

2021 Spring ECE606 - Homework 3 Solutions

Problem 1 (a)

Schrodinger's equations is as follows:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + U(x)\psi = E\psi \quad (1)$$

A numerical solution to the Schrodinger equation can be accomplished using the Taylor series expansion of ψ around $\pm a$, the unit step for the numerical approximation through the finite difference method.

$$\psi(x_o + a) = \psi(x_o) + a \frac{d\psi}{dx} + \frac{a^2}{2!} \frac{d^2\psi}{dx^2} + \mathcal{O}(a)^3 \quad (2)$$

$$\psi(x_o - a) = \psi(x_o) - a \frac{d\psi}{dx} + \frac{a^2}{2!} \frac{d^2\psi}{dx^2} + \mathcal{O}(a)^3 \quad (3)$$

$$\psi(x_o + a) + \psi(x_o - a) = 2\psi(x_o) + a^2 \frac{d^2\psi}{dx^2} + \mathcal{O}(a)^3 \quad (4)$$

$$\frac{d^2\psi}{dx^2} a^2 = \psi(x_o + a) - 2\psi(x_o) + \psi(x_o - a) + \mathcal{O}(a)^3 \quad (5)$$

$$\frac{d^2\psi}{dx^2} = \frac{\psi(x_o + a) - 2\psi(x_o) + \psi(x_o - a)}{a^2} + \mathcal{O}(a) \quad (6)$$

(6) shows that the second derivative of ψ can be approximated with an order of error a . Letting $t = \frac{\hbar^2}{2ma^2}$ we find that (1) resolves to

$$-ta^2 \left(\frac{\psi_{i-1} - 2\psi_i + \psi_{i+1}}{a^2} \right) + U_i \psi_i = E\psi_i \quad (7)$$

$$-t\psi_{i-1} + (2t + U_i)\psi_i - t\psi_{i+1} = E\psi_i \quad (8)$$

Problem 1 (b)

The numerical dispersion can be found by plugging the ansatz into (8)

$$Ee^{ikx_j} = (U + 2t)e^{ikx} - te^{ikx_{j+1}} - te^{ikx_{j-1}} \quad (9)$$

$$E = (U + 2t) - te^{ika} - te^{-ika} \quad (10)$$

$$E = U + 2t - t(e^{ika} + e^{-ika}) \quad (11)$$

$$E = U + 2t - 2t \cos(ka) \quad (12)$$

$$E = U + 2t [1 - \cos(ka)] \quad (13)$$

Problem 1 (c)

Now consider the Taylor series expansion of $\cos(x)$ at 0:

$$\cos(x) \approx 1 + \frac{\sin(0)}{1!}x - \frac{\cos(0)}{2!}x^2 + \dots \quad (14)$$

This shows that $\cos(x) \approx 1 - x^2/2$ for values of x around 0. Thus, if a is sufficiently small then it is approximately parabolic

$$E \approx U + \frac{\hbar^2 k^2}{2m} \quad (15)$$

Problem 1 (d)(e)

The group velocity is

$$v_g = \frac{1}{\hbar} \frac{dE}{dk} \quad (16)$$

$$v_g = \frac{2ta}{\hbar} \sin(ka) \quad (17)$$

$$v_g = \frac{\hbar}{ma} \sin(ka) \quad (18)$$

The effective mass is determined from the second derivative of the dispersion with respect to energy:

$$\frac{d^2 E}{dk^2} = 2ta^2 \cos(ka) \quad (19)$$

Therefore the effective mass is:

$$m^* = \frac{m}{\cos(ka)} \quad (20)$$

For the parabolic relation the group velocity is:

$$v_g = \frac{\hbar k}{m} \quad (21)$$

And the effective mass is:

$$m^* = m \quad (22)$$

Problem 1 (f)

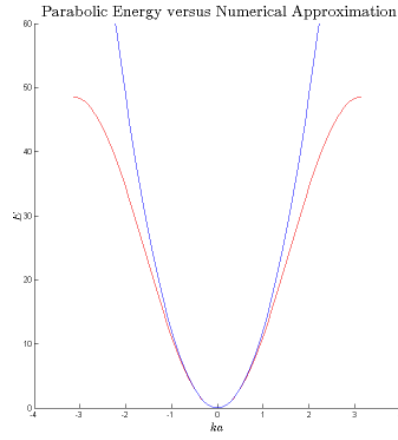


Figure 1: Parabolic energy versus numerical approximation

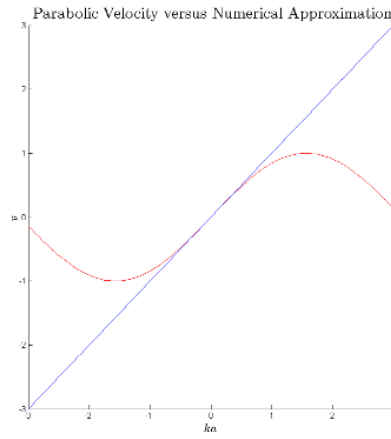


Figure 2: Parabolic velocity versus numerical approximation

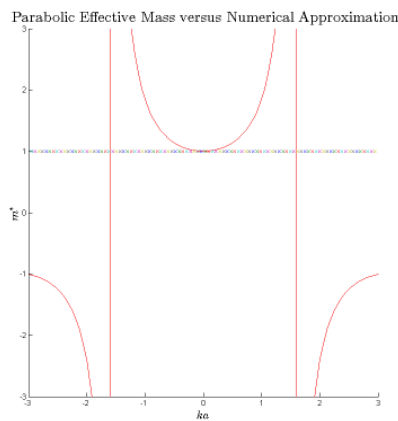


Figure 3: Parabolic mass versus numerical approximation

Figure 1 shows the relationship between the numerical approximation shown in red and the parabolic relationship shown in blue. As discussed in part b), the dispersion is approximated well for small values of a .

