrix} y_k^\top & y_{k+1}^\top & \cdots & y_{k+\delta-1}^\top & y_{k+\delta}^\top \end{bmatrix}^\top.
\]
(2)
From the dynamics (1), we obtain
\[
Y_{k:k+\delta} = \Theta_\delta x_k + \Gamma_\delta W_{k:k+\delta},
\]
where \( W_{k:k+\delta} = [w_k^\top \ w_{k+1}^\top \ \cdots \ w_{k+\delta-1}^\top \ w_{k+\delta}^\top]^\top \), and
\[
\Theta_\delta = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\delta-1} \end{bmatrix}, \quad \Gamma_\delta = \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & D \end{bmatrix}.
\]

3.2. Proposed Observer Architecture

Let \( \delta \in \mathbb{N} \) be a constant specifying the maximum delay
(in number of time-steps) that can be tolerated for esti-
mat ing the state of the system. The proposed observer
has the form
\[
\dot{x}_{k+1} = Q\dot{x}_k + LY_{k:k+\delta},
\]
(3)
where \( \dot{x}_k \in \mathbb{R}^{n_x} \) is an estimate of the state \( x_k \) at the
\( k \)-th time instant, and \( Q \in \mathbb{R}^{n_x \times n_x} \), \( L \in \mathbb{R}^{n_x \times (\delta+1)n_y} \) are
observer gain matrices to be designed.

Note that the observer updates an estimate of the state
\( x_k \) based on the measurements of the system from time-
step \( k \) to \( k+\delta \). When \( \delta = 0 \) the observer is in the form of a
**predictor** (as it estimates \( x_{k+1} \) based on \( y_k \)). When \( \delta = 1 \),
the observer estimates the state \( x_{k+1} \) using measurements
up to \( y_{k+1} \), and when \( \delta > 1 \), the observer is analogous to a
**fixed-lag smoother** from the filtering literature [28]. Note
that one can equivalently view this observer as providing
an estimate of the state \( x_{k-\delta+1} \) based on measurements
from time-step \( k-\delta \) to the current time-step \( k \). Throughout
the paper, we will use the form (3) for notational con-
vienience.

3.3. Observer Error Dynamics

Let the observer error at the \( k \)-th time step be defined as
\( e_k = \dot{x}_k - x_k \). Then from (1) and (3), the error dynamics are
governed by
\[
e_{k+1} = \dot{e}_{k+1} = LQe_k + (L\Theta_\delta - A + Q)x_k + (L\Gamma_\delta - \Phi_\delta)W_{k:k+\delta}.
\]
(4)
where \( \Phi_\delta \triangleq \begin{bmatrix} B & 0 & \cdots & 0 \end{bmatrix} \).
(5)

We define a **performance output**
\[
z_k \triangleq He_k
\]
(6)
where \( z_k \in \mathbb{R}^{n_z} \), and \( n_z \leq n_x \). This performance output
is employed to select subsets/linear combinations of error
3.4. Sufficient Conditions for Observer Design with Guaranteed Performance

In this section, we provide sufficient conditions for the existence of an observer of the form (3), whose error dynamics are $\ell_\infty$-stable with a specified performance level $\gamma$. We begin with the following lemma pertaining to $\ell_\infty$-stability of general discrete-time systems of the form (7a).

**Lemma 1.** Consider the error system (7a) with performance output (7b). Let $V(e_k) : \mathbb{R}^{n_x} \mapsto [0, \infty)$ satisfy
\[
\chi_1 \|e_k\|^2 \leq V(e_k) \leq \chi_2 \|e_k\|^2
\]
for some $\chi_1, \chi_2 > 0$ and every $e_k \in \mathbb{R}^{n_x}$. Let $\Delta V_k \equiv V(e_{k+1}) - V(e_k)$. Suppose there exist scalars $\alpha \in (0, 1), \mu_1, \mu_2 > 0$ such that
\[
\Delta V_k \leq -\alpha (V(e_k) - \mu_1 d_k^T d_k) \quad (10a)
\]
\[
\|\psi(e_k)\|^2 \leq \mu_2 V(e_k). \quad (10b)
\]
Then the dynamics (7a) with performance output (7b) are $\ell_\infty$-stable with performance level $\gamma = \sqrt{\mu_1 \mu_2}$ with respect to the disturbance input sequence $\{d_k\}$.

