A Decidable Logic for Tree Data-Structures with Measurements

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Abstract. We present DRYAD_{dec}, a decidable logic that allows reasoning about tree data-structures with measurements. This logic supports user-defined recursive measure functions based on Max or Sum, and recursive predicates based on these measure functions, such as AVL trees or red-black trees. We prove that the logic's satisfiability is decidable. The crux of the decidability proof is a small model property which allows us to reduce the satisfiability of DRYAD_{dec} to quantifier-free linear arithmetic theory which can be solved efficiently using SMT solvers. We also show that DRYAD_{dec} can encode a variety of verification and synthesis problems, including natural proof verification conditions for functional correctness of recursive tree-manipulating programs, legality conditions for fusing tree traversals, synthesis conditions for conditional linear-integer arithmetic functions. We developed the decision procedure and successfully solved 220+ DRYAD_{dec} formulae raised from these application scenarios, including verifying functional correctness of programs manipulating AVL trees, red-black trees and treaps, checking the fusibility of height-based mutually recursive tree traversals, and counterexample-guided synthesis from linear integer arithmetic specifications. To our knowledge, DRYAD_{dec} is the first decidable logic that can solve such a wide variety of problems requiring flexible combination of measure-related, data-related and shape-related properties for trees.

1 Introduction

Logical reasoning about tree data-structures has been needed in various application scenarios such as program verification [32,24,26,42,49,4,14], compiler optimization [17,18,9,44] and webpage layout engines [30,31]. One particular class of desirable properties is the measurements of trees such as the size or height. For example, one may want to check whether a compiler optimizer always reduces the size of the program in terms of the number of nodes in the AST, or a tree balancing routine does not increase the height of the tree. These measurements are usually tangled with other shape properties and arithmetic properties, making logical reasoning very difficult. For example, an AVL tree should be sorted (arithmetic property) and height-balanced (shape property based on height), or a red-black tree of height 5 should contain at least 10 nodes (two measurements combined).

Most existing logics for trees either give up the completeness, aiming at mostly automated reasoning systems [5,16,39,4], or disallow either data properties [32,58,28] or tree measurements [24,25]. There do exist some powerful automatic verification systems that are capable of handling all of data, shape and tree measurements, such as VCDryad [26,42,36] and Leon [49,50]. However, the underlying logic of VCDryad cannot reason about the properties of AVL trees or red-black trees in a decidable fashion. In other words, they can verify the functional correctness of programs manipulating AVL trees or red-black trees, but they do not guarantee to provide a concrete counterexample to disprove a defective program. Leon [49,50] does guarantee decidability/termination for a small and brittle fragment of their specification language, which does not capture even the simplest measurement properties. For example, consider a program that inserts a new node to the leftmost path of a full tree: Skipping lines 2 and 3, the program recursively finds the leftmost leaf of the input tree and inserts a newly created node to the left. The **requires** (line 2) and **ensures** (line 3) clauses describe the simplest properties regarding the size of the tree: if the input tree is a nonempty full tree, the returned tree after running the program should not be a full tree and should contain at least 2 nodes. Note that the full-treeness *full** and the tree-size *size** can be defined recursively in VCDryad or Leon in a similar manner. However, none of VCDryad or Leon can verify the program below in a decidable fashion (see explanation in Section 5).

```
1  loc insertToLeft(Node t)
2    requires full*(t) ∧ size*(t) ≥ 1
3    ensures ¬full*(ret) ∧ size*(ret) ≥ 2
4    {
5      if (t.l == nil) t.l = new Node();
6      else t.l = insertToLeft(t.l);
7      return t;
8    }
```

In this work, our aim is to develop a decidable logic for tree data-structures that combines shape, data, and measurement. The decidability for such a powerful logic is highly desirable, as the decision procedure will guarantee to construct either a proof or witness trees as a disproof, which can benefit a wide variety of techniques beyond deductive verification, e.g., syntax-guided synthesis or test generation.

The decidable logic we set forth in this paper stems from the DRYAD logic, an expressive tree logic proposed along with a proof methodology called Natural Proofs [26,42]. DRYAD allows the user to define recursive definitions that can be unfolded exhaustively for arbitrarily large trees. Natural proofs, as a lightweight, automatic but incomplete proof methodology, restricts the unfolding to the footprint of the program only, then encodes the unfolded formula to decidable SMT-solvable theories using predicate abstraction, i.e., treating the remaining recursive definitions as uninterpreted. The limited unfolding and predicate abstraction make the procedure incomplete.

In this paper, we identify DRYAD$_{dec}$, a fragment of DRYAD, and show that its satisfiability is decidable. The fragment limits both user-defined recursive definitions and formulae with carefully crafted restrictions to obtain the *small model property*. With a given DRYAD$_{dec}$ formula, one can analytically compute a bound up to which all recursive definitions should be unfolded, and the small

$dir \in Loc$ Fields	$G \in Loc$ Field Groups	$x, y \in Loc$ Variables	$K : Int$ Constant
$f \in Int$ Fields	r : *Intermittence*	$j, k \in Int$ Variables	$q \in Boolean$ Variables

$$\text{Increasing } Int \text{ function} : \mathit{mif}^*(x) \stackrel{def}{=} \texttt{ite}\Big(\texttt{isNil}(x),\ -\infty,\ \max\big(\{\mathit{mif}^*(x.dir) | dir \in Dir\} \cup \{it[x]\}\big)\Big)$$

$$\text{Decreasing } Int \text{ function} : \mathit{mdf}^*(x) \stackrel{def}{=} \texttt{ite}\Big(\texttt{isNil}(x),\ \infty,\ \min\big(\{\mathit{mdf}^*(x.dir) | dir \in Dir\} \cup \{it[x]\}\big)\Big)$$

$$\text{Increasing } IntSet \text{ function} : \mathit{sf}^*(x) \stackrel{def}{=} \texttt{ite}\Big(\texttt{isNil}(x),\ \emptyset,\ \big(\textstyle\bigcup_{dir} \mathit{sf}^*(x.dir)\big) \cup ST[x]\Big)$$

$$\text{Measure function Max-based} : \mathit{lif}^*(x) \stackrel{def}{=} \texttt{ite}\Big(\texttt{isNil}(x),\ 0,\ \max_{dir \in Dir} \mathit{lif}^*(x.dir)\ + \texttt{ite}^r(\upsilon[x], 1, 0)\Big)$$

$$\text{Measure function Sum-based} : \mathit{eif}^*(x) \stackrel{def}{=} \texttt{ite}\Big(\texttt{isNil}(x),\ 0,\ \sum_{dir \in Dir} \mathit{eif}^*(x.dir)\ + \texttt{ite}^r(\upsilon[x], 1, 0)\Big)$$

$$\text{General predicate} : gp^*(x) \stackrel{def}{=} \texttt{ite}\Big(\texttt{isNil}(x),\ \texttt{true},\ \big(\bigwedge_{dir} gp^*(x.dir)\big) \wedge \varphi[x.dir,\ x.f]\Big)$$

(φ may involve other general predicates or increasing functions that only have positive coefficients, or decreasing functions that only have negative coefficients.)

$$\text{Measure-related predicate} : mp^*(x) \stackrel{def}{=} \texttt{ite}\Big(\texttt{isNil}(x),\ \texttt{true},\ \big(\bigwedge_{dir} mp^*(x.dir)\big) \wedge \varphi[x.dir,\ x.f]\Big)$$

(φ may involve anything allowed for general predicates and one Max-based measure function lif^* in the form of $\mathit{lif}^*(x.dir_1) - \mathit{lif}^*(x.dir_2) \geq K$)

$$\text{Local } Int \text{ Term:}\quad it, it_1, it_2, \ldots ::= K \mid x.f \mid t_1 + t_2 \mid -t \mid \texttt{ite}(\upsilon, t_1, t_2)$$
$$\text{Local } Set \text{ Term:}\quad ST, ST_1, ST_2, \ldots ::= \emptyset \mid \{it\} \mid ST_1 \cup ST_2 \mid ST_1 \cap ST_2$$
$$\text{Local Formula:}\quad \upsilon, \upsilon_1, \upsilon_2, \ldots ::= it_1 \geq 0 \mid \upsilon_1 \wedge \upsilon_2 \mid \upsilon_1 \vee \upsilon_2 \mid \neg\upsilon$$

Fig. 1: Templates of DRYAD$_{dec}$ Functions and Predicates

model property ensures that a fixed number of unfolding is sufficient and guarantees completeness. The DRYAD$_{dec}$ logic features the following properties: a) allows user-defined and mutually recursive definitions to describe the functional properties of AVL trees, red-black trees and treaps; b) the satisfiability problem is *decidable*; c) experiments show that the logic can be used to encode and solve a variety of practical problems, including *correctness verification, fusibility checking and syntax-guided synthesis*. To the best of our knowledge, DRYAD$_{dec}$ is the first decidable logic that can reason about a flexible mixture of sophisticated data, shape and measure properties of trees.

2 A decidable fragment of DRYAD

DRYAD is a logic for reasoning about tree data-structures, first proposed by Madhusudan *et al.* [26]. DRYAD can be viewed as a variant of first-order logic extended with least fixed points. The syntax of DRYAD is free of quantifiers but supports user-provided recursive functions for describing properties and measurements of tree data structures. Each recursive function maps trees to a boolean value, an integer or a set of integers, and is defined recursively in the following form: $F^*(x) \stackrel{def}{=} \texttt{ite}(\texttt{isNil}(x),\ F_{base},\ F_{ind})$, where F_{base} stands for the value of the base case, i.e., x is *nil*, and F_{ind} recursively defines the value of $F^*(x)$ based on the local data fields and subtrees of x. DRYAD is in general undecidable and Mad-

$$\begin{array}{rl}
\textit{Int}\ \text{Term:}\quad t, t_1, t_2, \ldots ::= & K \mid j \mid \textit{mif}^*(x) \mid \textit{mdf}^*(x) \mid t_1 + t_2 \mid -t \mid \texttt{ite}(l, t_1, t_2) \\
\textit{IntSet}\ \text{Term:}\quad S, S_1, S_2, \ldots ::= & \emptyset \mid \{t\} \mid \textit{sf}^*(x) \mid S_1 \cup S_2 \mid S_1 \cap S_2 \\
\text{Measure-related Formula}\ \psi ::= & \textit{lif}^*(x) - \textit{lif}^*(y) \geq K \mid \textit{eif}^*(x) - \textit{eif}^*(y) \geq K \mid \\
& \textit{lif}^*(y) \geq K \mid \textit{eif}^*(y) \geq K \mid \textit{mp}^*(x) \\
& (x \text{ is related to } \textit{lif}^*, \textit{eif}^*, \text{ or } \textit{mp}^*, \text{ respectively.}) \\
\text{Negatable Formula:}\quad l ::= & q \mid t \geq 0 \mid t \in S \mid \psi \mid \texttt{isNil}(x) \mid \textit{gp}^*(x) \mid \neg l \\
\text{Formula:}\quad \varphi, \varphi_1, \varphi_2, \ldots ::= & l \mid S_1 \not\subseteq S_2 \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2
\end{array}$$

(Every variable x can be related to only one measure function.)

Fig. 2: Syntax of DRYAD$_{dec}$ logic

husudan *et al.* [26] present an automatic but incomplete procedure for DRYAD based on a methodology called *Natural Proofs*.

In this paper, we carefully crafted a decidable fragment of DRYAD, called DRYAD$_{dec}$, which is amenable for reasoning about the measurement of trees.

2.1 Syntax

The templates for recursive functions and predicates allowed in DRYAD$_{dec}$ are shown in Figure 1 and the syntax of DRYAD$_{dec}$ is presented in Figure 2. To simplify the presentation, these figures show unary functions and predicates only, i.e., those recursively defined over a single tree. DRYAD$_{dec}$ also supports recursive functions and predicates with multiple arguments, which are amenable to define data structures characterizing loop invariants, such as list segments, tree-with-a-hole, etc.

Overall, DRYAD$_{dec}$ allows seven categories of recursive functions or predicates with various types, constraints on their definitions and forms of occurrence in a formula. Figure 3 gives several common examples of recursive definitions expressible in DRYAD$_{dec}$. We explain the intuition behind each category below:

Increasing or decreasing *Int* function[1] defines the maximum or minimum value of $it\big[x\big]$, where x is the location being unfolded in the tree. The local term $it\big[x\big]$ is an integer term defined only based on the local data fields of x. The most common example is $it\big[x\big] = x.\mathsf{key}$; then the function gives the maximum or minimum key stored in a tree. These increasing/decreasing functions can be combined using standard arithmetic connectives to form atomic formulae.