Figure 2 shows a sketch of the relationship between the numerical approximation shown in red and the parabolic relationship shown in blue. For values around 0, $\sin(ka)$ is approximately equal to 0 which leads to a close approximation to the actual velocity in this region.

Figure 3 shows a sketch of the relationship between the numerical approximation shown in red and the parabolic relationship shown as the x ticks. For values around 0, $\cos(ka)$ is approximately equal to 1 which leads to a close approximation to the actual effective mass in this region.

Problem 1 (g)

Consider the effective mass at the three values of ka using the finit difference scheme:

At $ka = 0$ the effective mass is $m^* = m$ which mirrors the parabolic relation. The particle (assuming it is negatively charged like an electron) will accelerate in the positive-x direction towards an infinite velocity.

At $ka = \pi/2$, the particle will have an infinite mass and therefore experience no acceleration.

At $ka = -\pi$, the particle will have an effective mass of $m^* = -m$. Therefore the particle will accelerate in the negative-x direction towards an infinite velocity.

Problem 2 (a)

The bandgap between the first and second bands corresponding to $U = 2$ eV is 1.28 eV.

Problem 2 (b)

At the higher potential barrier energy ($U = 4$ eV), the bandgap increased to 2.46 eV.

Problem 2 (c)

The bands became more narrow. The lower energy band went from 0.019 to 0.002 eV. The upper energy band went from 0.448 to 0.125 eV.

Problem 2 (d)

The effective mass of a lattice particle enables a semiclassical analysis of its behavior in a solid. For instance, $F = ma$ is the standard force equation for a classical particle. Consider a one dimensional acceleration on a object. If the acceleration on a particle is in the positive-x direction, a negative effective mass means that the force on the particle will be in the opposite direction. In other words, a negative effective mass means the particle will move in the opposite direction that would be expected with a classical mass.

Problem 3 (a)

Si is a Diamond crystal.

Problem 3 (b)

There are a total of 8 atoms in a unit cell of Si: 4 internal, $(1/8)*8$ corners, and $(1/3)*6$ on the faces.

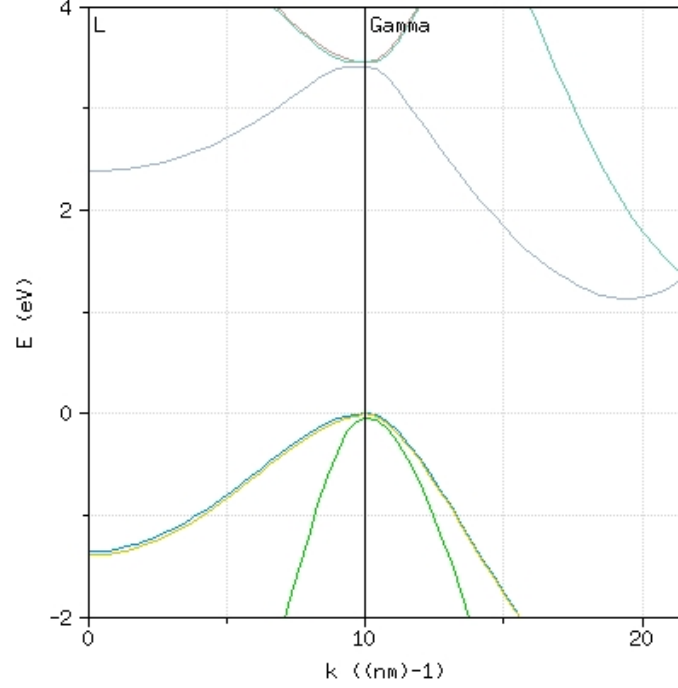


Figure 4: Bulk Central Bands for Si

Problem 3 (c)(d)

Please see Figure 4. The energy at the $\mathcal{L}$ point is 2.38 eV, Γ point is 3.40 eV, and $\mathcal{X}$ point is 1.34 eV. The lowest energy point is at $k = 19.6$, so the bandgap (indirect) of Si is around 1.13 eV.

Problem 3 (e)

The three bands below the 0 eV mark are identified as the valence bands. They are classified, top to bottom, as the heavy hole band, light hole band, and split-off band.

Problem 3 (f)(g)

The spacing between the heavy and light hold bands with the split-off band is 0.05 eV. The split-off band arises as a result of spin-orbit coupling.

Problem 3 (h)

For the longitudinal effective masses, the one at $\mathcal{L}$ point ($m_l = 3.41 m_0$) is much larger than $\mathcal{X}$ point's one ($m_l = 0.905 m_0$). Oppositely, the transverse effective mass at $\mathcal{L}$ ($m_t = 0.173 m_0$) is smaller than $\mathcal{X}$ point ($m_t = 0.201 m_0$).