**Proof.** From the difference equation (10a), we write
\[
V(e_k) \leq (1 - \alpha)^k V(e_0) + \alpha \mu_1 \|d\|_{\infty}^2 \sum_{j=0}^{k-1} (1 - \alpha)^j
\]
\[
\leq (1 - \alpha)^k V(e_0) + \mu_1 \|d\|_{\infty}^2, \quad (11)
\]
for any $k \geq 0$, since $0 < \alpha < 1$. This implies global exponential stability when $d$ is identically zero, satisfying property (P1) of Definition 1.

Furthermore, for an initial error of zero, we have $V(e_0) = 0$, which indicates $V(e_k) \leq \mu_1 \|d\|_{\infty}^2$. From the above, (6) and (10b), we get $\|\psi(e_k)\|^2 \leq \mu_2 V(e_k) \leq \mu_1 \mu_2 \|d\|_{\infty}^2$, which proves (P2). Finally, for any $e_k \in \mathbb{R}^{n_x}$, from (10b) and (11), we obtain
\[
\|\psi(e_k)\|^2 \leq \mu_2 (1 - \alpha)^k V(e_0) + \mu_1 \mu_2 \|d\|_{\infty}^2.
\]

As $z_k = \psi(e_k)$, this implies $\limsup_{k \to \infty} \|z_k\|^2 \leq \mu_1 \mu_2 \|d\|_{\infty}^2$, which verifies property (P3). Since (P1)-(P3) are verified, we conclude that the error dynamics (7a) are $\ell_\infty$-stable with performance level $\sqrt{\mu_1 \mu_2}$. \(\square\)

We are now ready to state the following main theorem of this section. This will later enable the design of the observer gains $Q$ and $L$ (see (8)) while providing performance guarantees.

**Theorem 1.** Consider a general discrete-time linear error system of the form
\[
e_{k+1} = Ae_k + Bd_k, \quad (12a)
\]
\[
z_k = He_k, \quad (12b)
\]

which are similar to the notion of input-to-state stability (ISS), see for example [30, 31, 32]. However, the definition differs from ISS by extending the notion of stability to a performance output and considering global ultimate output response guarantees as presented in (P3). Our notion of $\ell_\infty$-stability differs from the notion of $\mathcal{H}_\infty$ stability in that we assume boundedness of the peak value $\|d\|_{\infty}$ of the disturbance input sequence $\{d_k\}$, rather than the $\ell_2$ norm, $\|d\|_2 = \sqrt{\sum_{k=1}^{\infty} d_k^T d_k}$.

Formally, our objective is to construct the observer (3) so that the resulting observer error dynamics (8) with performance output (6) is guaranteed to be $\ell_\infty$-stable operating at a specified performance level with respect to the disturbance input sequence $\{W_{k:k+\delta}\}$.
with unknown disturbance input $d_k$ and performance output $z_k$. Suppose there exists a matrix $P = P^T > 0$ and scalars $\mu > 0$, $\alpha \in (0, 1)$ that satisfy the matrix inequalities

$$
\begin{bmatrix}
A^T \!\! P A + (\alpha - 1)P & A^T \!\! P B \\
B^T \!\! P A & B^T \!\! P B - \alpha I
\end{bmatrix} \preceq 0 \quad (13a)
$$

$$
\mu P - H^T \!\! H \succeq 0. \quad (13b)
$$

Then error system (12a) with performance output (12b) is $\ell_\infty$-stable with performance level $\gamma = \sqrt{\mu}$ with respect to the disturbance input sequence $\{d_k\}$.