Increasing *IntSet* function defines the union of all set terms $ST\big[x\big]$ for any location x under the tree, where $ST\big[x\big]$ is a set of local integer terms defined only based on the local data fields of x. The most typical example is the function representing the set of all keys w.r.t. the data field key, where $ST\big[x\big] = \{x.\mathsf{key}\}$. These *IntSet* functions can be combined with regular *Int* terms arbitrarily to form *IntSet* terms in DRYAD$_{dec}$, which can be further used to construct atomic

[1] Intuitively, a DRYAD$_{dec}$ function is increasing/decreasing if its value monotonically increases/decreases when the input tree expands. The monotonicity will be formally defined in Section 3.1

formulae for set-inclusion and subset relationship. The only restriction is that the subset checking $S_1 \subseteq S_2$ can occur *negatively only*.

There are two types of **measure functions**. Intuitively, they recursively define Max- and Sum-based measurements of a tree or tree segment, respectively. For each node x under the tree, it counts towards the measurement, i.e., the height/size being increased by 1, if and only if a local formula $\upsilon\big[x\big]$ is satisfied. In Figure 1, this conditional value is written as $\mathtt{ite}^r(\upsilon\big[x\big], 1, 0)$, where r is an integer constant called *intermittence*. For example, the black height for red black trees can be defined with intermittence 2: $\mathtt{ite}^2(x.\mathit{color} = \mathtt{black}, 1, 0)$. The intermittence's semantics will be explained in Section 2.3. Specifically, when $\upsilon\big[x\big] \stackrel{def}{=} \mathtt{true}$ and $r = 1$, the corresponding Max- and Sum-based functions define the regular tree height and size, respectively. In this paper, we denote them as *height** and *size**.

A measure-related *Int* term can be a measure function $f^*(x)$ only, or a difference of form $f^*(x_1) - f^*(x_2)$. A measure-related *Int* term can be compared with a constant K. For example, one can specify two trees with the same height using $\mathit{height}^*(x_1) - \mathit{height}^*(x_2) = 0$.

General predicate is satisfied by trees (x) if and only if a local constraint φ is satisfied between any location in x. Notice that φ may involve other non-measure-related functions or predicates (with some restrictions as shown in Figure 1). For example, the *sorted** property can be defined based on *max** and *min** (see the definition of *sorted** in Figure 3).

Measure-related predicate is similar to general predicates. In addition to everything allowed in the definition of general predicates, a measure-related predicate is allowed to involve a single measure-related function in the difference form. For example, an *avl**-tree requires the *height**-difference between two subtrees is at most one (see the definition of *avl** in Figure 3).

2.2 Syntactic Restrictions for Decidability

As we have mentioned before, the syntax of DRYAD$_{dec}$ is carefully crafted for decidability. Besides the specific syntactical restrictions delineated above for the definitions in each category of recursive functions or predicates, DRYAD$_{dec}$ also restricts how variables, functions and predicates can be related to each other. As shown in Figure 2, a variable x is considered related to a measure function if x occurs in a measure-related predicate or in the difference form $f^*(x) - f^*(y)$. One important restriction of DRYAD$_{dec}$ is that a location variable can be related to only one measure function. For example, DRYAD$_{dec}$ cannot express a single-path tree: $\mathit{height}^*(x) = \mathit{size}^*(x)$.

Insight Behind the Syntax. Intuitively, the DRYAD$_{dec}$ syntax characterizes the class of formulae *independent* to the height/size of the tree. Hence non-measure functions such as *min** or *max** can occur unrestrictedly in the logic, as their values are only determined by the "witness nodes". For measure functions such as height or size, obviously they are determined by the height/size of the tree; that's why we allow only differences between measure functions such as

$height^*(x_1) - height^*(x_2)$, as the difference is *unchanged* if we tailor both the two trees rooted by x_1 and x_2 at the same time. Likewise for subset relation, the negation of subset relation $S_1 \nsubseteq S_2$ can also be captured by a "witness node" which is in the set of S_1 but not in the set of S_2 whereas the subset relation $S_1 \subseteq S_2$ is determined by all elements in two sets. Therefore, $S_1 \nsubseteq S_2$ is allowed whereas $S_1 \subseteq S_2$ is not as $S_1 \subseteq S_2$ is not ensured to be unchanged through tailoring. To conclude, we try to maximize the logic without losing decidability.

Capabilities And Limitations. DRYAD$_{dec}$ can express all standard tree-based data structures such as lists, trees, lists of trees, etc., and some limited non-tree data structures such as doubly linked lists or cyclic lists. However, DRYAD (and inherently DRYAD$_{dec}$) is unable or not natural to express non-tree data structures, e.g., DAGs or overlaid data structures. The main restrictions from DRYAD to DRYAD$_{dec}$ are twofold. First, only Max- and Sum-based measure functions are allowed. For example, DRYAD$_{dec}$ cannot define the length of the leftmost path of a tree. Second, properties involving multiple measure functions are not allowed. For example, as red-black trees are defined using black-height, DRYAD$_{dec}$ cannot describe the real height of a red-black tree.

Category	Name	Definition
Measure Function (Max-based)	$height^*$	$\texttt{ite}\big(\texttt{isNil}(x), 0, \texttt{max}\big(height^*(x.left), height^*(x.right)\big) + 1\big)$
	bh^*	$\texttt{ite}\big(\texttt{isNil}(x), 0, \texttt{max}\big(bh^*(x.left), bh^*(x.right)\big) + \texttt{ite}^2(x.isBlack, 1, 0)\big)$
Measure Function (Sum-based)	$size^*$	$\texttt{ite}\big(\texttt{isNil}(x), 0, size^*(x.left) + size^*(x.right) + 1\big)$
Non-Measure Function	max^*	$\texttt{ite}\big(\texttt{isNil}(x), -\infty, \texttt{max}\big(max^*(x.left), max^*(x.right), x.key\big)\big)$
	min^*	$\texttt{ite}\big(\texttt{isNil}(x), \infty, \texttt{min}\big(max^*(x.left), min^*(x.right), x.key\big)\big)$
	$keys^*$	$\texttt{ite}\big(\texttt{isNil}(x), \emptyset, keys^*(x.left) \cup keys^*(x.right) \cup \{x.key\}\big)$
Measure-related Predicate	avl^*	$\texttt{ite}\big(\texttt{isNil}(x),\ \texttt{true},\ avl^*(x.left) \land avl^*(x.right)$ $\land 1 \geq height^*(x.left) - height^*(x.right) \geq -1\big)$
	rbt^*	$\texttt{ite}\big(\texttt{isNil}(x),\ \texttt{true},\ rbt^*(x.left) \land rbt^*(x.right)$ $\land bh^*(x.left) = bh^*(x.right)\big)$
General Predicate	$sorted^*$	$\texttt{ite}\big(\texttt{isNil}(x),\ \texttt{true},\ sorted^*(x.left) \land sorted^*(x.right)$ $\land max^*(x.left) < x.key < min^*(x.right)\big)$
	$treap^*$	$\texttt{ite}\big(\texttt{isNil}(x),\ \texttt{true},\ treap^*(x.left) \land treap^*(x.right)$ $\land max\_key^*(x.left) < x.key < min\_key^*(x.right)$ $\land max\_prt^*(x.left) < x.prt \land max\_prt^*(x.right) < x.prt\big)$

Fig. 3: List of recursive definitions

2.3 Semantics

The semantics of DRYAD$_{dec}$ is consistent with the semantics of DRYAD defined in [26], which is interpreted on program heaps. A heap consists of a finite set of locations with the same layout. Each location contains a set of pointer fields *Dir* and a set of data fields *DF*. In addition, there is a set of location variables *LV*, a set of integer variables *IV*, and a special location *nil* where the pointer fields can point to. We call $\Sigma = (Dir, DF, LV, IV)$ a signature for the DRYAD$_{dec}$ logic, and call the heap w.r.t. Σ a Σ-heap. The formal definition is as below:

Definition 1. *Let* $\Sigma = (Dir, DF)$*. A* Σ*-heap is a tuple* (N, pf, df) *where:*

– *N is a finite set of locations; $nil \in N$ is a special location;*
– *$pf : (N \setminus \{nil\}) \times Dir \to N$ is a function defining the pointer fields;*
– *$df : (N \setminus \{nil\}) \times DF \to \mathbb{Z}$ is a function defining the data fields.* □

A recursive definition $f^*(x)$ can be interpreted on a Σ-heap (N, pf, df) by mapping x to a location n_x in the heap. As f^* is a recursive definition, $f^*(x)$ is *undefined* if n_x is not the root of a tree; otherwise it is evaluated inductively using the recursive definition of f^*. Notice that the evaluation is only determined by a subset of N that is reachable from n_x. If a heap T's locations form a tree, we use $f^*(T)$ to represent the interpretation of $f^*(x)$ with x mapped to the root of T. We simply call T a *Σ-tree*. We denote n as $root(T)$, and the subtree rooted by $n.dir$ as $T.dir$.

A Dryad$_{dec}$ formula $\varphi(\bar{x}, \bar{j}, \bar{r})$ can be interpreted on a Σ-heap by mapping every *Loc* variable in $\bar{x}$ to a location in the heap and mapping every *Int* variable in $\bar{j}$ and *IntSet* variable in $\bar{r}$ to the corresponding sort. The mapping is valid only if every *Loc* variable maps to the root of a tree in the heap; otherwise the interpretation is undefined.

Most logical connectives and recursive functions/predicates are interpreted as one can expect. In addition, measure functions have a special intermittence constraint. Recall that any measure function f^*'s definition comes with an intermittence r occurred in form of $\mathtt{ite}^r(\upsilon[x], 1, 0)$. The intermittence is a positive integer indicating *how often* the local formula $\upsilon[x]$ should be satisfied in the trees. Formally, f^* is defined on a tree T only if the following intermittence constraint is satisfied: for *any* node x in T and its $(r-1)$ immediate ancestors, there is a node w within these r nodes such that $\upsilon[w]$ is true.

Notice that a satisfiable φ with m *Loc* variables $x_1, \ldots, x_m$ can always be satisfied by a heap consisting of m disjoint trees $T_1, \ldots, T_m$ by mapping every x_i to the root of T_i. In the rest of the paper, we focus on checking satisfiability and consider only these disjoint-tree models.

3 Proof of Decidability

In this section, we prove that the satisfiability problem of Dryad$_{dec}$ is decidable. The crux of the proof is the *small model property*: Given a Dryad$_{dec}$ formula φ, it is satisfiable only if it is satisfied by a model of bounded size. The main idea is to show that if φ is satisfied by a model larger than the bound, one can tailor the model to obtain a smaller model which preserves the satisfiability (Theorem 1).

Intuitively, the value of an increasing/decreasing *Int* function or increasing *IntSet* function always relies on a *witness node*. For example, if an increasing *Int* function mif^* is defined w.r.t. a local term it within any tree T, there is a witness node w s.t. $mif^*(T) = it[w]$ and $it[w] \geq it[u]$ for any other node u. Then these function values can be preserved as long as these witness nodes are retained in the tailored model (Lemma 6).

The most challenging part is that the value of a measure-related function will become smaller. Nonetheless, we prove that one can tailor the tree appropriately such that the height/size is reduced by exactly 1 while all relevant

$d : \lvert Dir \rvert$	n : # *Int* Variables	m : # *Loc* Variables
P : # General Predicates	M : # lif^* -related Predicates	C : Balance Bound
D_{ht} : Height Bound	D_{sz} : Size Bound	D_{sub} : Subtractive Bound
F : # Increasing/Decreasing *Int* Fuctions	E : # Increasing *IntSet* Fuctions	

Fig. 4: Denotations for metrics

recursive predicates are still preserved. Then as these measure functions only occur in the form $f^*(x_1) - f^*(x_2)$, both $f^*(x_1)$ and $f^*(x_2)$ will be reduced by 1 simultaneously and the difference will remain unchanged. Moreover, we prove the tailoring guarantees that the evaluation of other functions and predicates are not affected (Lemmas 7 and 8).

3.1 Preliminaries

We start with some formal definitions and lemmas. The proofs for these lemmas can be found at the project website [1].