Problem 3 (i)

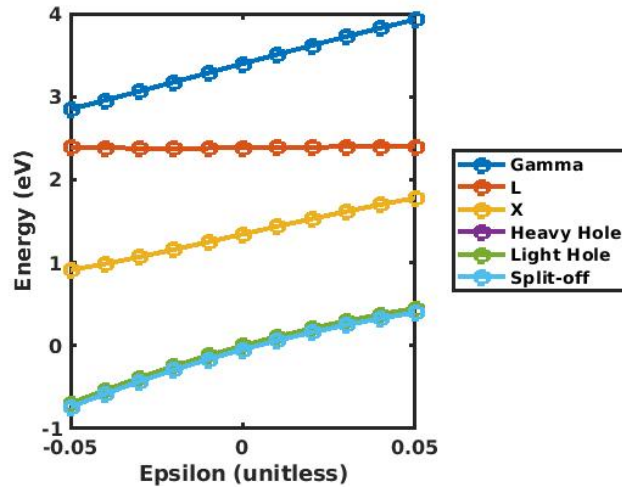


Figure 5: Band Edge vs Strain

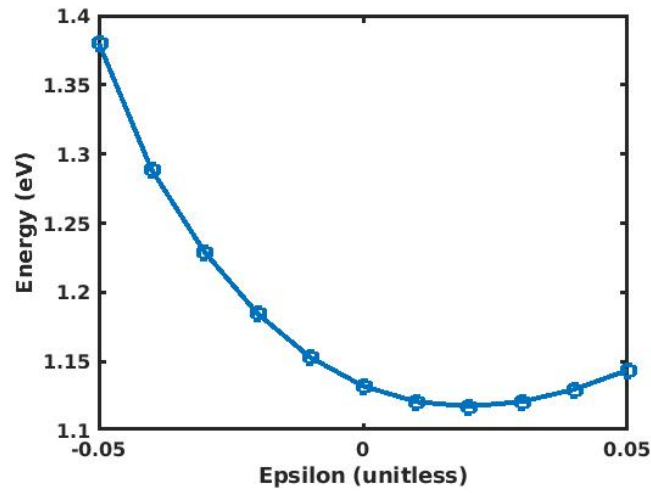


Figure 6: Bandgap vs Strain

We can calculate the bandgap of strained silicon by computing the energy difference between the band edges at the point with lowest energy and the valence band (heavy hole or light hole). Fig 6 shows that the bandgap starts to decrease from $\epsilon = -5\%$ but increases again when the strain is larger than 2 %.

Problem 3 (j)

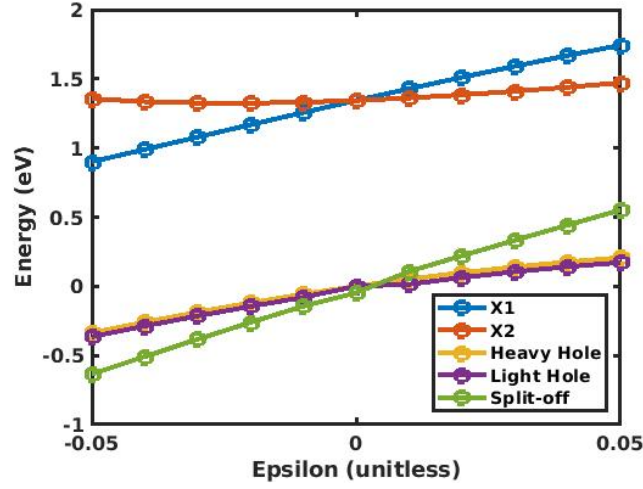


Figure 7: Band Edge vs Strain

Since the strain is applied along $[1\ 0\ 0]$, the band edge at X1 $\{1\ 0\ 0\}$ changes as a function of applied strain. On the other hand, the strain effect has a smaller impact on the band edge at X2 $\{0\ 1\ 0\}$. For bulk Silicon, X valleys are six-fold degenerate.

According to Fig 7, we can observe the strain-induced energy shift. The band edge at split-off band increases from $\epsilon = -5\%$ to $\epsilon = 5\%$ and it becomes higher than the band edges of heavy hole and light hole when the applied strain is positive. So actually the split-off band plays the role of heavy hole band in the valence dispersion when positive strain is exerted. Some similarities can also be observed in other valence bands.

