Problem Set 1

Spring 18

Due: Friday, February 16th, 11:59pm.

1. Propositional Logic

   In 1976, Appel and Haken proved the *four color theorem*: every finite planar graph is 4-colorable (the vertices can be colored with four colors such that every two adjacent vertices are colored differently). Use the compactness theorem for propositional logic to extend the theorem to *infinite* planar graphs, i.e., even a planar graph has infinite vertices, it is still 4-colorable.

   Hint: Show that for arbitrary planar graph G, the 4-colorability of G can be encoded to the satisfiability of a propositional formula φ_G. Then use the compactness theorem to argue φ_G is still satisfiable even if G becomes infinite.

2. First-Order Logic

   Prove that first-order logic cannot express infinity, i.e., there does not exists a FOL formula φ such that for any first-order structure, $M \models \varphi$ if and only if $|M|$ is infinite (or finite).

   Hint: use the compactness theorem for FOL.

3. First-Order Theories

   Use the Quantifier-Elimination method for the theory of rationals to construct quantifier-free formulae for the following quantified formulae:

   (a) $\forall y(3 < x + 2y \vee 2x + y < 3)$

   (b) $\forall x\Big(\big(\exists y(x = 2y)\big) \rightarrow \big(\exists y(3x = 2y)\big)\Big)$

4. Floyd-Hoare Verification

   The following program purports to compute the factorial of a nonnegative integer n. Prove the programs partial correctness (i.e. that if it halts, it computes the factorial of n, for any nonnegative input n), by giving a Floyd-style proof. Do this by giving an inductive invariant at every point in the program.

```
int r, t;
r = 1;
t = n;
while (t > 0) {
    r = r * t;
    t = t - 1;
}
return r;
```