Normalization. We normalize a DRYAD_{dec} formula φ through repeatedly applying the following steps until no rule can be applied:

1. For every `ite`-expression $E_{ite} = \texttt{ite}(l, t_1, t_2)$ in φ, rewrite φ to $(l \wedge \varphi[t_1/E_{ite}]) \vee (\neg l \wedge \varphi[t_2/E_{ite}])$;
2. For every literal $S_1 \not\subseteq S_2$, introduce a fresh integer variable w as a witness, and replace the literal with $w \in S_1 \wedge w \notin S_2$;
3. For every atomic formula of the form $t \in A \cap B$ or $t \in A \cup B$, replace it with $t \in A \wedge t \in B$ or $t \in A \vee t \in B$, respectively;
4. For every atomic formula of the form $t_1 \in \{t_2\}$, replace it with $t_1 = t_2$;
5. For every atomic formula $t \in S$ where t is a non-variable expression, introduce a fresh integer variable j and replace $t \in S$ with $j \in S \wedge j = t$;
6. For every literal $\mathit{lif}^*(x) - \mathit{lif}^*(y) \not\geq K$ or $\mathit{eif}^*(x) - \mathit{eif}^*(y) \not\geq K$, replace it with $\mathit{lif}^*(y) - \mathit{lif}^*(x) \geq 1 - K$ or $\mathit{eif}^*(y) - \mathit{eif}^*(x) \geq 1 - K$.

We denote the normalized formula constructed from φ as $Norm(\varphi)$. The first two steps remove the `ite`-expressions and the $\not\subseteq$ relations from the formula. Steps 3–5 make sure that set terms occur in the form of $j \in \mathit{sf}^*(x)$ only. Step 6 makes sure differences between measure functions occur positively only. To check the satisfiability of φ, one can always normalize the formula first, as the normalization process preserves satisfiability, which can be trivially proved:

Lemma 1. *For any DRYAD_{dec} formula φ, φ and $Norm(\varphi)$ are equisatisfiable.*

Formula Metrics. The size bound for the small model property will be determined by a set of metrics regarding the signature Σ, the formula φ and the set of recursive definitions it relies on. For the rest of the paper, we fix the denotation for these metrics, as shown in Figure 4. Besides simple counting of functions or predicates, these metrics also include the bounds on various kinds of constants involved in the formula. Specifically, we define the following four bounds:

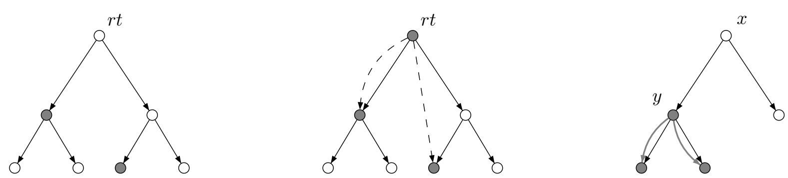
Definition 2 (Balance Bound). *For any Max-based measure function lif^*, the* balance bound C *is the* maximal *constant in the set:* $\{\texttt{ite}(K > 0, K, 1 - K) \mid lif^*(t) - lif^*(t') \geq K$ *occurred in the definition of a lif^*-related predicate*$\}$.

Definition 3 (Subtractive Bound). *The* subtractive bound D_{sub} *of a formula φ is the* maximal *constant in the set:* $\{\texttt{max}(K, 0) \mid lif^*(x) - lif^*(y) \geq K$ *or* $eif^*(x) - eif^*(y) \geq K$ *occurred positively in* $\varphi\}$.

Definition 4 (Height Bound). *The* height bound D_{ht} *of a formula φ is the* maximal *constant in the set:* $\{rK \mid lif^*(y) \geq K$ *occurred positively in φ and r is the intermittence of* $lif^*\}$.

Definition 5 (Size Bound). *The* size bound D_{sz} *of a formula φ is the* maximal *constant in the set:* $\{(\frac{d^r-1}{d-1}) \cdot K + 1 \mid eif^*(y) \geq K$ *occurred positively in φ and r is the intermittence of* $eif^*\}$.

Remark: For all of the above bounds, if the corresponding set is empty, we define the bound to be 0.



(a) T: a binary-tree heap S: the set of shaded nodes

(b) $tailor_S(T)$: the tailored tree represented by shaded nodes and dashed edges

(c) critical nodes and critical paths

Fig. 5: A binary tree example of tailored trees and critical nodes and paths

Tailored Tree and Monotonicity. As a key concept in the decidability proof, the small model is formalized via *tree tailoring*: a tree model can be tailored to obtain a smaller model.

Definition 6 (Tailored tree). *Let $T = (N, pf, df)$ be tree, and let $S \subset N$ be a subset, then the tailored tree $tailor_S(T)$ can be defined as (N', pf', df'), where (i) $N' = S \cup \{lca(S') \mid S' \subseteq S\}$ where $lca(S')$ is the lowest common ancestor of S'; (ii) $pf'(x, dir) = lca(N' \cap T_x.dir)$ for any $x \in N'$ and $dir \in Dir$, where $T_x.dir$ is the subtree of T rooted by $x.dir$; and (iii) $df' = df|_{N' \times DF}$.*

Note that N' is LCA-closed, the lowest common ancestor $lca(N' \cap T.dir)$ defined by $pf'(x, dir)$ always belongs to N'. As an example, Figure 5a shows a tree-shaped heap T and a subset S of nodes (the shaded ones); Figure 5b shows the tailored tree $tailor_S(T)$ constructed from S. The edges of the tailored tree are represented using dashed edges.

Now with tailored tree formally defined, we can prove the *monotonicity* of non-measure functions/predicates, a very important property for our decidability proof. We prove the following three lemmas.

Lemma 2 (Monotonicity for increasing/decreasing function). *Let mif^* (or mdf^*) be an increasing (or decreasing) function w.r.t. Σ. Let T be a Σ-tree and let $tailor_S(T)$ be the tailored tree w.r.t. a subset of nodes S. Then $mif^*(T) \geq mif^*(tailor_S(T))$ (or $mdf^*(T) \leq mdf^*(tailor_S(T))$).*

Lemma 3 (Monotonicity for increasing *IntSet* function). *Let T be a Σ-tree and let $tailor_S(T)$ be the tailored tree w.r.t. a subset of nodes S. Then for any increasing set function sf^*, $sf^*(tailor_S(T)) \subseteq sf^*(T)$.*

Lemma 4 (Monotonicity for general predicate). *Let T be a Σ-tree and let $tailor_S(T)$ be the tailored tree w.r.t. a subset of nodes S. Then for any general predicate gp^*, $gp^*(T)$ implies $gp^*(tailor_S(T))$.*

Critical Path. While measure-related functions/predicates do not have witness nodes, their evaluation can be determined by a set of paths, which we call *critical paths*.

Definition 7 (Critical Node and Critical Path). *Let $T = (N, pf, df)$ be a nonempty Σ-tree and $y \in N$ be a node. Let lif^* be a Max-based measure function. Then y is a* critical node *of T w.r.t. lif^* if one of the following conditions holds:*

1. *$lif^*(y) \geq lif^*(z)$ for any other sibling node z;*
2. *there is a measure-related predicate mp^* whose recursive definition involves a subformula of the form $lif^*(x.dir_1) - lif^*(x.dir_2) \geq K$, and there is a node $x \in N$ such that:*
   - *either $K \geq 1$, $lif^*(x.dir_1) - lif^*(x.dir_2) = K$, $y = x.dir_2$ and $x.dir_1$ is a critical node;*
   - *or $K \leq 0$, $lif^*(x.dir_1) - lif^*(x.dir_2) = K - 1$, $y = x.dir_1$ and $x.dir_2$ is a critical node.*

For the second case, we also call y a critical child *of x. Moreover, a* critical path *w.r.t. lif^* is a path from a child of T to a leaf consisting of critical nodes only.*

As an example, Figure 5c shows a binary tree rooted by x. The shaded nodes are critical nodes and curved edges are two critical paths w.r.t. *height**. (See definition of *height** in Figure 3.)

Lemma 5 (Length bound for critical paths). *Let lif^* be a Max-based function with intermittence r and with a local constraint υ, let T be a d-ary tree. Then for any critical path of T w.r.t. lif^*, the number of nodes satisfying υ on the path is at least $\lfloor \frac{lif^*(T)-1}{(d-1)Cr+1} \rfloor$, where C is the balance bound of lif^*.*

3.2 Tailorability

The tailorability of various functions/predicates is the crux of guaranteeing the small model property, which in turn guarantees the decidability. As mentioned before, non-measure functions/predicates can be easily preserved as long as the tailoring does not affect witness nodes.

Lemma 6 (Tailorability for non-measure functions and general predicates). *Let $T = (N, pf, df)$ be a tree, $S \subset N$ be a subset of nodes.*

Then if the height of T is greater than $P + F + |S|$, there is a tailored tree T' of T such that

(i) T' *contains all nodes of* S*;*
(ii) $f^*(T') = f^*(T)$ *for any increasing/decreasing Int function* f^**;*
(iii) $gp^*(T') \leftrightarrow gp^*(T)$ *for any general predicate* gp^**.*

Proof. See [1]. □

For a Max-based function, a large tree can be tailored by removing exactly one node from every critical path; hence the function value is reduced by 1. Similarly, Sum-based functions can also be reduced by 1 through tailoring.

Lemma 7 (Tailorability for Max-based function). *Let $T = (N, pf, df)$ be a d-ary tree, $S \subset N$ be a subset of nodes. Let lif^* be a Max-based measure function with intermittence r and balance bound C. Then if $lif^*(T) > (P + M + F + |S| + 1) \cdot ((d-1)Cr + 1)$, there is a tailored tree T' of T such that*

(i) T' *contains all nodes of* S*;*
(ii) $f^*(T') = f^*(T)$ *for any increasing/decreasing Int function* f^**;*
(iii) $gp^*(T') \leftrightarrow gp^*(T)$ *for any general predicate* gp^**;*
(iv) $lif^*(T') = lif^*(T) - 1$*;*
(v) $mp^*(T') \leftrightarrow mp^*(T)$ *for any* lif^**-related predicate* mp^**.*

Proof. Let the definition of lif^* be $\texttt{ite}\Big(\texttt{isNil}(x),\ 0,\ \ldots\ +\ \texttt{ite}^r(\upsilon\big[x\big], 1, 0)\Big)$ Consider an arbitrary critical path w.r.t. lif^* in T. By Lemma 5, the number of nodes in the path satisfying the local constraint υ from the definition of lif^* is at least

$$\lfloor \frac{lif^*(T) - 1}{(d-1)Cr + 1} \rfloor \geq \lfloor \frac{(P + M + F + |S| + 1) \cdot ((d-1)Cr + 1)}{(d-1)Cr + 1} \rfloor = P{+}M{+}F{+}|S|{+}1$$

Let $\mathcal{N}$ be the set including all these nodes. We denote a node in $\mathcal{N}$ as n_j if it is the j-th highest one in the set. For each j, consider the set of nodes $\mathcal{N}_j \stackrel{def}{=} \{n \mid n \prec n_j \wedge n \not\preceq n_{j+1}\}$, where $n \prec n_j$ denotes that n is a descendant of n_j. Intuitively, $\mathcal{N}_j$ is the root or a descendant of a sibling of n_{j+1}. Notice that there are at least $P + M + F + |S| + 1$ such sets and they are all disjoint, i.e., there is a set of at least $P + M + F + 1$ nodes such that for every node j in the set, $\mathcal{N}_j \cap S = \emptyset$. Furthermore, consider the witness node for every $f^*(T)$, where f^* is an increasing or decreasing *Int* function, among the remaining at least $P + M + F + 1$ nodes, at least $P + M + 1$ ones are nodes for which corresponding set $\mathcal{N}_j$ does not contain any witness nodes. Moreover, as the number of all predicates is $P + M$, there is at least one node l such that n_l and n_{l+1} [2] agree on the evaluation of all general predicates and lif^*-related predicates.

[2] Let n_{l+1} be *nil* if $|\mathcal{N}| \leq l$.

Now we can replace the subtree rooted by n_l with the subtree rooted by n_{l+1} to form a tailored tree T_l. Notice that T_l holds the first three properties for the desired tailored tree:

1. T_l retains all nodes of S, as $\mathcal{N}_j \cap S = \emptyset$.
2. $f^*(T_l) = f^*(T)$ for any increasing or decreasing f^*.
3. $gp^*(T_l)$ if and only if $gp^*(T)$ for any general predicate gp^*.

The reason for properties (i) and (ii) to hold is straightforward. For property (iii), consider three situations:

1. if $gp^*(T)$ is true, so is $gp^*(T_l)$ by Lemma 4.
2. if $gp^*(T)$ is false and $gp^*(n_l)$ is true, then T does not satisfy gp due to a path not affected by the tailoring. Hence $gp^*(T_l)$ remains false.
3. if $gp^*(T)$ is false and $gp^*(n_l)$ is false, by our assumption about l, n_l and n_{l+1} agree on the evaluation of all general predicates. Hence $gp^*(n_{l+1})$ is also false. Then by Lemma 4, $gp^*(T_l)$ is also false.

Moreover, as n_l and n_{l+1} agree on all predicates, the tailoring also preserves any *lif**-related predicate *mp*.

