Problem Set 1
Spring 18

Due: Friday, February 16th, 11:59pm.

1. Propositional Logic

In 1976, Appel and Haken proved the four color theorem: every finite planar graph is 4-colorable (the vertices can be colored with four colors such that every two adjacent vertices are colored differently). Use the compactness theorem for propositional logic to extend the theorem to infinite planar graphs, i.e., even a planar graph has infinite vertices, it is still 4-colorable.

Hint: Show that for arbitrary planar graph $G$, the 4-colorability of $G$ can be encoded to the satisfiability of a propositional formula $\varphi_G$. Then use the compactness theorem to argue $\varphi_G$ is still satisfiable even if $G$ becomes infinite.

2. First-Order Logic

Prove that first-order logic cannot express infinity, i.e., there does not exists a FOL formula $\varphi$ such that for any first-order structure, $M \models \varphi$ if and only if $|M|$ is infinite (or finite).

Hint: use the compactness theorem for FOL.

3. First-Order Theories

Use the Quantifier-Elimination method for the theory of rationals to construct quantifier-free formulae for the following quantified formulae:

(a) $\forall y (3 < x + 2y \lor 2x + y < 3)$
(b) $\forall x \left( (\exists y (x = 2y)) \rightarrow (\exists y (3x = 2y)) \right)$

4. Floyd-Hoare Verification

The following program purports to compute the factorial of a nonnegative integer $n$. Prove the programs partial correctness (i.e. that if it halts, it computes the factorial of $n$, for any nonnegative input $n$), by giving a Floyd-style proof. Do this by giving an inductive invariant at every point in the program.

```c
int r, t;
r = 1;
t = n;
while (t > 0) {
    r = r * t;
    t = t - 1;
}
return r;
```