Proof. We choose a Lyapunov function candidate of the form $V_k \triangleq \mathcal{V}(e_k) = e_k^T \!\! P e_k$. This satisfies the Rayleigh inequality $\lambda_{\min}(P)\|e_k\|^2 \leq V_k \leq \lambda_{\max}(P)\|e_k\|^2$ for any $e_k \in \mathbb{R}^{n_x}$. Thus, (9) is verified.

Then, evaluating the function difference on the trajectories of the dynamics (12a), we get

$$
\Delta V_k = e_{k+1}^T \!\! P e_{k+1} - e_k^T \!\! P e_k
= e_k^T (A^T \!\! P A - \alpha I) e_k + 2e_k^T A^T \!\! P B d_k + d_k^T B^T \!\! P B d_k.
$$

A congruence transformation of (13a) with $[e_k^T \quad w_k^T]^T$ yields

$$
0 \geq e_k^T A^T \!\! P A e_k - (1 - \alpha)e_k^T \!\! P e_k + 2e_k^T A^T \!\! P B d_k - \alpha d_k^T d_k + d_k^T B^T \!\! P B d_k = \Delta V_k + \alpha \Delta V_k - \alpha d_k^T d_k.
$$

This implies $\Delta V_k \leq -\alpha^2 (V_k - d_k^T d_k)$, which satisfies (10a) with $\mu_1 = 1$. From (13b) we get

$$
\|z_k\|^2 = \|H e_k\|^2 \leq \mu e_k^T \!\! P e_k = \mu V_k,
$$

which satisfies (10b) with $\mu_2 = \mu$. Applying Lemma 1 concludes the proof.

It is important to note that the matrix inequalities presented in (13) are not LMIs in $A$, $B$, $P$, $\mu$ and $\alpha$. We devote the following section to the synthesis of LMIs for efficient computation of the observer gains.

4. Synthesis of LMIs for Delayed Observer Design

In this section, we return to the observer (3), with error dynamics (8), and use Theorem 1 to design the observer gains $Q$ and $L$ while guaranteeing $\ell_\infty$-stability with specified performance with respect to the cumulative disturbance input sequence $\{W_{k,k+\delta}\}$.

4.1. LMI conditions

We begin with the following design theorem.

**Theorem 2.** Fix $\alpha \in (0, 1)$. Suppose there exist matrices $P = P^T > 0$, $Z$ and a scalar $\mu > 0$ that satisfy the linear matrix inequalities

$$
\begin{bmatrix}
(\alpha - 1)P & * & * \\
0 & -\alpha I & * \\
PA - Z \Theta_\delta & Z \Gamma_\delta - P \Phi_\delta & -P
\end{bmatrix} \preceq 0 \quad (14a)
$$

Then the observer (3) characterized by gains $L = P^{-1}Z$ and $Q = A - L \Theta_\delta$ with performance output (6) has $\ell_\infty$-stable error dynamics operating at a performance level $\gamma = \sqrt{\mu}$ with respect to the cumulative disturbance input sequence $\{W_{k,k+\delta}\}$.

Proof. We prove this theorem by demonstrating that the linear matrix inequalities (14) are equivalent to the matrix inequalities (13) in Theorem 1.

For brevity, let $P := P$, $A := A - L \Theta_\delta$ and $B := L \Gamma_\delta - \Phi_\delta$. Then we see that the observer error dynamics (8) with performance output (6) are of the form in Theorem 1 with the disturbance input $d_k := W_{k,k+\delta}$. We take Schur complements of (14a) and recall $Z = PL$. Since $P > 0$, this yields

$$
0 \geq \begin{bmatrix}
(\alpha - 1)P & 0 & 0 \\
0 & -\alpha I & 0 \\
PA & B^T \!\! P A & B^T \!\! P B - \alpha I
\end{bmatrix} \preceq 0
$$

which is equivalent to (13a). Also note that (13b) is equivalent to (14b) by the Schur complement lemma by replacing $H$ with $H$. This concludes the proof.