This tailoring also removes exactly one node from $\mathcal{N}$ for the critical path we are considering. One can continue this tailoring for other critical paths until all critical paths have been shortened and the value of *lif** is reduced by 1. We claim that the resulting tree is just the desired tailored tree T'. As each tailoring guarantees the first three properties, we only need to show the last two properties. Property (iv) is obvious: all critical paths of z are shortened and *lif*$^*(z)$ is reduced by 1. For Property (v), we prove it by a bottom-up induction for any node z under which a tailoring took place. The evaluation of any *lif**-related predicate $mp^*(z)$ is not affected: if the subtree under z replaced another subtree rooted by z', $mp^*(z)$ if and only if $mp^*(z')$ is true; otherwise, there was a separate tailoring for each critical child of z. Therefore

- by induction hypothesis, $mp^*(z.dir)$ is preserved for any mp^* and any *dir*;
- local *Int* terms are not affected, as z is unchanged during the tailoring;
- for any increasing or decreasing function f^* and any child $T.dir$, the value of $f^*(T.dir)$ is preserved during every tailoring and still unchanged;
- similarly, $gp^*(T.dir)$ for any general predicate gp^* is unchanged;
- for any critical child $T.dir$, *lif*$^*(T.dir)$ only occurs in subtractive formulae in the recursive definition for *lif*$^*(x)$. Notice that *lif*$^*(T.dir)$ is decreased by 1 and so is any other critical *lif*$^*(T.dir')$, the evaluation of these subtractive formulae will be unaffected.

$\square$

Lemma 8 (Tailorability for Sum-based function). *Let* $T = (N, pf, df)$ *be a* d*-ary tree,* $S \subset N$ *be a subset of nodes. Let eif** *be a Sum-based measure function with intermittence* r*. Then if eif*$^*(T) > 2 \cdot (|S| + F + 2^P) - 1$*, there is a tailored tree* T' *of* T *such that*

(i) T' *retains all nodes of* S*;*
(ii) $f^*(T') = f^*(T)$ *for any increasing/decreasing Int function* f^**;*
(iii) $gp^*(T') \leftrightarrow gp^*(T)$ *for any general predicate* gp^**;*
(iv) $eif^*(T') = eif^*(T) - 1$*;*

Proof. Let the definition of eif^* be $\texttt{ite}\Big(\texttt{isNil}(x),\ 0,\ \ldots\ +\ \texttt{ite}^r(\upsilon\big[x\big], 1, 0)\Big)$. Let $\mathcal{N}$ be the set including all nodes satisfying υ. Note that $|\mathcal{N}| = eif^*(T) \geq 2 \cdot (|S| + F + 2^P)$. Consider those nodes in $\mathcal{N}$ but not above two other nodes in $\mathcal{N}$ from two different branches: $\mathcal{N}' \stackrel{def}{=} \{n \mid n \in \mathcal{N}, \nexists n_1, n_2, dir_1, dir_2 : dir_1 \neq dir_2 \wedge n_1 \prec n.dir_1 \wedge n_2 \prec n.dir_2\}$. Similar to the proof of Lemma 7, for each node $n \in \mathcal{N}'$, T can be tailored by removing the subtree rooted by n or replaced with its subtree preserving all nodes from $\mathcal{N}'$. We denote the set of removed nodes $\mathcal{N}_n$. Moreover, it is not hard to see that $|\mathcal{N}'| \geq \lceil \frac{|\mathcal{N}|+1}{2} \rceil \geq |S| + F + 2^P + 1$.

Now we remove from $\mathcal{N}'$ every node n such that $\mathcal{N}_n \cap S \neq \emptyset$ or $\mathcal{N}_n$ contains the witness node for $f^*(T)$ for a increasing or decreasing *Int* function f^*. Let the set of the remaining nodes in $\mathcal{N}'$ be $\mathcal{N}''$. As the number of removed nodes from $\mathcal{N}'$ is at most $|S| + F$, $|\mathcal{N}''| \geq 2^P + 1$. Therefore there are at least two nodes $n_1, n_2 \in \mathcal{N}''$ such that n_1 and n_2 agree on all general predicates. If n_1 and n_2 are on the same path and n_1 is above n_2, then we tailor $\mathcal{N}_{n_1}$; otherwise we tailor $\mathcal{N}_{n_2}$. WLOG, assume the tailoring replaces n_2 with n_2' and forms T'. The tailoring satisfies all desired properties:

1. T' retains all nodes of S as $\mathcal{N}_{n_2}$ does not contain any node of S.
2. T and T' agree on all increasing/decreasing functions as all witness nodes are retained.
3. T and T' also agree on all general predicates: for any gp^*, if n_2 and n_2' agree on gp^*, the preservation can be propagated up to the root of T. Otherwise, $gp^*(n_2)$ is false and $gp^*(n_2')$ is true. Notice that n_1 and n_2 are not on the same path in this situation – otherwise n_2' is between n_2 and n_1 and does not satisfy gp^*. Then n_1 is not affected by the tailoring and $gp^*(n_1)$ remains false and propagates up to the root: $gp^*(T)$ remains false.
4. By the definition of $\mathcal{N}$, n_2 is the only node in $\mathcal{N}_{n_2}$ that satisfies the local constraint ϵ; hence $eif^*(T') = eif^*(T) - 1$.

□

3.3 Decidability

Now we are ready to show the small model property for $\text{D}\text{RYAD}_{dec}$.

Theorem 1. *Let* φ *be a* Σ*-formula in* DRYAD_{dec}*. Then there is a height bound* h_φ *such that* φ *is satisfiable if and only if it can be satisfied by trees with height at most* h_φ*.*

Proof. According to Lemma 1, we assume φ is normalized and satisfiable. Consider any m disjoint trees T_1 through T_m satisfying φ. For any T_i, we construct a subset of nodes S_i as follows: for every literal $j \in sf^*(x_i)$ where $sf^*(x)$ is an *IntSet* function recursively defined as $\texttt{ite}\Big(\texttt{isNil}(x),\ \emptyset,\ \big(\bigcup_{dir} sf^{*}(x.dir)\big) \cup ST\big[x\big]\Big)$, there must be a witness node y such that $j \in T_i\big[y\big]$. We add y to S_i. For a fixed location variable x_i, there are up to En atomic formulae of the form $j \in sf^{*}(x_i)$. Hence there are up to En nodes in the subset S_i constructed for T_i.

Now if x_i is related to a Max-based measure function, we claim the following height bound: $h_\varphi = (En+P+M+F+1)\cdot((d-1)Cr+2)+D_{ht}+(m-1)D_{sub}-1$. for a set of variables J including x_i. We define J recursively as the smallest set satisfying the following properties:

- x_i belongs to J;
- if $lif^{*}(x_1) - lif^{*}(x_2) \geq K$ occurs in φ and the inequation is tight, i.e., the model we are considering satisfies $lif^{*}(x_1) - lif^{*}(x_2) = K$, then x_2 belongs to J if x_1 does.

Similarly, if x_i is related to a Sum-based measure function eif^{*}, we claim the following size bound: $U_\varphi = 6(En + F + 2^P) - 3 + 2D_{sz} + 2(m-1)D_{sub}$. Note that the size bound is trivially a height bound as well. The proofs for the two bounds h_φ and U_φ can be found at [1].

If x_i is not related to any measure function, we claim a height bound $En + P + F$. When T_i's height is greater than the bound, by Lemma 6, it can be tailored to T_i' and have all set-inclusions, non-measure *Int* functions and general predicates preserved.

Now we obtain a tree T_i' with strictly fewer nodes. By assumption, T_i is the smallest model and T_i' should not satisfy φ. In the rest of the proof, we will show they do satisfy φ; and the contradiction concludes the proof.

As φ is quantifier-free, we only need to show that for any literal in φ, if T_i satisfies it, so does T_i'. We prove this for each type of literals:

Measure-related Predicate. For any measure-related predicate $mp^{*}(x_j)$ in φ, x_j must be involved in a Max-based measure function lif^{*} or not involved in any measure function. Replacing T_j with T_j' guarantees that $lif^{*}(T_j') = lif^{*}(T_j) - 1$, and according to Lemma 7, $mp^{*}(T_j) = mp^{*}(T_j')$.

Measure-related Inequation. For any atomic formula $f^{*}(x_i) - f^{*}(y) \geq K$ affected by the tailoring, the second rule for the construction of J guarantees that x_i is in J. If $f^{*}(x_i) - f^{*}(y)$ is strictly greater than K or less than K, as the value of $f^{*}(x_i)$ is reduced by only 1 in the course of shrinking, the inequation is still satisfied or unsatisfied. Otherwise, $f^{*}(x_i) - f^{*}(y) = K$, then y is also contained in J. In that case, $f^{*}(x_i)$ is also reduced by 1. Hence $f^{*}(x_i) - f^{*}(y) \geq K$ will remain satisfied or unsatisfied in T_i'.

For any atomic formula $f^{*}(x) \geq K$ affected by the tailoring, the tailoring only happens when $f^{*}(x)$ is greater than K before the tailoring. We have shown above that $f^{*}(x) \geq K$ is still satisfied after each tailoring. Hence the satisfiability is preserved.

For any atomic formula $f^{*}(y) \leq K$, the tailorings will make it easier to be satisfied.

Non-measure predicate or function. By Lemma 7 and Lemma 8, any tailoring described above does not affect the evaluation of any non-measure predicate or function, including any general predicate and increasing/decreasing function.

Set Inclusion. For any $j \in sf^*(x_j)$ in φ satisfied by T_j, if it occurs positively, the witness node is in S and will be preserved during the tailoring from T_j to T'_j; hence it is satisfied by T'_j as well. If T_j does not satisfy $j \in sf^*(x_j)$, as the set $sf^*(x_j)$ becomes smaller during the tailoring (by Lemma 3), T'_j does not satisfy $j \in sf^*(x_j)$.

`isNil` **predicate and other boolean variables.** These are not affected by tree tailoring and obviously unchanged.

□

Corollary 1. *The satisfiability problem of* DRYAD_{dec} *is decidable. For a fixed signature* Σ *and a fixed set of recursive functions, the problem is in NEXPTIME.*

Proof. Given a DRYAD_{dec} formula φ with maximum constant bound D (including subtractive, size and height bounds), by Theorem 1, a minimal satisfying model of the normalized formula consists of m disjoint trees, each of which has a bounded height $\mathcal{O}(n+mD)$, i.e., there are up to $2^{\mathcal{O}(n+mD)}$ nodes in the smallest model. Hence one can unfold every recursive function/predicate in the formula for $2^{\mathcal{O}(n+mD)}$ times and leave them uninterpreted. The resulting formula is equisatisfiable with φ and obviously decidable as it is in the theory of quantifier-free uninterpreted functions and linear integer arithmetic (QF_UFLIA), which is NP-complete. As the size of the QF_UFLIA formula is $2^{\mathcal{O}(n+mD)}$, the satisfiability of DRYAD_{dec} is decidable and is in NEXPTIME.

If Σ does not involve any Max-based measure function, then the size of the tree and the QF_UFLIA formula is bounded by $\mathcal{O}(n + mD)$, and the time complexity becomes NP-complete. □

4 Experiments

To demonstrate the expressivity of DRYAD_{dec} and the efficiency of the decision procedure, we implemented the decision procedure and solved 220+ DRYAD_{dec} formulae. These formulae encode various problems from three verification/synthesis scenarios: natural proof verification, fusion of recursive tree traversals, and synthesis of CLIA functions. The implementation is SMT-based: for each formula, we first analytically computed the height bound; then the decision procedure encoded the DRYAD_{dec} formula to a QF_UFLIA formula with the computed bound, and invoked an SMT solver to solve the formula.

Applications. The first set of 61 DRYAD_{dec} formulae is for program verification. We aim to verify the functional correctness of five tree-manipulating programs, i.e., every routine should ensure that the returned tree after insertion remains a corresponding data-structure. We have described insertToLeft in Section 1; BST-insert, Treap-insert, AVL-insert and RBT-insert are self-explanatory. We manually broke down each program into basic blocks and wrote all of the

Natural Proof Verification Conditions (NPVC) following the NPVC-generation algorithm adapted from [26]. For sanity checking, we also manually implanted some artificial bugs to the programs and created the corresponding NPVCs.

The second set of 48 formulae is for checking the fusibility of recursive tree traversals. Fusion of tree traversals arises in numerous settings [37,31,44,43,17,18,47,27,8] for performance concern. One of the crucial parts for this fusion process is to check the *fusibility* of two traversals, i.e., if there exists a fused traversal that has identical behavior with the original two traversals. We used DRYAD$_{dec}$ to check all possible fusions of two pairs of traversals: a pair of height-based, mutually recursive traversals and another pair of a post-order traversal execute before a pre-order traversal. Neither can be handled by state-of-the-art checkers [48]. Please find more details of encoding fusibility to DRYAD$_{dec}$ at [1].