MATLAB Code

```
clear all

strain = linspace(-0.05,0.05,11);

hh_band_edge = [-0.69645, -0.537137, -0.387544, -0.248064, -0.118883, -1.77704e-06, 0.108724,
0.207562, 0.296875, 0.377097, 0.448712];
lh_band_edge = [-0.69645, -0.537137, -0.387544, -0.248064, -0.118883, -1.77704e-06, 0.108724,
0.207562, 0.296875, 0.377097, 0.448712];
sh_band_edge = [-0.74088, -0.582066, -0.433005, -0.294084, -0.165477, -0.0471794, 0.0609613,
0.159219, 0.24796, 0.327624, 0.398699];
gamma_band_edge = [2.84858, 2.95538, 3.06488, 3.17588, 3.28739, 3.39856, 3.5087, 3.61724,
3.72374, 3.82783, 3.92927];
X_band_edge = [0.910891, 0.984483, 1.06762, 1.15688, 1.2495, 1.34325, 1.43638, 1.52753, 1.61569,
1.70011, 1.78029];
L_band_edge = [2.39332, 2.38228, 2.37732, 2.3767, 2.37896, 2.38287, 2.38744, 2.39189, 2.39558,
2.39805, 2.39895];

lowest_band_edge = [0.683659, 0.751349, 0.841599, 0.936614, 1.03404, 1.13218, 1.22955, 1.32501,
1.41767, 1.50692, 1.59232];

FIG = figure(1)
hold on
set(gca,'FontWeight','bold','FontSize',16)
set(gca,'LineWidth',2)
xlabel('Epsilon (unitless)','FontWeight','bold','FontSize',16)
ylabel('Energy (eV)','FontWeight','bold','FontSize',16)
xt=get(gca, 'XTick');
set(gca, 'FontSize', 12)
plot(strain,gamma_band_edge,'-o','LineWidth',3,'MarkerSize',8)
plot(strain,L_band_edge,'-o','LineWidth',3,'MarkerSize',8)
plot(strain,X_band_edge,'-o','LineWidth',3,'MarkerSize',8)
plot(strain,hh_band_edge,'-o','LineWidth',3,'MarkerSize',8)
plot(strain,lh_band_edge,'-o','LineWidth',3,'MarkerSize',8)
plot(strain,sh_band_edge,'-o','LineWidth',3,'MarkerSize',8)
legend('Gamma','L','X','Heavy Hole','Light Hole','Split-off','Location','southeast')
box on
hold off

bandgap = lowest_band_edge - hh_band_edge
FIG = figure(2)
hold on
set(gca,'FontWeight','bold','FontSize',16)
set(gca,'LineWidth',2)
xlabel('Epsilon (unitless)','FontWeight','bold','FontSize',16)
ylabel('Energy (eV)','FontWeight','bold','FontSize',16)
xt=get(gca, 'XTick');
set(gca, 'FontSize', 12)
plot(strain,bandgap,'-o','LineWidth',3,'MarkerSize',8)
plot(strain,lowest_band_edge,'-o','LineWidth',3,'MarkerSize',8)
plot(strain,hh_band_edge,'-o','LineWidth',3,'MarkerSize',8)
box on
hold off

hh_band_edge = [-0.337374, -0.259999, -0.187322, -0.119643, -0.0571651, -1.77704e-06, 0.0518092,
0.0982965, 0.139545, 0.175684, 0.206883];
lh_band_edge = [-0.365059, -0.287668, -0.214683, -0.145942, -0.0794711, -1.77704e-06, 0.0142426,
```

```

0.0627802, 0.104663, 0.140984, 0.172171];
sh_band_edge = [-0.63545, -0.50846, -0.383592, -0.261387, -0.144097, -0.0471794, 0.107483, 0.220335,
0.332351, 0.442514, 0.550516];
X1_band_edge = [0.900211, 0.989326, 1.07886, 1.16809, 1.2564, 1.34325, 1.42821, 1.51091, 1.59105,
1.66839, 1.74276];
X2_band_edge = [1.35475, 1.33531, 1.32561, 1.32444, 1.33066, 1.34325, 1.36125, 1.38379, 1.41009,
1.43942, 1.47117];

FIG = figure(3)
hold on
set(gca,'FontWeight','bold','FontSize',16)
set(gca,'LineWidth',2)
xlabel('Epsilon (unitless)','FontWeight','bold','FontSize',16)
ylabel('Energy (eV)','FontWeight','bold','FontSize',16)
xt=get(gca, 'XTick');
set(gca, 'FontSize', 12)
plot(strain,X1_band_edge,'-o','LineWidth',3,'MarkerSize',8)
plot(strain,X2_band_edge,'-o','LineWidth',3,'MarkerSize',8)
plot(strain,hh_band_edge,'-o','LineWidth',3,'MarkerSize',8)
plot(strain,lh_band_edge,'-o','LineWidth',3,'MarkerSize',8)
plot(strain,sh_band_edge,'-o','LineWidth',3,'MarkerSize',8)
legend('X1','X2','Heavy Hole','Light Hole','Split-off','Location','southeast')
box on
hold off

```

2022 Spring ECE606 - Homework 4 Solutions

Problem 1 (a)

The eigen energies of a 1D infinite potential square well can be written as

$$E_n = \frac{\hbar^2 k_n^2}{2m^*} \quad (1)$$

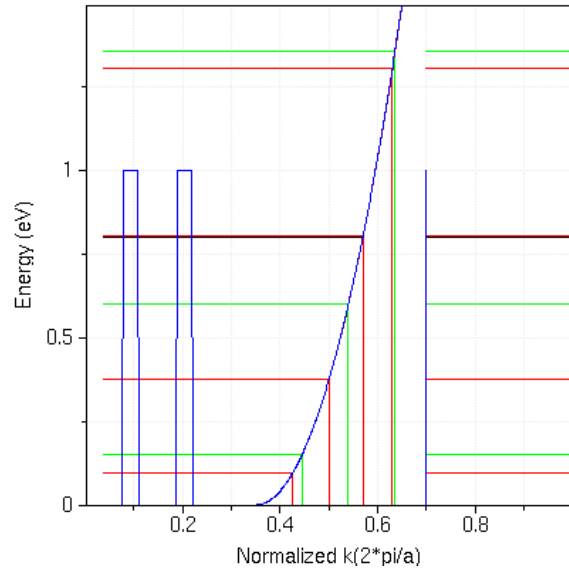
where $k_n = n\pi/L$. So the lowest three eigen energies are E_1 , E_2 , and E_3 .

$$E_1 = \frac{\hbar^2 \pi^2}{2m_0 m_e L^2} = 0.4 \frac{1}{0.1 \times 5^2} = 0.16 \text{ eV} \quad (2)$$

$$E_2 = E_1 \times 2^2 = 0.64 \text{ eV} \quad (3)$$

$$E_3 = E_1 \times 3^2 = 1.44 \text{ eV} \quad (4)$$

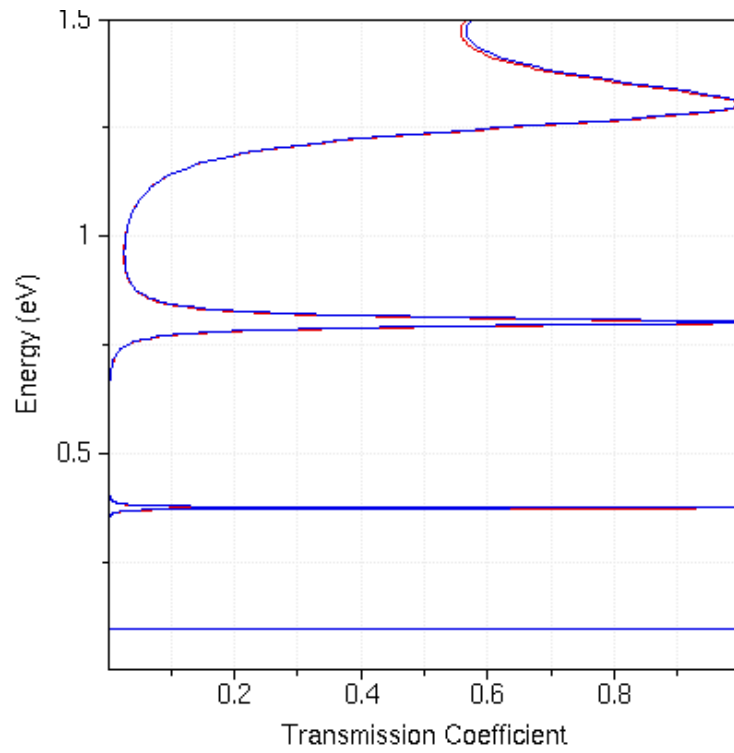
Problem 1 (b)