**Remark 3.** Note that a feasible solution to the conditions (14) implies that the $\ell_\infty$-gain of the performance output $z_k$ with respect to the cumulative disturbance input sequence $\{W_{k,k+\delta}\}$ is upper bounded by $\gamma$. Hence, the $\ell_\infty$-gain of $z_k$ with respect to the disturbance input sequence $\{w_k\}$ is upper bounded by $\gamma \sqrt{\delta + 1}$.

4.2. When is $\gamma$ finite?

Next, we present necessary and sufficient conditions for the proposed observer to exist for a finite performance level $\gamma$.

**Theorem 3.** There exists a $\gamma \in (0, \infty)$ such that properties (P1)–(P3) in Definition 1 hold if and only if the pair $(A, C)$ is detectable.

Proof. (⇐) We begin by recalling the observer error dynamics (8). The solution to this difference equation is given by

$$
e_k = Q^k e_0 + \sum_{j=0}^{k-1} Q^{k-1-j} (L \Gamma_\delta - \Phi_\delta) W_{j+1,j+\delta}.
$$
If the pair \((A, C)\) is detectable, this implies that there exists a matrix \(L\) such that \(Q = A_L - L\Theta_d\) is Schur. Hence, there exist scalars \(K > 0\) and \(0 < \nu < 1\) such that \(\|Q^k\| \leq K\nu^k\) for all \(k \in \mathbb{N}\) [33, 34]. Hence,

\[
\|e_k\| \leq K\nu^k\|e_0\| + \|LT_\delta - \Phi_d\| \sqrt{\nu + 1}\|w\|_{\infty}^k \sum_{j=0}^{k-1} \nu^{k-1-j}.
\]

Therefore,

\[
\limsup_{k \to \infty} \|e_k\| \leq \|LT_\delta - \Phi_d\| \left( \frac{\sqrt{\nu + 1}}{1 - \nu} \right) \|w\|_{\infty}.
\]

Since \(z_k = H e_k\), we get \(\limsup_{k \to \infty} \|z_k\| \leq \gamma \|w\|_{\infty}\), where

\[
\gamma \triangleq K\|H\|\|LT_\delta - \Phi_d\| \frac{\sqrt{\nu + 1}}{1 - \nu}.
\]

\((\implies)\) The fact that detectability of \((A, C)\) is necessary for an observer to satisfy the properties \((P1)-(P3)\) follows from standard arguments: if the system is not detectable, let the disturbance input sequence \(\{u_k\}\) be identically zero and choose the initial state to be in the undetectable subspace. This causes the output of (1) to be identically zero for all time while the state remains bounded away from zero. This scenario is indistinguishable from the case where the state is identically zero, and thus property \((P3)\) cannot be satisfied for both scenarios simultaneously.

**Remark 4.** Since the sequence \(\gamma^*_\delta\) is decreasing and lower bounded by zero, it is a convergent sequence.

The following corollary provides conditions for which the limit of the sequence \(\gamma^*_\delta\) is zero.

**Corollary 1.** If

\[
\text{rank} \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = n_x + n_w (16)
\]

for all \(s \in \mathbb{C}\) such that \(|s| \geq 1\), then there exists a \(\delta_0 \leq n_x\) such that \(\gamma^*_\delta = 0\) for all \(\delta \geq \delta_0\).

**Proof.** We know from [23] that if (16) is satisfied, then there exists a \(\delta_0 \leq n_x\) and an \(L\) such that \(LT_\delta = \Phi_d\) and \(L\Theta_d = A - Q\) for some stable matrix \(Q\) and every \(\delta \geq \delta_0\). The inequality (14a) then becomes

\[
\begin{bmatrix} (\alpha - 1)P & * & * \\ 0 & -\alpha I & * \\ PA - Z\Theta_d & 0 & -P \end{bmatrix} \preceq 0.
\]

Since \(\alpha > 0\), this is equivalent to

\((A - L\Theta_d)^\top P(A - L\Theta_d) + (\alpha - 1)P \preceq 0\)

by replacing \(Z = PL\) and taking Schur complements. Therefore, for any \(\mu_\delta > 0\), we can scale \(P\) by a positive scalar to ensure that \(\Xi^*_\delta \leq 0\) in conjunction with

\[
\begin{bmatrix} P & H^\top \\ H \end{bmatrix} \succeq 0.
\]

This implies \(\gamma^*_\delta = 0\) for all \(\delta \geq \delta_0\). This concludes the proof.