The last set of 112 formulae is for synthesizing Conditional Linear Integer Arithmetic (CLIA) functions. The goal is to synthesize a sequence of arithmetic operations that implements an unknown function described by a formula. DRYAD$_{dec}$ formulae are created by our in-house Syntax Guided Synthesis SyGuS synthesizer [13] as queries raised from the Counter-Example Guided Inductive Synthesis (CEGIS) algorithm. We adopted 23 benchmarks from the 2017 SyGuS [2] competition, for which the queries fall into DRYAD$_{dec}$. The detail of the CEGIS algorithm and the DRYAD$_{dec}$ encoding can be found at [1].

Scenario	Signature	E	P	M	F	r	C	D_{sub}	Bound
BST mutation	$bst^*, keys^*, max^*, min^*$	1	1	0	2	0	0	0	$n+3$
Treap mutation	$treap^*, prts^*, max\_prt^*$ $keys^*, max\_key^*, min\_key^*$	2	1	0	3	0	0	0	$n+4$
AVL mutation	$height^*, avl^*$	0	0	1	0	1	2	3	$3m-2$
RBT mutation	bh^*, rbt^*	0	0	1	0	2	1	3	$3m-2$
CLIA	$\{exp^*_{spec_f,F} \mid \emptyset \subset F \subseteq G\}$	0	$2^{\lvert G\rvert}-1$	0	0	0	0	0	$\lvert G\rvert \cdot \lvert spec_f\rvert$
Fusion	$dp^*, schd^*$	0	2	0	F	0	0	0	$F+2$

Table 1: Height/Size bounds for different scenarios (Metrics defined in Figure 4)

Bound Optimization. We implemented the decision procedure with a set of optimizations. The height/size bound derived in Theorem 1 is general and loose, affecting the decision procedure's scalability. We developed many optimization strategies for different situations. Every strategy is automatically applied when the corresponding condition is satisfied. Table 1 shows the best bounds we obtain for each scenario after all applicable optimizations. Below we explain the main optimization strategies we developed.

To check the satisfiability of a formula φ, we first converted φ to the Disjunctive Normal Form (DNF) and computed the height/size bound for each disjunct separately, as φ is satisfiable if and only if one of the disjuncts is satisfiable. This helps us compute a better bound in many situations, as for each disjunct, at least one or more factors used in the bound computation, e.g., n, m, D_{ht}, D_{sz} and D_{sub}, can be reduced.

Analyzing how variables occur in φ can also be helpful. For example, the number of location variables m only contribute to the bound with the term

$(m-1)D_{sub}$. This term is concise only if there is a chain of variables $x_1, \ldots, x_m$ such that for any $i < m$, there is a literal $lif^*(x_i) - lif^*(x_{i+1}) \geq K$ in φ with a positive K. Hence the number m can be improved to $|V| + 2$ where $V = \{x \mid$ there are y_1, y_2 and positive K_1, K_2 such that $lif^*(x) - lif^*(y_1) \geq K_1$ and $lif^*(y_2) - lif^*(x) \geq K_2$ occur in $\varphi\}$.

Moreover, when a location variable is involved in the regular *height** function, the local constraint υ is true and trivially satisfied by all nodes. Hence in the proof of Theorem 1, the claim $lif^*(x_i) - L \leq En + P + M + F$ can be improved to $lif^*(x_i) = 0$. As the intermittence r is trivially 1, the height bound can be improved to $(En + P + M + F + 1) \cdot ((d-1)C + 1) + D_{ht} + (m-1)D_{sub}$.

We also observed that the definitions of *avl** and *rbt** do not involve any positive constant, e.g., there is no formula $lif^*(x.dir) - lif^*(x.dir') \geq K$ with positive K. For these measure-related predicates, if they only occur positively in a DRYAD_{dec} formula φ, the height bound computed in Lemma 7 can be improved, because we only need to tailor those paths with maximum number of nodes satisfying the corresponding measure function lif^*'s local constraint υ. Once all of these paths are tailored, the value of lif^* is reduced by 1; moreover, these tailorings make the measure-related predicates easier to be satisfied. Hence the balancedness factor $(d-1)Cr + 1$ can be skipped and the height bound for Lemma 7 becomes $P + F + |S|$; the height bound for lif^*-related variables in Theorem 1 also can be improved to $(2En + 2P + 2F + 1) + D_{ht} + (m-1)D_{sub}$.

For CLIA synthesis, with a set of counterexamples G, there are $2^{|G|} - 1$ predicates and the height bound should be $2^{|G|} - 1$ according to Theorem 1. However, we can easily show an alternative bound which is usually better: $|G| \cdot |spec_f|$ where $|spec_f|$ is the number of distinct f-terms in φ, e.g., those terms of the form $f(v_1, \ldots, v_n)$: no matter how large a decision tree T is, concretizing the $|spec_f|$ terms for each counterexample will lead to up to $|spec_f|$ leaf nodes and the whole set G will lead to up to $|G| \cdot |spec_f|$ leaf nodes in T. Let this set of leaves be S and we can tailor T to $tailor_S(T)$, which is of height up to $|G| \cdot |spec_f|$ and does not affect the evaluation of any f-term.

Performance. Our implementation leverages Z3 [33], a state-of-the-art SMT solver as the backend QF_UFLIA solver. The experiments were conducted on a server with a 40-core, 2.2GHz CPU and 128GB memory running Fedora 26.

Table 2 summarizes the experimental results on correctness verification and tree traversal fusion. For each DRYAD_{dec} formula, we report the formula size, the analytically computed height bound, the size of the encoded Z3 constraint in KB, the time spent by Z3 in seconds ($\perp$ for timeout to 30 mins) and the satisfiability result. Bounds computed from Theorem 1 and corresponding Z3 running time are shown in parentheses if Bounds computed from Theorem 1 are not equal to the optimized bounds. For the program verification examples, the NPVCs generated from different basic blocks vary in their sizes, but share the same height bound. Experiments show that the height bound is critical for the performance of our decision procedure. Our bound optimization can significantly decrease the bounds, making the decision procedure scale well to solve all benchmarks. Table 3 lists the names of CLIA synthesis problems, each

Category	Formulae	Dryad$_{dec}$ size	Bound (Unoptimized)	Z3 size (KB)	Time (s) (Unoptimized)	Satisfiable?
BST_insert	nil, rec_l_pre, rec_r_pre rec_l_post, rec_r_post	≤48	5(11)	≤161	<1 (⊥)	N
	rec_r_post_bug	48	5(11)	161	0.3 (100.5)	Y
Treap_insert	nil, rec_l_pre, rec_r_pre, rec_l_prt_le, rec_r_prt_le, rec_l_r_rtt, rec_r_l_rtt	≤108	7(17)	≤1,696	<12 (⊥)	N
	rec_l_prt_le_bug,	88	7(17)	1,172	0.7 (89.8)	Y
AVL_insert (balancedness)	nil, rec_l_pre, rec_r_pre, rec_l_no_rtt, rec_l_r_rtt, rec_r_no_rtt, rec_r_l_rtt, rec_l_lr_rtt, rec_r_rl_rtt, rec_l_df_0, rec_r_df_0	≤197	7(10)	≤399	<1 (<6)	N
	rec_r_rl_rtt_bug	197	7(10)	399	2.7 (63.2)	Y
AVL_insert (sortedness)	nil, rec_l_pre, rec_r_pre, rec_l_no_rtt,, rec_r_no_rtt, rec_l_r_rtt, rec_r_l_rtt, rec_l_lr_rtt, rec_r_rl_rtt	≤134	5(11)	≤271	<1 (⊥)	N
RBT_insert (balancedness)	nil, rec_l_pre, rec_r_pre, rec_l_l_blk, rec_l_r_rd, rec_l_ll_rd, rec_l_all_blk, rec_r_r_blk, rec_r_l_rd, rec_r_rr_rd, rec_r_all_blk, rec_l_lr_rd, rec_r_rl_rd	≤150	7(10)	≤464	<1 (<6)	N
	l_r_rd_bug	142	7(10)	279	0.4 (9.4)	Y
RBT_insert (sortedness)	nil, rec_l_pre, rec_l_l_blk, rec_r_pre, rec_r_r_blk, rec_l_r_rd, rec_l_lr_rd, rec_l_ll_rd, rec_l_all_blk, rec_r_l_rd, rec_r_rl_rd, rec_r_rr_rd, rec_r_all_blk	≤136	5(11)	≤271	<1 (⊥)	N
InsertToLeft	nil, rec_pre, rec_post	≤28	7	≤216	<1	N
Fusion (post_pre)	schd_lrab, schd_rlab	4	5	84	<1	N
	schd_lrba, schd_rlba	4	5	84	<1	Y
	unfusible_schd(20)	4	6	<216	<1	Y
Fusion (mutl_rec)	schd_lra1b2, schd_rla1b2, schd_lrb2a1, schd_rlb2a1	4	7	604	<3	N
	unfusible_schd(20)	4	9	<3,304	<7	Y

Table 2: Performance for program verification and fusibility checking

followed by the number of formulae raised to solve it, the Dryad$_{dec}$ formula size and synthesis time. All queries for CLIA synthesis are solved in negligible time.

5 Related Work

It is well known that the First-Order Logic (FOL) of finite graphs is undecidable [51], and the decidability can only be obtained by restricting the logic or the class of graphs. There is a rich literature on logics over tree-like structures [7,21].

PALE [32] has been developed to verify all structures that can be expressed using graph types [21], by reducing problems to the MONA system [12]. Nonethe-

Category	Fornulae	DRYAD$_{dec}$ size	Time(s)
Multiple functions	fg_fivefuncs(3), fg_sixfuncs(3), fg_sevenfuncs(3), fg_eightfuncs(3), fg_ninefuncs(3), fg_tenfunc1(3), fg_tenfunc2(3)	<279	<1
Polynomial	fg_polynomial1(3), fg_polynomial2(3), fg_polynomial3(3), fg_polynomial4(4)	<60	<1
Other CLIA	fg_max2(7), fg_VC22_a(17)	<2,227	<1
INV	ex11-new(18), ex11(17), ex14_simp(3), ex14_vars(3), formula22(1), formula25(1), formula27(1), treax1(3), trex1_vars(3), vsend(4)	<936	<1

Table 3: Performance for SyGuS benchmarks synthesis

less, PALE and other similar techniques [11,29,57] do not reason with the data stored in the structure. Separation logic [35,45] has been a popular logic for reasoning with heap structures. Many decidable fragments have been identified. There has been significant efforts on decidable logic for structure properties of list-like structures. SLP [34] and SeLoger [6,10] are designed to check validity of the entailment problem for separation logic over pointers and lists. Iosif *et al.* [14] extend separation logic with recursive definitions to define structures of bounded tree-width, and guarantee the decidability by classical MSO reasoning.

The last decade has seen logics for reasoning about both the structure properties and data properties. The LISBQ [22] logic used in the HAVOC system is a well known decidable logic; it obtains decidability by syntactically restricting the reachability predicates and universal quantification. The CSL [3] logic is designed in a similar vein, with a different set of syntactic restrictions that allow it to express doubly-linked lists. Neither LISBQ nor CSL can handle basic tree data-structures such as binary search trees. AF^R [15] is also a decidable fragment of first-order logic with transitive closure for list-like structures. The GRIT logic [40,41] is capable to handle tree structures; its decidability is obtained by reducing the separation logic to a decidable fragment of first order logic. GRIT is decidable for reasoning local data properties, such as sortedness, but measurements of trees cannot be expressed. The STRAND logic [24,25] combines a powerful tree logic with an arbitrary data-logic. If the underlying data-logic is decidable, a fragment of STRAND is also decidable. As the first decidable logic for binary search trees, a main limitation of STRAND is it cannot express any tree measurement. In other words, AVL trees or red black trees cannot be defined. The underlying logic in the type checker Catalyst [19] is decidable but Catalyst cannot handle measurements either. In contrast, combining term algebra and Presburger arithmetic [58,28] yields decidable theories that can model tree balancedness of red black trees, but not sortedness.

More recently, several automatic verification systems for heap-manipulating programs have been developed. Liquid Types [46,20] handle measurements by folding or unfolding the recursive definitions systematically and then treat the refined types as uninterpreted functions. As the number of unfolding or folding needed is unbounded, the system has to give up either termination or completeness. Inherited the approach from Liquid Types, LiquidHaskell [53,54,55,52] cannot guarantee termination and completeness at the same time either. Apart from DRYAD and natural proofs, by which our decidable logic is inspired, [49,50] and [4] exploit recursive definitions and proof tactics that unfold the definitions tactically. These approaches can handle arbitrary combinations of data, shape and measurement properties for trees, but give up general decidability, as mentioned in Section 1 and explained below.