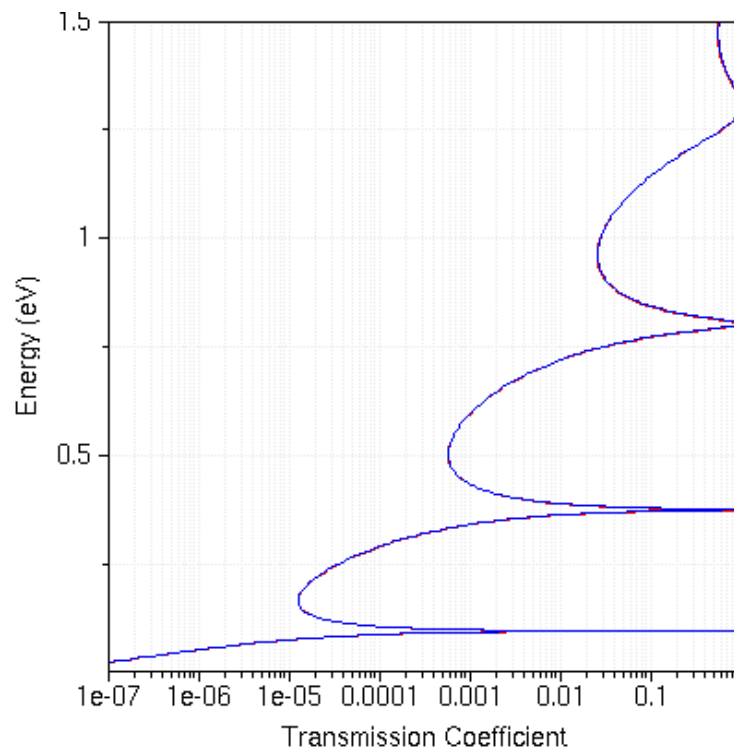
The green lines are energy levels of infinite well, and the red lines are energy levels of the double barrier. Eigen values in the double barriers are relatively lower for two reasons. First, the wave function can penetrate into the barrier, so the effective well becomes wider. Second, state in finite double barriers has finite lifetime or resonance width.

Problem 1 (c)

Below shows the transmission value in linear scale.

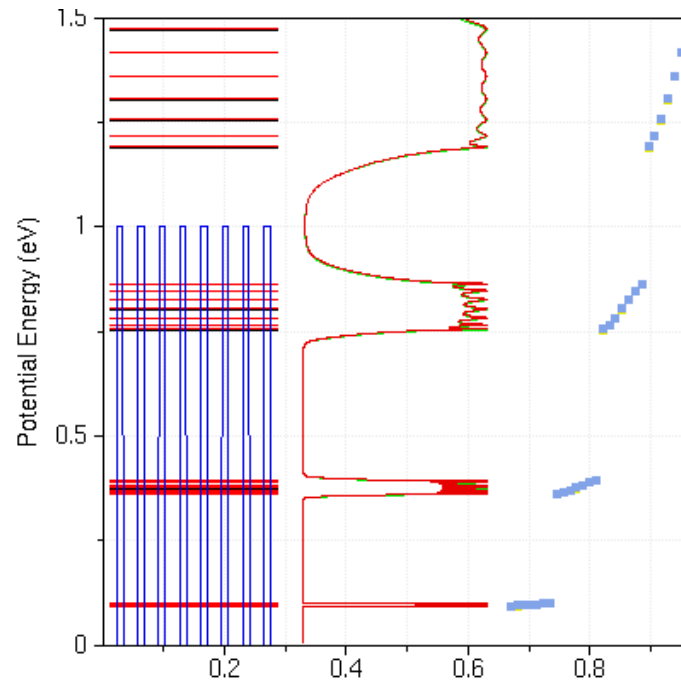


as well as in log scale.