### 5. Exogenous Input Reconstruction

Next, we provide sufficient conditions for the asymptotic reconstruction of the exogenous unknown inputs. By Assumption 2, the matrix \(G\) has full column rank. Hence, there exists a left-inverse \(G^\dagger\) that satisfies \(G^\dagger G = I\). Specifically, one could use the Moore-Penrose pseudo-inverse \(G^\dagger := (G^\top G)^{-1}G^\top\). From the plant dynamics (1), we deduce

\[
w_k = G^\dagger \begin{bmatrix} x_{k+1} - Ax_k \\ y_k - Cx_k \end{bmatrix}.
\]

Let an estimate of the unknown input \(w_k\) be given by

\[
\hat{w}_k = G^\dagger \begin{bmatrix} \hat{x}_{k+1} - A\hat{x}_k \\ y_k - C\hat{x}_k \end{bmatrix}.
\]
We present an upper bound for the unknown input reconstruction error for any observer which satisfies the conditions in Theorem 2.

**Theorem 5.** Let $G \triangleq [B^T \ D^T]^T$ and $G^\dagger$ be a left inverse of $G$. Suppose that there exist matrices $P = P^T > 0$, $Z$ and scalars $\alpha \in (0,1)$, $\mu > 0$ that satisfy

$$
\begin{bmatrix}
(\alpha - 1)P & * & * \\
0 & -\alpha I & * \\
PA - Z\Theta_\delta & Z\Gamma_\delta - P\Phi_\delta & -P
\end{bmatrix} \preceq 0 \quad (19a)
$$

$$
\begin{bmatrix}
P & * \\
I & \mu I
\end{bmatrix} \succeq 0. \quad (19b)
$$

Then the unknown input reconstruction error is bounded by

$$
\limsup_{k \to \infty} \|\hat{w}_k - w_k\| \leq \gamma \|\bar{w}\|_\infty, \quad (20)
$$

where $\gamma = \|G^\dagger\|(1 + \|A\| + \|C\|) \sqrt{(1+\delta)\mu}$.

**Proof.** From (17) and (18), the unknown input estimation error is given by

$$
\hat{w}_k - w_k = G^\dagger \begin{bmatrix}
\hat{x}_{k+1} - Ax_k - x_{k+1} + Ax_k \\
y_k - C\hat{x}_k - y_k + Cx_k
\end{bmatrix}
= G^\dagger \begin{bmatrix}
e_{k+1} - Ae_k \\
-Ce_k
\end{bmatrix}
= G^\dagger \begin{bmatrix}
0 \\
-e_k
\end{bmatrix}
= \begin{bmatrix}
\alpha I \\
\mu I
\end{bmatrix} \begin{bmatrix}
e_k
\end{bmatrix}.
$$

Applying the triangle inequality to the above, we obtain

$$
\|\hat{w}_k - w_k\| \leq \|G^\dagger\| \left(\|e_{k+1}\| + \|A\|\|e_k\| + \|C\|\|e_k\|\right) \quad (21)
$$

From (19b), we note that $H = I$, that is, the performance output is $z_k = e_k$. Applying Theorem 2, a feasible solution to the LMI conditions (19) implies that $\limsup_{k \to \infty} \|e_k\| \leq \sqrt{\mu} \|W\|_\infty$. Hence, we can rewrite (21) as $\limsup_{k \to \infty} \|\hat{w}_k - w_k\| \leq \gamma \|\bar{w}\|_\infty$, which concludes the proof.

The following corollary provides a condition which ensures arbitrarily accurate asymptotic estimation of the unknown input.