Recall the insertToLeft example we described in Section 1. To reason about the recursively defined full-treeness and tree-size in Leon, one has to define an ad hoc abstraction function α that maps trees to the domain (Int, Boolean), whose first and second elements represent the tree size and full-treeness, respectively. Then Leon can decidably verify the insertToLeft example only if α is *sufficiently surjective* (see Definition 7 of [49]), which is not the case. To show α is not sufficiently surjective, it suffices to find a positive integer p such that for an arbitrarily large tree t with $\alpha(t) = (i, b)$, the property $|\alpha^{-1}(i, b)| > p$ cannot be characterized by a linear arithmetic formula $M_{i,b}(c)$. Now let t be an arbitrarily large non-full tree such that $\alpha(t) = (i, \mathit{false})$. Notice that i, as the first part of the abstraction, represents the size of the tree t and is arbitrarily large, too. Then the term $|\alpha^{-1}(i, b)|$ essentially means the number of different non-full trees with size i. As the total number of binary trees of size i can be computed combinatorially as $\frac{(2i)!}{(i+1)!\cdot i!}$ and there is a single full tree when $i = 2^k - 1$ for some k. Hence, the property $|\alpha^{-1}(i, \mathit{false})| > p$ can be essentially captured by the following formula

$$M_{i,\mathit{false}} \equiv \frac{(2i)!}{(i+1)! \cdot i!} - \mathrm{ite}(\exists k : i = 2^k - 1, 1, 0) > p$$

Obviously, this $M_{i,\mathit{false}}$ is too complicated and not equivalent to any linear arithmetic formula. Therefore, the abstract domain (Int, Boolean) representing size and full-treeness is not sufficiently surjective and hence cannot be reasoned by Leon in a decidable fashion.

The more recent following work [23,38] either only handle tree with bounded size in a decidable fashion or can only verify the red-black properties and the black-height of the tree, i.e., they cannot verify the functional correctness of AVL or red-black trees manipulating programs. A more recent work [56] related to Liquid Types also shows decidability for transparent formulae; but the formulae handled in our experiments are usually non-transparent.

Acknowledgments This material is based upon work supported by the National Science Foundation under Grant No. 1837023.

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A Proofs for Decidability

Lemma 9 (Bound for Max-Based Function). *For any Σ-tree T and for any Max-based measure function lif* w.r.t. Σ with intermittence r, $\lfloor height^*(T)/r \rfloor \leq lif^*(T)$.*

Proof. We prove $\lfloor height^*(T)/r \rfloor \leq lif^*(T)$ by induction on T's height h.

Base case: If $h < r$, the lower bound is 0. As lif^* is nondecreasing, $lif^*(T) \geq lif^*(nil) = 0$.

Induction step: If $h \geq r$, assume the lemma holds for any tree whose height is up to $h - 1$, then there must exist a subtree T' of T whose height is $h - r$, and we have $\lfloor height^*(T')/r \rfloor \leq lif^*(T')$. Notice that $root(T')$ is the r-th generation descendant of $root(T)$, and by the definition of intermittence, $lif^*(T) \geq lif^*(T') + 1 \geq \lfloor height^*(T')/r \rfloor + 1 = \lfloor height^*(T)/r \rfloor$. □

Lemma 10 (Bound for Sum-Based Function). *Let $\Sigma = (Dir, DF)$ with $|Dir| = d$. Then for any Sum-based measure function eif* w.r.t. Σ with intermittence r, $\lceil \left(\frac{d-1}{d^r-1}\right) \cdot size^*(T) \rceil - 1 \leq eif^*(T)$.*

Proof. Let the definition of eif^* be $\texttt{ite}\Big(\texttt{isNil}(x),\ 0,\ \ldots\ +\ \texttt{ite}^r(\upsilon\big[x\big], 1, 0)\Big)$ We first consider the case that $\upsilon\big[T\big]$, i.e., the root of T satisfies the local property υ, and claim that $\left(\frac{d-1}{d^r-1}\right) \cdot size^*(T) \leq eif^*(T)$ by induction on T's size s:

Base case: If $s \leq \frac{d^r-1}{d-1}$, the lower bound is 1. As $eif^*(T)$ is the number of nodes satisfying υ, which is the case for the root, obviously $eif^*(T) \geq 1$.

Induction step: If $s > \frac{d^r-1}{d-1}$, consider the *maximal* subtrees of T satisfying υ, formally,

$$\mathcal{M} \stackrel{def}{=} \{S \mid S \prec T \wedge \upsilon[S] \wedge \nexists R.(S \prec R \prec T \wedge \upsilon[R])\}$$

Note that the subtrees in $\mathcal{M}$ are all disjoint, and due to the intermittence r, $|\mathcal{M}| \leq d^r$. Moreover, the number of nodes above all subtrees from $\mathcal{M}$ is at most $\frac{d^r-1}{d-1}$, formally $|\mathcal{L}| \leq \frac{d^r-1}{d-1}$ where

$$\mathcal{L} \stackrel{def}{=} \{R \mid \forall S \in \mathcal{M}. R \not\preccurlyeq S\}$$

Then by induction, every tree S in the set satisfies $\left(\frac{d-1}{d^r-1}\right) \cdot size^*(S) \leq eif^*(S)$, hence

$$eif^*(T) = \Big(\sum_{S \in \mathcal{M}} eif^*(S)\Big) + 1 \geq \Big(\sum_{S \in \mathcal{M}} \Big(\frac{d-1}{d^r-1}\Big) \cdot size^*(S)\Big) + 1 = \Big(\frac{d-1}{d^r-1}\Big) \cdot \sum_{S \in \mathcal{M}} size^*(S) + 1$$

$$= \Big(\frac{d-1}{d^r-1}\Big) \cdot (size^*(T) - |\mathcal{L}|) + 1 \geq \Big(\frac{d-1}{d^r-1}\Big) \cdot \Big(size^*(T) - \frac{d^r-1}{d-1}\Big) + 1 = \Big(\frac{d-1}{d^r-1}\Big) \cdot size^*(T)$$

Now for arbitrary tree T, we can also construct the sets $\mathcal{M}$ and $\mathcal{L}$. The only difference is that root does not necessarily satisfy v and increase eif^*:

$$\mathit{eif}^*(T) = \sum_{S \in \mathcal{M}} \mathit{eif}^*(S) \geq \sum_{S \in \mathcal{M}} \big(\frac{d-1}{d^r-1}\big) \cdot \mathit{size}^*(S) = \big(\frac{d-1}{d^r-1}\big) \cdot \sum_{S \in \mathcal{M}} \mathit{size}^*(S)$$

$$= \big(\frac{d-1}{d^r-1}\big) \cdot (\mathit{size}^*(T) - |\mathcal{L}|) \geq \big(\frac{d-1}{d^r-1}\big) \cdot (\mathit{size}^*(T) - \frac{d^r-1}{d-1}) = \big(\frac{d-1}{d^r-1}\big) \cdot \mathit{size}^*(T) - 1$$

As $\mathit{eif}^*(T)$ must be an integer, we have $\mathit{eif}^*(T) \geq \lceil \big(\frac{d-1}{d^r-1}\big) \cdot \mathit{size}^*(T) \rceil - 1$. □

Proof of Lemma 2

Proof. We prove by induction on the structure of T for the case of mif^* only; the proof for the case of mdf^* is similar. If T is nil, the claim is trivially true. Otherwise, let the root of T be rt and assume the lemma is true for any subtree of T. Then for any valid subset of nodes S, consider two situations:

– If rt is not the LCA of S, then all nodes of S come from a single subtree of T, say $T.\mathit{dir}$. Then $\mathit{tailor}_S(T)$ is also a tailored subtree of $T.\mathit{dir}$. By assumption and referring to the definitions in Figure 1,

$$\mathit{mif}^*(T) = \mathtt{max}\big(\{\mathit{mif}^*(T.\mathit{dir}) | \mathit{dir} \in G\} \cup \{\mathit{it}\big[rt\big]\}\big) \geq \mathit{mif}^*(T.\mathit{dir}) \geq \mathit{mif}^*(\mathit{tailor}_S(T))$$

– If rt is the LCA of S, then $\mathit{tailor}_S(T)$ consists of the root of T and $\mathit{tailor}_{S_{\mathit{dir}}}(T.\mathit{dir})$ for every dir. Then compare $\mathit{mif}^*(T)$ and $\mathit{mif}^*(\mathit{tailor}_S(T))$:

$$\mathit{mif}^*(T) = \mathtt{max}\big(\{\mathit{mif}^*(T.\mathit{dir}) | \mathit{dir} \in G\} \cup \{\mathit{it}\big[rt\big]\}\big)$$
$$\mathit{mif}^*(\mathit{tailor}_S(T)) = \mathtt{max}\big(\{\mathit{mif}^*(\mathit{tailor}_S(T).\mathit{dir}) | \mathit{dir} \in G\} \cup \{\mathit{it}\big[rt\big]\}\big)$$

Notice that every element of the former max-set is pairwise greater than or equal to the corresponding element in the latter one. As the local part $\mathit{it}\big[rt\big]$ only depends on the local fields of rt, there is no change between T and $\mathit{tailor}_S(T)$. Moreover, by assumption, $\mathit{mif}^*(T.\mathit{dir}) \geq \mathit{mif}^*(\mathit{tailor}_{S_{\mathit{dir}}}(T.\mathit{dir}))$ for any dir. Therefore, $\mathit{mif}^*(T) \geq \mathit{mif}^*(\mathit{tailor}_S(T))$.

□

Proof of Lemma 3

Proof. According to the template for increasing set functions, the definition of sf^* is of the form $\mathit{sf}^*(x) \stackrel{\mathit{def}}{=} \mathtt{ite}\Big(\mathtt{isNil}(x),\ \emptyset,\ \big(\bigcup_{\mathit{dir}} \mathit{sf}^*(x.\mathit{dir})\big) \cup ST\big[x\big]\Big)$. Based on the definition, for any tree T, obviously $\mathit{sf}^*(T)$ can be characterized as $\{ST\big[x\big] \mid x \text{ is a node in } T\}$. Hence as $\mathit{tailor}_S(T)$ consists of a subset of nodes from T, $\mathit{sf}^*(\mathit{tailor}_S(T)) \subseteq \mathit{sf}^*(T)$. □

Proof of Lemma 4

Proof. We prove by induction on the structure of T. If T is *nil*, the claim is trivially true. Otherwise, let the root of T be rt and assume the claim is true for any subtree of T. Then for any valid subset of nodes S, consider two situations:

- If rt is not the LCA of S, then all nodes of S come from a single subtree of T, say $T.dir$. Then $tailor_S(T) = tailor_S(T.dir)$. For any general predicate gp^*, by assumption and referring to the definitions in Figure 2,

$$gp^*(T) \Rightarrow gp^*(T.dir) \Rightarrow gp^*(tailor_S(T.dir)) \Rightarrow gp^*(tailor_S(T))$$

- If rt is the LCA of S, then $tailor_S(T)$ consists of the root of T and $tailor_S(T.dir)$ for every dir. Assuming $gp^*(T)$, to prove $gp^*(tailor_S(T))$, we need to show

$$\big(\bigwedge_{dir} gp^*(tailor_S(T.dir))\big) \wedge \varphi\big[tailor_S(T.dir),\ T.\boldsymbol{f}\big]$$

  By the inductive hypothesis, $gp^*(T) \Rightarrow gp^*(T.dir) \Rightarrow gp^*(tailor_S(T.dir))$ for any dir.
  What remains is to prove $\varphi\big[tailor_S(T.dir),\ T.\boldsymbol{f}\big]$. Notice that for any increasing function mif^*, by its definition and Lemma 2, $mif^*(tailor_S(T.dir)) \leq mif^*(T.dir)$; similarly $mdf^*(tailor_S(T.dir)) \geq mdf^*(T.dir)$ for any decreasing function mdf^*. Hence for any $nt \geq 0$ in φ, the inequation is easier to be satisfied when any occurrence of $mif^*(T.dir)$ (or $mdf^*(T.dir)$) is replaced with $mif^*(tailor_S(T.dir))$ (or $mdf^*(tailor_S(T.dir))$). Moreover, for any general predicate p^*, it can only occur in φ positively, and by inductive hypothesis, $p^*(tailor_S(T.dir))$ is true because $p^*(T.dir)$ is true.