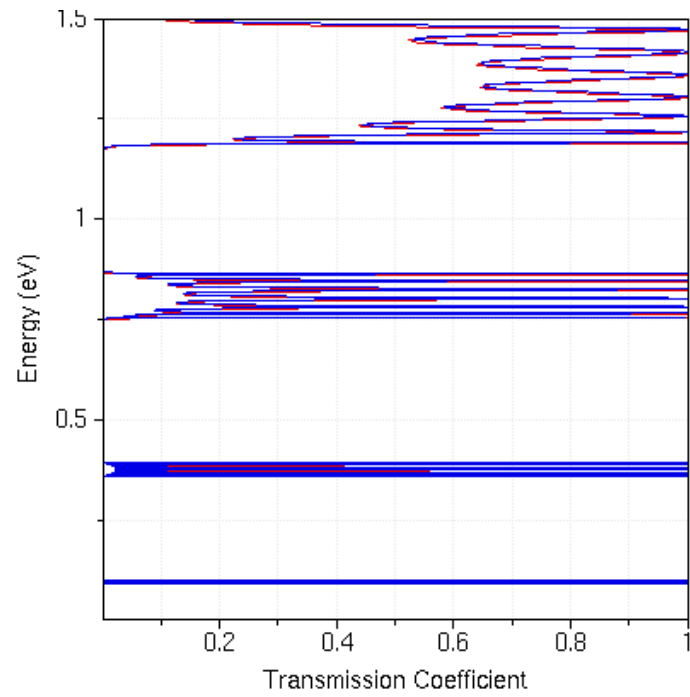


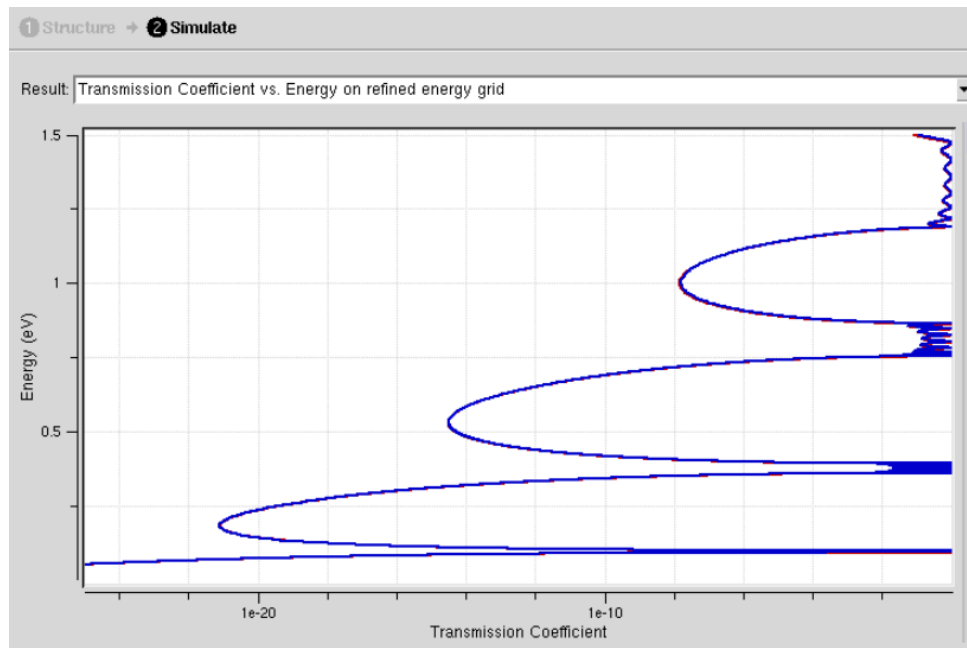
Problem 1 (d)

There are seven states per band.



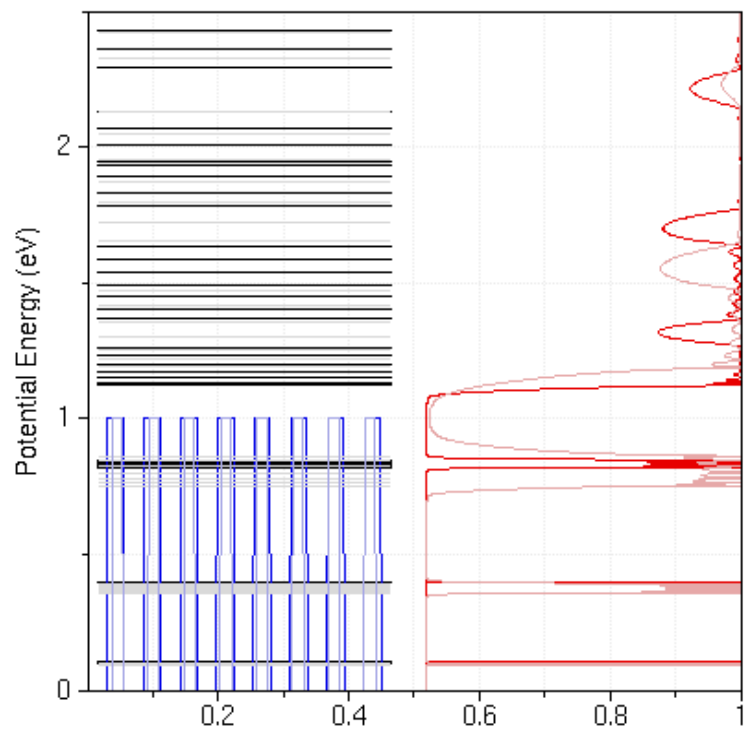
The transmission plots in linear and log scale are given as follows





Problem 1 (e)

In the following diagram, the light red curve represents the transmission of the 2nm-wide barrier, while the solid red curve shows the 4nm barrier's transmission.



Problem 2 (a)

$$\vec{a}_2 \times \vec{a}_3 = L \begin{pmatrix} \sqrt{3}/2 - 0 \\ 0 - 1.5 \\ 0 \end{pmatrix} = L \begin{pmatrix} \sqrt{3}/2 \\ -3/2 \\ 0 \end{pmatrix} \quad (5)$$

Therefore

$$\vec{b}_1 = 2\pi \frac{L(\sqrt{3}/2, -3/2, 0)}{L^2(3/2, -\sqrt{3}/2, 0)(\sqrt{3}/2, -3/2, 0)} = \frac{2\pi}{L} \frac{(\sqrt{3}/2, -3/2, 0)}{3\sqrt{3}/2} \quad (6)$$

$$\vec{b}_1 = \frac{2\pi}{L} \begin{pmatrix} 1/3 \\ -1/\sqrt{3} \\ 0 \end{pmatrix} \quad (7)$$

$$\vec{a}_3 \times \vec{a}_1 = L \begin{pmatrix} 0 - -\sqrt{3}/2 \\ 3/2 - 0 \\ 0 - 0 \end{pmatrix} = L \begin{pmatrix} \sqrt{3}/2 \\ 3/2 \\ 0 \end{pmatrix} \quad (8)$$