**Corollary 2.** Suppose the plant (1) satisfies the minimum phase condition (16). Then there exists a sufficiently large delay $\delta_0$ and an observer of the form (3) which satisfies

$$
\limsup_{k \to \infty} \|\hat{w}_k - w_k\| = 0.
$$

**Proof.** From Corollary 1, we know that if the system (1) is minimum phase, then there exists a $\delta_0$ sufficiently large that ensures a feasible solution to (19) and the $\ell_\infty$-gain $\gamma_\delta$ is identically zero for all $\delta \geq \delta_0$. Applying Theorem 5 concludes the proof.

6. **Numerical Example**

We consider an unstable discrete-time system with

$$
A = \begin{bmatrix}
0.71 & -0.28 & 0.4 & 0.29 \\
0.02 & 0.18 & 0.03 & -0.08 \\
-0.7 & 0.34 & -0.42 & -0.69 \\
-0.23 & -0.2 & -0.2 & -0.36
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & -1 \\
0 & 0
\end{bmatrix},
$$

$$
C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}^T, \quad D = \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}.
$$

Note that the matching condition rank $CB = \text{rank } B$ does not hold and the matrix $A$ is selected to be non-invertible to make the problem more challenging, as discussed in [24]. We select $H = I$, fix $\alpha = 0.5$, and choose a delay of $\delta = 3$ samples. Then we construct the matrices $\Theta_\delta$, $\Gamma_\delta$ and $\Phi_\delta$ and solve the LMIs presented in (14) to obtain $\gamma \|W_{k+\delta}\|_\infty = 2\gamma \|\bar{w}\|_\infty = 0.2319$. The computed observer gains are:

$$
Q = \begin{bmatrix}
0 & 0.005 & 0 & 0 & 0 & 0 & 0.005 \\
0.001 & 0.103 & 0.033 & -0.078 & 0 & -0.082 \\
0.01 & 0.715 & -0.419 & -0.689 & 0 & -0.628 \\
0.001 & -0.14 & -0.2 & -0.363 & 0 & 0.062 \\
0.003 & 0.01 & 0 & 0 & 0 & 0.01 \\
-0.001 & -0.101 & -0.033 & 0.078 & 0 & 0.083
\end{bmatrix},
$$

and

$$
L = \begin{bmatrix}
0 & -0.005 & 1 & 0 \\
0.018 & 0.082 & 0 & 0 \\
-0.713 & -0.372 & 0 & 0 \\
-0.234 & -0.062 & 0 & 0 \\
-0.003 & -0.01 & 0 & 0 \\
-0.018 & -0.083 & 0 & 1
\end{bmatrix}.
$$

Figure 1: (Top) Delayed observer error norm with $\delta = 3$. (Middle) Unknown input reconstruction error norm with $\delta = 3$. (Bottom) Variation of performance level $\gamma$ with increasing delay $\delta$. The red continuous lines denote the computed bound $\gamma \|W\|_\infty = 0.2319$. The dashed red lines denote the estimate of the unknown input $w_k$. 

![Graphs showing performance metrics](image-url)
The observer error norm is illustrated in Figure 1 with a randomly selected initial condition
\[ x_0 = \begin{bmatrix} -6.353 & -1.913 & 3.243 & 4.129 \end{bmatrix}^\top. \]

The observer is initialized at zero, that is, \( \hat{x}_0 = 0 \), and the unknown input acting on the system is
\[ w_k = \begin{bmatrix} \text{sawtooth}(0.5k) \ & \text{square}(0.8k) \end{bmatrix}^\top. \]

As seen from Figure 1 the proposed observer performs in accordance with the guarantees provided in Theorem 2. The observer estimation error norm and unknown input reconstruction error norms are shown in Figure 1. It is clear that the simulated observer satisfies the error bound computed from Theorem 2. We also verify our result in Theorem 4 by solving (14) with \( \alpha = 0.5 \) and \( H = I_4 \).

7. Conclusions

We developed a scheme for state estimation and unknown input reconstruction for discrete-time linear systems with unknown inputs. We presented robustness guarantees for the observation error system and provided sufficient conditions for the reconstruction of unknown disturbance inputs up to a pre-defined accuracy. Applications of this work include attack detection in cyber-physical systems and fault detection and isolation.

References