□

Proof of Lemma 5

Proof. For any two adjacent nodes x and $x.d$ in a critical path, there are two cases:

- if $x.d$ is a critical node due to the first situation of Definition 7, i.e., $lif^*(x.dir) \geq lif^*(x.dir')$ for any other child of x, then by the definition of Max-based functions, $lif^*(x.dir) \geq lif^*(x) - 1$.
- if $x.dir$ is a critical node due to the second or third situation of Definition 7, there is another critical child $x.dir'$ and a constant K such that $lif^*(x.dir') - lif^*(x.dir) = \texttt{ite}(K > 0, K, 1 - K)$, which is bounded by C, i.e., $lif^*(x.dir) \geq lif^*(x.dir') - C$.

As the number of children of x is d, we have

$$lif^*(x.dir) \geq lif^*(x.dir') - C \geq lif^*(x.dir'') - 2C \geq \cdots \geq lif^*(x) - 1 - (d-1)C$$

Specifically, if x does not satisfy υ, $lif^*(x.dir) \geq lif^*(x) - (d-1)C$. By the definition of intermittence r, the distance between x and its closest descendant on the path satisfying υ, say y, is $r-1$. Then $lif^*(y) \geq \cdots \geq lif^*(x) - (d-1)Cr - 1$. Hence the number of nodes from the critical path satisfying υ must be at least $\lfloor \frac{lif^*(T)-1}{(d-1)Cr+1} \rfloor$. □

Proof of Lemma 6

Proof. Let $\{n_0, \ldots, n_h\}$ be a longest path in T with n_0 being the root of T, n_h being the *nil* node and h being the height of T. For each i, let $\mathcal{N}_i \stackrel{def}{=} \{n \mid n \prec n_i \wedge n \not\preceq n_{i+1}\}$. Let $\mathcal{I}$ be the set of numbers i such that

1. n_i belongs to the longest path;
2. $\mathcal{N}_i \cap S = \emptyset$; and
3. $\mathcal{N}_i$ does not contain the witness node for any non-measure function $f^*(T)$.

According to the assumption, $|\mathcal{I}| \geq P+1$, i.e., there is an integer $j \in \mathcal{I}$ such that n_j and n_{j+1} [3] agree on any general predicate gp^*. Hence we replace the subtree rooted by n_j with the subtree rooted by n_{j+1} to form a tailored tree T'. Notice that the set of removed nodes are exactly the set $\mathcal{N}_j$, which does not contain any witness node or any node from S, T' satisfies the first two properties. Moreover, the third property is also satisfied as n_j and n_{j+1} agree on all general predicates and it is not hard to see that the preservation can propagate up to the root of T. □

Partial Proof of Theorem 1: Bound for Max-Based Functions

Proof. The proof is by contradiction. Assume T_i is a tree whose height is greater than h_φ. Consider the longest path from root to a leaf. Obviously the path contains more than h_φ nodes. Let the number of nodes on the path satisfying *lif**'s local constraint υ be L.

Now we claim $height^*(x_i) - L \leq En + P + M + F$. Otherwise, there are more than $En + P + M + F$ nodes on the path not satisfying υ. Similar to the proof of Lemma 6, we can make an argument to show that there are at least two nodes $p \succ p'$ such that: a) they agree on all predicates; b) replacing p with the direct child of p on the path does not remove any witness node for set-inclusion or non-measure functions. For any node $q \succ p$, as p does not satisfy υ, $lif^*(q)$ is not affected for any node above p. Moreover, for any *lif**-related predicate mp^*, as $mp^*(p) = mp^*(p')$, $mp^*(p)$ is preserved during the tailoring, and the preservation is propagated up to q. This process can continue until $height^*(x_i) - L \leq En + P + M + F$.

Now we can assume that $L > (En+P+M+F+1) \cdot ((d-1)Cr+1) + D_{ht} + (m-1)D_{sub}$. By definition,

$$lif^*(x_i) \geq L > (En + P + M + F + 1) \cdot ((d-1)Cr + 1) + D_{ht} + (m-1)D_{sub}$$

Now we define a set of variables J recursively as the smallest set satisfying the following properties:

- x_i belongs to J;

[3] if n_j is a leaf, then let n_{j+1} be the empty tree.

– if $\mathit{lif}^*(x_1) - \mathit{lif}^*(x_2) \geq K$ occurs in φ and the inequation is tight, i.e., the model we are considering satisfies $\mathit{lif}^*(x_1) - \mathit{lif}^*(x_2) = K$, then x_2 belongs to J if x_1 does.

Notice that for the first case, $\mathit{lif}^*(x_i)$ is greater than both $(En + P + M + F + 1) \cdot ((d-1)Cr+1) + D_{ht} + (m-1)D_{sub}$. For the second case, if $\mathit{lif}^*(x_1)$ is greater than a bound B, $\mathit{lif}^*(x_2) = \mathit{lif}^*(x_1) - K \geq B - K \geq B - D_{sub}$. As there are at most m variables in J, for any $\mathit{lif}^*(x_j) \in J$,

$$\mathit{lif}^*(x_j) \geq \mathit{lif}^*(x_i) - (m-1)D > (En + P + M + F + 1) \cdot ((d-1)Cr + 1) + D_{ht}$$

Then by Lemma 7, T_j can be tailored to a smaller tree T_j' such that $\mathit{lif}^*(T_j') = \mathit{lif}^*(T_j) - 1$. Notice that for any satisfied constraint of the form $\mathit{lif}'^*(T_j) \geq K$, by Lemma 9 and Definition 4, $\mathit{lif}'^*(T_j') \geq \lfloor \frac{height(T_j')}{r} \rfloor \geq \lfloor \frac{\mathit{lif}^*(T_j')}{r} \rfloor \geq \lfloor \frac{D_{ht}}{r} \rfloor \geq K$. We replace every such T_j with T_j'. □

Partial Proof of Theorem 1: Bound for Sum-Based Functions

Proof. Assume T_i is a tree whose size is greater than U_φ. Let the local constraint for eif^* be υ and consider the set of nodes

$$\mathcal{N} = \{n \mid \nexists n_1, n_2, dir_1, dir_2 : dir_1 \neq dir_2 \wedge n_1 \prec n.dir_1 \wedge n_2 \prec n.dir_2 \wedge \upsilon(n_1) \wedge \upsilon(n_2)\}$$

Notice that $|\mathcal{N}| \geq \lceil \frac{U_\varphi + 1}{2} \rceil > 3(En + F + 2^P) - 1 + D_{sz} + (m-1)D_{sub}$.

Now we assume there are are at most $En+F+2^P$ nodes in $\mathcal{N}$ not satisfying υ. Otherwise note that each node can be replaced with its unique child containing nodes satisfying υ without affecting the evaluation of eif^* at all. Then similar to the proof of Lemma 8, we can find at least two nodes n_1, n_2 from the set such that tailoring either n_1 or n_2 does not remove any witness node, and tailoring one of them does not affect any general predicate gp^*. With this tailoring the size of the tree is reduced but $\mathit{eif}^*(x_i)$ is unaffected as none of the nodes satisfying υ is removed. This process can continue until the number of nodes in $\mathcal{N}$ not satisfying υ is no more than $En + F + 2^P$.

Hence we can assume that there are more than $2(En + F + 2^P) - 1 + D_{sz} + (m-1)D_{sub}$ nodes in $\mathcal{N}$ satisfying φ, hence $\mathit{eif}^*(x_i) > \mathit{eif}^*(x_j) > 2En + 2F + 2^{P+1} - 1 + D_{sz}$.

Now we construct the set of monotonic terms M in a similar way as for lif^*. With a similar argument, for any $x_j \in M$, $\mathit{eif}^*(x_j) > 2En+2F+2^{P+1}-1+D_{sz}$.

Therefore by Lemma 8, T_j can be tailored to a smaller tree T_j' such that $\mathit{eif}^*(T_j') = \mathit{eif}^*(T_j) - 1$. Notice that for any satisfied constraint of the form $\mathit{eif}'^*(T_j) \geq K$, by Lemma 10 and Definition 5, $\mathit{eif}'^*(T_j') \geq \lceil \frac{r-1}{d^r-1} \cdot size^*(T_j') \rceil - 1 \geq \lceil \frac{r-1}{d^r-1} \cdot \mathit{eif}^*(T_j') \rceil - 1 \geq \lceil \frac{r-1}{d^r-1} \cdot D_{sz} \rceil - 1 \geq K$. We replace every such T_j with T_j'. □

Category	Name	Definition
General Predicate (for Tree Fusion)	dp^*	$\texttt{ite}\big(\texttt{isNil}(x),\ \texttt{true},\ dp^*(x.left) \wedge dp^*(x.right)$ $\wedge\ (\neg(x.ts\_a1 > 0 \wedge x.ts\_a2 > 0) \vee x.ts\_a1 < x.ts\_a2)$ $\wedge\ (\neg(x.ts\_b1 > 0 \wedge x.ts\_b2 > 0) \vee x.ts\_b1 < x.ts\_b2)$ $\wedge\ (\neg(max\_ts\_b1^*(x.left) > 0)$ $\quad \vee x.ts\_a1 > max\_ts\_b1^*(x.left) \vee x.ts\_a2 > max\_ts\_b1^*(x.left))$ $\wedge\ (\neg(max\_ts\_b1^*(x.right) > 0)$ $\quad \vee x.ts\_a1 > max\_ts\_b1^*(x.right) \vee x.ts\_a2 > max\_ts\_b1^*(x.right))$ $\wedge\ (\neg(max\_ts\_b2^*(x.left) > 0)$ $\quad \vee x.ts\_a1 > max\_ts\_b2^*(x.left) \vee x.ts\_a2 > max\_ts\_b2^*(x.left))$ $\wedge\ (\neg(max\_ts\_b2^*(x.right) > 0)$ $\quad \vee x.ts\_a1 > max\_ts\_b2^*(x.right) \vee x.ts\_a2 > max\_ts\_b2^*(x.right))$ $\wedge\ (\neg(max\_ts\_a1^*(x.left) > 0)$ $\quad \vee x.ts\_b1 > max\_ts\_a1^*(x.left) \vee x.ts\_b2 > max\_ts\_a1^*(x.left))$ $\wedge\ (\neg(max\_ts\_a1^*(x.right) > 0)$ $\quad \vee x.ts\_b1 > max\_ts\_a1^*(x.right) \vee x.ts\_b2 > max\_ts\_a1^*(x.right))$ $\wedge\ (\neg(max\_ts\_a2^*(x.left) > 0)$ $\quad \vee x.ts\_b1 > max\_ts\_a2^*(x.left) \vee x.ts\_b2 > max\_ts\_a2^*(x.left))$ $\wedge\ (\neg(max\_ts\_a2^*(x.right) > 0)$ $\quad \vee x.ts\_b1 > max\_ts\_a2^*(x.right) \vee x.ts\_b2 > max\_ts\_a2^*(x.right))$ $\wedge\ (\neg(x.ts\_a1 > 0) \vee ((\neg\texttt{isNil}(x.left) \vee max\_ts\_b2(x.left) > 0)$ $\qquad \wedge(\neg\texttt{isNil}(x.right) \vee max\_ts\_b2(x.right) > 0)))$ $\wedge\ (\neg(x.ts\_b2 > 0) \vee ((\neg\texttt{isNil}(x.left) \vee max\_ts\_a1(x.left) > 0)$ $\qquad \wedge(\neg\texttt{isNil}(x.right) \vee max\_ts\_a1(x.right) > 0))))\big)$
	$schd^*_{lra1b2}$	$\texttt{ite}\big(\texttt{isNil}(x),\ \texttt{true},\ schd^*_{lra1b2}(x.left) \wedge schd^*_{lra1b2}(x.right)$ $\wedge\ x.ts\_a1 < x.ts\_b2$ $\wedge\ \texttt{max}(max\_ts\_a1^*(x.left), max\_ts\_b2^*(x.left))$ $\quad < \texttt{min}(min\_ts\_a1^*(x.right), min\_ts\_b2^*(x.right))$ $\wedge\ \texttt{max}(max\_ts\_a1^*(x.left), max\_ts\_a1^*(x.right)$ $\qquad , max\_ts\_b2^*(x.left), max\_ts\_b2^*(x.right)) < \texttt{min}(x.ts\_a1, x.ts\_b2)\big)$
	$schd^*_{rla1b2}$	$\texttt{ite}\big(\texttt{isNil}(x),\ \texttt{true},\ schd^*_{rla1b2}(x.left) \wedge schd^*_{rla1b2}(x.right)$ $\wedge\ x.ts\_a1 < x.ts\_b2$ $\wedge\ \texttt{max}(max\_ts\_a1^*(x.right), max\_ts\_b2^*(x.right))$ $\quad < \texttt{min}(min\_ts\_a1^*(x.left), min\_ts\_b2^*(x.left))$ $\wedge\ \texttt{max}(max\_ts\_a1^*(x.left), max\_ts\_a1^*(x.right)$ $\qquad , max\_ts\_b2^*(x.left), max\_ts\_b2^*(x.right)) < \texttt{min}(x.ts\_a1, x.ts\_b2)\big)$
	$schd^*_{\ldots}$	$\ldots$
General Predicate (for CLIA Synthesis)	$exp^*_{spec_f,G}$	$\texttt{ite}\Big(\texttt{isNil}(x),\ \texttt{true},\ exp^*_{spec_f,G}(x.left) \vee exp^*_{spec_f,G}(x.right)$ $\quad \vee\big(\texttt{isNil}(x.left) \wedge \texttt{isNil}(x.right) \bigwedge_{\boldsymbol{e}\in G} spec_{f\leftarrow eval(x)}(\boldsymbol{e})\big)$ $\quad \vee\big(\big(\neg\texttt{isNil}(x.left) \vee \neg\texttt{isNil}(x.right)\big)\wedge$ $\qquad \bigvee_{\emptyset\subset F\subset G} \big(\bigwedge_{\boldsymbol{e}\in F} eval_{\boldsymbol{e}}(x) \geq 0 \wedge \bigwedge_{\boldsymbol{e}\in G\setminus F} eval_{\boldsymbol{e}}(x) < 0$ $\qquad\qquad \wedge\ exp^*_{spec_f,F}(x.left) \wedge exp^*_{spec_f,G\setminus F}(x.right)\big)\ \big)\Big)$