Therefore

$$\vec{b}_2 = 2\pi \frac{L(\sqrt{3}/2, 3/2, 0)}{L^2 3\sqrt{3}/2} \quad (9)$$

$$\vec{b}_2 = \frac{2\pi}{L} \begin{pmatrix} 1/3 \\ 1/\sqrt{3} \\ 0 \end{pmatrix} \quad (10)$$

Problem 2 (b)

The Wigner-Seitz algorithm was used to construct the unit-cell corresponding to the first Brillouin zone of Graphene as shown in Figure 1. The diagram was constructed using the MATLAB code included in the appendix.

The perpendicular bisectors of each edge of the reciprocal lattice were found by finding the midpoint $(a/2, b/2)$ and rotating the vector $x = (a, b)^T$ by 90 degrees using the rotation matrix. This produced perpendicular bisectors $x_\perp$:

$$x_\perp = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} a/2 \\ b/2 \end{bmatrix} \quad (11)$$

The intersection point of the intersecting perpendicular bisectors was found by solving the relation below with $x_\perp$ and $y_\perp$ representing two intersecting bisectors:

$$x_\perp = y_\perp \quad (12)$$

$$s \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a/2 \\ b/2 \end{bmatrix} = t \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} c/2 \\ d/2 \end{bmatrix} \quad (13)$$

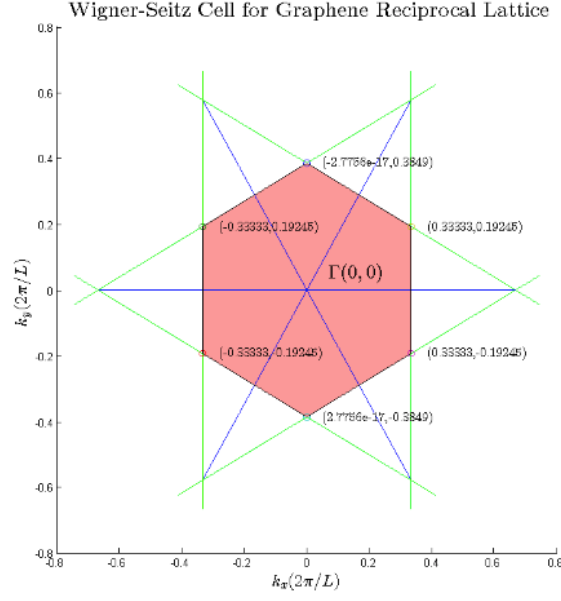


Figure 1: First Brillouin Zone of Graphene

$$\begin{bmatrix} x_1 & -y_1 \\ x_2 & -y_2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} c/2 - a/2 \\ d/2 - b/2 \end{bmatrix} \quad (14)$$

One of the solutions s or t can be used to find the point of intersection between the two bisectors. The corners are located, clockwise, at $(1/3, 1/3\sqrt{3})$, $(1/3, -1/3\sqrt{3})$, $(0, -2/3\sqrt{3})$, $(-1/3, -1/3\sqrt{3})$, $(-1/3, 1/3\sqrt{3})$, and $(0, 2/3\sqrt{3})$ in units of $2\pi/L$ respectively. The Γ point is located at $(0,0)$ as represented in the figure.

Problem 2 (c)(d)

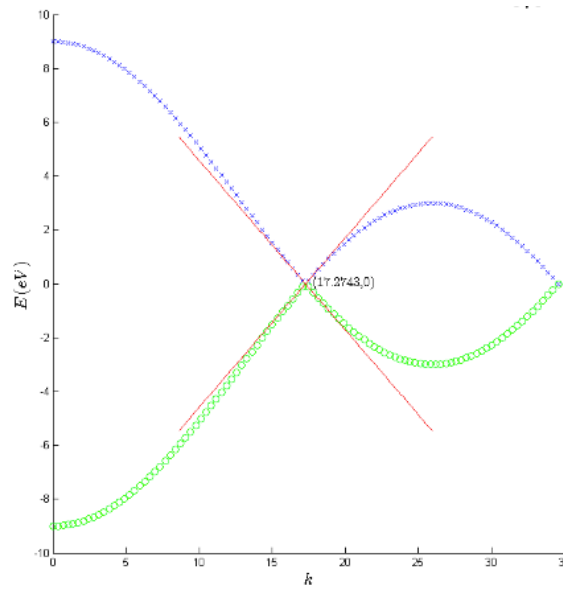


Figure 2: Graphene Dispersion Relation from Γ to $(0, \frac{4\pi}{L} \frac{1}{3\sqrt{3}})$

The dispersion relation for graphene is shown in Figure 2 (shown beyond the top point). An interatomic distance of $L = 0.14$ nm was used [1]. The point chosen was along the $k_x = 0$ and

$k_y : 0 \rightarrow \frac{2\pi}{L} \frac{2}{3\sqrt{3}}$ direction.

The effective mass and velocity in the vicinity of the corner at $E = 0$ can be found as follows, letting $a = \sqrt{3}Lk_y/2$:

$$E = \pm t \sqrt{1 + 4 \cos(0) \cos(a) + 4 \cos^2(a)} = \pm t \sqrt{(2 \cos(a) + 1)^2} = \pm t(1 + 2 \cos(a)) \quad (15)$$

Differentiating to find the velocity and mass:

$$v = \frac{1}{\hbar} \frac{dE}{dk} = \pm \frac{da}{dk} \frac{2t}{\hbar} \sin(a) \quad (16)$$

$$m = \frac{1}{\hbar^2} \frac{d^2 E}{dk^2} = \mp \left(\frac{da}{dk} \right)^2 \frac{2t}{\hbar^2} \cos(a) \quad (17)$$

These can be further resolved using the chosen Brillouin point

$$v = \pm \frac{\sqrt{3}}{2} L \frac{2t}{\hbar} \sin\left(\frac{\sqrt{3}}{2} L k_y\right) = \pm \frac{\sqrt{3}}{2} L \frac{2t}{\hbar} \sin\left(\frac{\sqrt{3}}{2} L \frac{4\pi}{L3\sqrt{3}}\right) = \pm \frac{\sqrt{3}tL}{\hbar} \sin(2\pi/3) = \pm \frac{3tL}{2\hbar} \quad (18)$$

And for the effective mass

$$m = \mp \frac{3L^2}{4\hbar^2} 2t \cos\left(\frac{2}{3}\pi\right) = \pm \frac{3tL^2}{\hbar^2} \quad (19)$$

Problem 2 (e)

Writing the dispersion relation as $E = \hbar v k$ in the corners is reasonable as shown by the red lines on Figure 2. At the dirac point, the dispersion relation is approximately linear and therefore corresponds well to the linear dispersion approximation using the velocity.

$$E = \hbar v k = \pm \frac{3tL}{2} k \quad (20)$$

Problem 2 (f)

As can be seen from Figure 2, there is a valley degeneracy of two (1/3 of a valley around each of the six points).

Problem 2 (g)

When considering a two dimensional k space, the number of states is equivalent to finding the number of minimal δk states of area $4\pi^2/(L_x L_y)$ that fit within a radius k circle.

$$N(k) = \frac{g\pi k^2}{4\pi^2/(L_x L_y)} = \frac{gL_x L_y k^2}{4\pi} \quad (21)$$

g is the degeneracy factor for the material under consideration. For graphene, the density of states does depend on energy with $k = E/\hbar v$:

$$N(E) = \frac{gL_xL_y}{4\pi} \frac{E^2}{\hbar^2v^2} \quad (22)$$

Differentiating and dividing by the unit area

$$D(E) = \frac{g}{2\pi} \frac{E}{\hbar^2v^2} = \frac{E}{\pi\hbar^2v^2} \quad (23)$$

A degeneracy factor of $g = 4$ was used to account for electron spin and valley degeneracy.

Problem 2 (h)

The energy dependent number of states $N(E)$ depends on the dispersion relation of the material. Consider first the parabolic band relation where $k = \sqrt{2mE/\hbar^2}$:

$$N(E) = \frac{gL_xL_y}{4\pi} \frac{2mE}{\hbar^2} \quad (24)$$

Differentiating this with respect to energy and dividing by the unit area produces the density of states per unit area

$$D(E) = \frac{gm}{2\pi\hbar^2} = \frac{m}{\pi\hbar^2} \quad (25)$$

A degeneracy factor of $g = 2$ was used to account for the electron spin. This shows that the 2D density of states does not depend on energy.

Problem 2 (i)

Graphene has a linear density of states while the parabolic band has no dependence on the energy. Conceptually, this implies that at lower energies a 2D parabolic device will conduct more than graphene, but as the energy increases graphene will scale linearly and begin conducting more than 2D device (ignoring differences in effective mass etc).

The electron density is found by multiplying the density of states by the occupation factor given by the Fermi function.

$$\rho(E) = D(E)f(E) \quad (26)$$

At a temperature of $T = 0$ the electron density obeys the same relative relationship with energy as the density of states. At $T = 0$ the Fermi function has no curvature and is either 0 or 1 depending on energy. As the temperature increases the Fermi function will begin to curve and as a result introduce a temperature dependence of the electron density in both the parabolic and graphene based devices.

ECE 60600 Homework2 Solution

1 Problem 1

(a) Because the potential is infinite for $x \leq 0$, we only need to deal with the two regions $0 < x < a$ and $x > a$.

$$\begin{aligned}\frac{d^2\psi_0}{dx^2} + k^2\psi_0 &= 0 & k &= \sqrt{2mE/\hbar^2} & 0 < x < a \\ \frac{d^2\psi_+}{dx^2} - \alpha^2\psi_+ &= 0 & \alpha &= \sqrt{2m(U_0 - E)/\hbar^2} & x > a\end{aligned}$$

(b) The general solutions, identical to Eqs (2.39b and c) in the text, are

$$\begin{aligned}\psi_0(x) &= A_0 \sin kx + B_0 \cos kx & 0 < x < a \\ \psi_+(x) &= A_+ e^{\alpha x} + B_+ e^{-\alpha x} & x > a\end{aligned}$$

(c) The appropriate boundary conditions are

$$\begin{aligned}\psi_0(0) &= 0 \\ \psi_+(+\infty) &= 0 \\ \psi_0(a) &= \psi_+(a) \\ \left. \frac{d\psi_0}{dx} \right|_a &= \left. \frac{d\psi_+}{dx} \right|_a\end{aligned}$$

(d) Application of the boundary conditions yields

$$\begin{aligned}B_0 &= 0 \\ A_+ &= 0 \\ A_0 \sin ka &= B_+ e^{-\alpha a} \\ kA_0 \cos ka &= -\alpha B_+ e^{-\alpha a}\end{aligned}$$

(e) Rewrite set of linear equations in (d) as matrix, we get

$$\begin{bmatrix} \sin(ka) & -e^{-\alpha a} \\ k \cos(ka) & \alpha e^{-\alpha a} \end{bmatrix} \begin{bmatrix} A_0 \\ B_+ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$