Fig. 6: List of recursive definitions for tree fusion and CLIA synthesis

```
1  A(n)
2    if (n == nil) return
3    B(n.l); B(n.r)
4    int ls = n.l ? 0 : n.l.s; int rs = n.r ? 0 : n.r.s
5    n.v = ls + rs + 1
6  B(n)
7    if (n == nil) return
8    A(n.l); A(n.r)
9    int lv = n.l ? 0 : n.l.v; int rv = n.r ? 0 : n.r.v
10   n.s = lv + rv
11 main(n)
12   A(n); B(n)
```

(a) Two Traversals of A Binary Tree

```
fused(n)
  if (n == nil) return;
  fused(n.l); fused(n.r)
  int ls = n.l ? 0 : n.l.s; int rs = n.r ? 0 : n.r.s
  n.v = ls + rs + 1
  int lv = n.l ? 0 : n.l.v; int rv = n.r ? 0 : n.r.v
  n.s = lv + rv
```

(b) Fused Traversals

Fig. 7: Fusing mutually recursive functions

B Verifying fusibility of Recursive Tree Traversals

We also use DRYAD$_{dec}$ to automatically verify the fusibility of recursive tree traversals, which is not possible with existing approaches.

The fusibility problem can be encoded to DRYAD$_{dec}$. We illustrate the encoding via an example.

To ensure validity of fusing two tree traversals, resulting fused traversal needs to satisfy the following requirements: a) fused traversal should perform exactly the same operations as unfused traversals; and b) fused traversal should not violate any data dependencies inferred from unfused traversals. Consider the program shown in Figure 7a. A and B are two mutually recursive functions manipulating binary trees. Besides distinct local updates, they call each other on the left and right children of the current node. The main function invokes A followed by B on the same node n. The two traversals can be fully fused to a single traversal as shown in Figure 7b. However, proving the fusibility here is beyond the capability of existing decidable logics.

With the two kinds of local updates in functions A and B, we can represent an arbitrary traversal using a tree with two fields, ts_a and ts_b, representing the timestamps of local updates on the current node from A (lines 4-5) and B (lines 9-10), respectively.

For example, if the tree consists of three nodes: root n, left child nl and right child nr, then the unfused traversals in Figure 7a guarantee $nl.ts_b < nr.ts_b < n.ts_a < nl.ts_a < nr.ts_a < n.ts_b$.

Now for an arbitrary tree T, the fusibility can be formulated as a DRYAD$_{dec}$ formula $schd^*(x) \wedge \neg dp^*(x)$ where both $schd^*$ and dp^* are general predicates. Intuitively, $schd^*$ characterizes the full fused schedule, dp^* captures data dependencies extracted from the unfused traversals. Their formal definitions can be found in Figure 6 in Appendix.

We used DRYAD$_{dec}$ to verify the validity of all possible fusions of two pairs of traversals: the fusion of mutually recursive traversals shown in Figure 7a and another pair of a post-order traversal execute before a pre-order traversal.

For each pair there are 24 possible fused schedules; each encoded to a separate $\textsc{Dryad}_{dec}$ formula.

C Synthesizing CLIA Functions

C.1 Background

The CLIA synthesis problem can be represented as a logical query $\exists f.\, \forall \boldsymbol{x}.\, spec_f(\boldsymbol{x})$ where f is a mapping from a sequence of integers to an integer or boolean value, and $spec_f(\boldsymbol{x})$ is a quantifier-free LIA formula involving the unknown function f. Intuitively, f is the function to be synthesized and $spec_f(\boldsymbol{x})$ is the specification for f. The synthesizer needs to find an implementation of f such that $spec_f(\boldsymbol{x})$ is satisfied for any input $\boldsymbol{x}$. The implementation is a conditional linear integer arithmetic (CLIA) term, which allows arbitrary LIA operations as well as ITE conditionals. The syntax of CLIA terms is defined in Figure 8.

The LIA synthesis problem is obviously undecidable as it needs to solve double-quantified formulae, and it cannot be solved by standard SMT solvers. A common approach to solve this problem is the Counter-Example Guided Inductive Synthesis (CEGIS) framework. The basic idea behind CEGIS is that a set of representative inputs G to the specification is usually sufficient to find a solution that works for all inputs. So the original equation can be reduced to a constraint of the following form:

$$\exists f.\ \bigwedge_{\boldsymbol{e} \in G} spec_f(\boldsymbol{e}) \tag{C.1}$$

The set G is usually initialized to contain a random value. If the constraint is satisfiable, the solver gets a candidate expression g and checks if it works for all inputs: $\forall \boldsymbol{x}.spec_g(\boldsymbol{x})$. If a counterexample is found, it is added to the set G and the process is repeated; otherwise the double-quantified formula is unsatisfiable, i.e., there is no solution for the synthesis problem.

In each iteration of the CEGIS loop, the verification phase can be solved by a standard SMT solver. Hence the CLIA synthesis problem is reduced to the problem of solving the single-quantified query (C.1) raised from each iteration. In the rest of this section, we show how to encode this example-based synthesis problem to $\textsc{Dryad}_{dec}$.

Int Const: c Int Var: x
Expr: $E, E_1, E_2 ::= c \mid x \mid E_1 + E_2 \mid E_1 - E_2 \mid$ if φ then e_1 else e_2
Cond: $\varphi, \varphi_1, \varphi_2 ::= E_1 \geq E_2 \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi$

Fig. 8: Conditional Linear Integer Arithmetic Terms

C.2 Decision Tree Representation

$$\begin{aligned}
\text{Int Const: } & c_0, c_1, c_2 \dots \\
\text{Atom Expr: } & e ::= c_0 + \sum_{1 \le i \le n} c_i \cdot x_i \\
\text{Atom Cond: } & \alpha ::= e \ge 0 \\
\text{Expr: } & E, E_1, E_2 ::= e \mid \textsf{if } \alpha \textsf{ then } E_1 \textsf{ else } E_2 \\
\text{Cond: } & \varphi, \varphi_1, \varphi_2 ::= \alpha \mid \textsf{if } \alpha \textsf{ then } \varphi_1 \textsf{ else } \varphi_2
\end{aligned}$$

Fig. 9: Decision tree normal form

To represent an n-ary function $f(x_1, \dots, x_n)$ in CLIA, we consider a *decision tree normal form* described in Figure 9. It is not hard to see that every CLIA function expressible in the Syntax of Figure 8 can be converted to this normal form. The proof relies on the fact that every atomic LIA equation or inequation can be rewritten to the form of $e \ge 0$, where e is a linear expression. Then the normal form expression can be represented as a binary tree in which every node contains $n+1$ integer fields $c_0, \dots, c_n$, representing the expression $c_0 + \sum_{1 \le i \le n} c_i \cdot x_i$. Each decision node (non-leaf node) tests whether the associated expression is nonnegative and proceeds to the "true" branch or "false" branch. Each leaf node determines the value of the function using the atomic expression it represents. For example, the binary max function

$$max2(x_1, x_2) \stackrel{def}{=} \textsf{if } x_1 \ge x_2 \textsf{ then } x_1 \textsf{ else } x_2$$

can be represented as the tree shown in Figure 10.

We denote the function represented by a decision tree T as *func*(T). Now for an iteration of the CEGIS loop with a query $\exists f.\ \bigwedge_{\boldsymbol{e} \in G} spec_f(\boldsymbol{e})$, the synthesis task can be expressed as: find a decision tree T such that $\bigwedge_{\boldsymbol{e} \in G} spec_{func(T)}(\boldsymbol{e})$.

C.3 Encoding LIA Synthesis to Dryad$_{dec}$

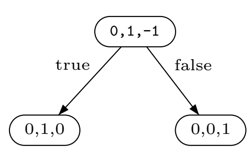

Fig. 10: Representation of the $max2$ function

Given a specification $spec_f(\boldsymbol{x})$ and a set of counterexamples G, we encode the query $\bigwedge_{\boldsymbol{e} \in G} spec_{func(T)}(\boldsymbol{e})$ as a general predicate $exp^*_{spec_f, G}$ in DRYAD$_{dec}$. The predicate is recursively defined and relies on other predicates with name $exp^*_{spec_f, F}$, where F is any nonempty subset of G. Intuitively, if T is a single leaf node, one can explicitly check whether $spec_f(\boldsymbol{e})$ is satisfied for each $\boldsymbol{e} \in G$: for each occurrence of $f(v_1, \dots, v_n)$ in the specification, it can be replaced with a local expression on T's root rt. If f is expected to return an integer, then the expression is simply the linear expression represented by the local fields: $eval(rt, v_1, \dots, v_n) = rt.c_0 + \sum_{1 \le i \le n}(v_i \cdot rt.c_i)$. If f is expected to return a boolean value, the *eval* expression checks whether the local linear expression is nonnegative: $eval(rt, v_1, \dots, v_n) = rt.c_0 + \sum_{1 \le i \le n}(v_i \cdot rt.c_i) \ge 0$. We repeatedly replace all occurrence of f in $spec_f(\boldsymbol{e})$ and denote the result as $spec_{f\leftarrow eval(rt)}(\boldsymbol{e})$. The conjunction of the results for every $\boldsymbol{e} \in G$ determines the value of $exp^*_{spec,G}(T)$.

If T is not a single leaf node, the root rt serves as a conditional and splits the example set G into two sets, depending on whether $eval(rt, \boldsymbol{e}) \geq 0$. Let $F \subseteq G$ be the subset satisfying the local condition, then T satisfies $spec_f$ for G if and only if $T.left$ and $T.right$ satisfy $spec_f$ for F and $G \setminus F$, respectively. Hence $exp^*_{spec_f,G}(T)$ can be recursively evaluated as $exp^*_{spec_f,F}(T.left) \wedge exp^*_{spec_f,G\setminus F}(T.right)$.

The precise definition of these predicates can be found in Figure 6 in Appendix. Notice that they are not standard general predicates allowed in DRYAD$_{dec}$, as the inductive case for $exp^*_{spec_f,G}(x)$ is not a conjunction including $exp^*_{spec_f,G}(x.left)$ and $exp^*_{spec_f,G}(x.right)$. Nonetheless, the negation of a predicate $exp^*_{spec_f,G}$ can all be defined as a standard general predicate $neg\text{-}exp^*_{spec_f,G}$, and every occurrence of $exp^*_{spec_f,G}(x)$ in the formula can be replaced with $\neg neg\text{-}exp^*_{spec_f,G}(x)$.

Proposition 1. *For any specification $spec_f(\boldsymbol{x})$, any set of counterexamples G and any decision tree T, $\bigwedge_{\boldsymbol{e}\in G} spec_{func(T)}(\boldsymbol{e})$ if and only if $exp^*_{spec_f,G}(T)$.*

In the k-th iteration of a CEGIS loop, the synthesis task is to produce a function satisfying the $k-1$ counterexamples generated in the first $k-1$ iterations. This can be encoded to a recursive predicate $\exp^*_{spec,G}(x)$ using above encoding method. We integrated the DRYAD$_{dec}$ decision procedure into a CEGIS-based CLIA function synthesizer, serving as the synthesizer, and obtained a DRYAD$_{dec}$ formula to solve for each iteration until a solution is found or there are too many counterexamples and the formula cannot be solved efficiently. The algorithm described above is served as part of our own SyGuS synthesizer. We adopted the benchmarks from CLIA track and INV track of 2017 SyGuS competition and solved the ones which fall into the algorithm.