This packet contains review material on the area of dynamics in the topics listed below. Solution videos for extensive set of examples related to these topics can be found on the course blog for ME 274: https://www.purdue.edu/dynamics. To view these videos, click on the “Lecture Example Videos” link on the right side of the blog page. To help you locate the appropriate solution videos, the chapter and section numbers for the videos are provided in square brackets below.

**Topics**

1. Particle kinematics [section 1A]
2. Rigid body kinematics [sections 2A-2C]
3. Kinetics equations for particles and rigid bodies
   - Newton’s 2nd law for particles [section 4A]
   - Newton-Euler equations for rigid bodies [section 5A]
   - Work-energy equation for particles [section 4B]
   - Work-energy equation for rigid bodies [section 5B]
   - Linear impulse-momentum equations for particles [section 4C]
4. Vibrations [sections 6A-6C]
1. ** PARTICLE KINEMATICS**

**Cartesian description**

**Path description**

**Polar description**

As always, the velocity and acceleration of point P are given by the first and second time derivatives, respectively, of the position vector \( \mathbf{r} \) for P:

\[
\mathbf{v} = \frac{d\mathbf{r}}{dt}
\]

\[
\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}
\]

The kinematic equations for the Cartesian, path and polar descriptions are derived in the following notes. The following figures showing the kinematic variables and unit vectors will be used in these derivations.
**Path description**

\[ \mathbf{v} = v \hat{e}_t \]

\[ \mathbf{a} = \dot{v} \hat{e}_t + \frac{v^2}{\rho} \hat{e}_n \]

- The velocity of a point is ALWAYS tangent to the path of the point. The magnitude of the velocity vector is the known as the scalar “speed” \( v \) of the point.

- The acceleration of the point has two components:
  - The component \( (v^2/\rho) \hat{e}_n \) is normal to the path. This is commonly referred to as the “centripetal” component of acceleration. This component is ALWAYS directed inward to the path (positive \( n \)-component) since \( v^2/\rho > 0 \).
  - The component \( \dot{v} \hat{e}_t \) is tangent to the path. The magnitude of this component is the “rate of change of speed” \( \dot{v} \) for the point.
    * When \( \dot{v} > 0 \) (increasing speed), the acceleration vector has a positive \( t \)-component (i.e., forward of \( \hat{e}_n \)).
    * When \( \dot{v} = 0 \) (constant speed), the acceleration vector has a zero \( t \)-component (i.e., \( \mathbf{a} \) is aligned with \( \hat{e}_n \)). Note that constant speed does NOT imply zero acceleration!
    * When \( \dot{v} < 0 \) (decreasing speed), the acceleration vector has a negative \( t \)-component (i.e., backward of \( \hat{e}_n \)). See figure below.
Polar description

\[ \ddot{v} = \dot{r} \hat{e}_r + r \ddot{\theta} \hat{e}_\theta \]
\[ \ddot{a} = \left( \ddot{r} - r \dot{\theta}^2 \right) \hat{e}_r + \left( r \ddot{\theta} + 2 \dot{r} \dot{\theta} \right) \hat{e}_\theta \]

- The values for the components of these vectors depend on your choice of the point O. Therefore, you need to carefully define your choice of point O at the beginning of the problem and stick with it throughout the problem.

- When the path of P is given as \( r = r(\theta) \) you will need to use the chain rule of differentiation to find the time derivatives \( \dot{r} = dr/dt \) and \( \ddot{r} = d^2r/dt^2 \) in terms of the time derivatives \( \dot{\theta} \) and \( \ddot{\theta} \).

- These vector expressions use the components of velocity and acceleration projected on the polar unit vectors \( \hat{e}_r \) and \( \hat{e}_\theta \). These projections are usually more difficult to determine than, say, the Cartesian projections and have less physical significance than the path component projections. However, in many applications involving observers of motion, the polar expressions are very useful.
2. **Rigid Body Kinematics**

\[ \ddot{v}_B = \ddot{v}_A + \dddot{\omega} \times \ddot{r}_{B/A} \]
\[ \ddot{a}_B = \dddot{a}_A + \dddot{\alpha} \times \dddot{r}_{B/A} + \dddot{\omega} \times [\dddot{\omega} \times \ddot{r}_{B/A}] \]

- The above two equations relate the kinematics of any two points A and B on the same rigid body while the body executes planar motion.

- The vectors \( \dddot{\omega} = \omega \dddot{k} \) and \( \dddot{\alpha} = \alpha \dddot{k} \) are the angular velocity and angular acceleration of the rigid body, respectively. These are properties of the motion of the rigid body and are the same regardless of which points A and B are used in the above equation. That is, a rigid body has one (and only one) angular velocity, and only one angular acceleration.

- The direction \( \dddot{k} \) for the angular velocity and acceleration vectors denotes the axis about which the body rotates. The signs of \( \omega \) and \( \alpha \) provide the sense of the rotational velocity and acceleration. Suppose that \( \dddot{k} \) points out of the page. Then, by the right-hand rule, a positive sign denotes counterclockwise motion, and a negative sign denotes clockwise rotation.

- If A is a fixed point (\( \dddot{v}_A = 0 \)), then from above we see that \( \dddot{v}_B = \dddot{\omega} \times \ddot{r}_{B/A} \). In this case, the body is rotating about point A. Since \( \dddot{\omega} \times \dddot{r}_{B/A} \) is perpendicular to \( \dddot{r}_{B/A} \), \( \dddot{v}_B \) is perpendicular to \( \dddot{r}_{B/A} \) when the body is rotating about point A.

- For general motion of a body with no prescribed fixed points, the point about which the body rotates is not immediately obvious. We will address this point later on when we work with instant centers.

**Interpretation**: The relative velocity vector \( \dddot{v}_{B/A} = \dddot{v}_B - \dddot{v}_A = \dddot{\omega} \times \dddot{r}_{B/A} \) is always perpendicular to the line connecting points A and B. However, the relative acceleration vector \( \dddot{a}_{B/A} = \dddot{a}_B - \dddot{a}_A = \dddot{\alpha} \times \dddot{r}_{B/A} - \omega^2 \dddot{r}_{B/A} \) is NOT perpendicular to the line connecting points A and B. How do we know that this is true? See the following figure.
Consider the wheel shown below that rolls along a rough, stationary horizontal surface with the center of the wheel O having a velocity and acceleration of \( v_O \) and \( a_O \), respectively. As the wheel moves, it is assumed that sufficient friction acts between the wheel and the fixed horizontal surface that the contact point C does not slip. The consequences of the “no slip” condition at C are:

\[
\begin{align*}
    v_{cx} &= 0 \\
    a_{cx} &= 0
\end{align*}
\]

(If either of the above is not true, then point C “slips” as the wheel moves.)
3. **Kinetics Equations for Particles and Rigid Bodies**

**Summary table**

<table>
<thead>
<tr>
<th>Method</th>
<th>Body model</th>
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<td><strong>Newton-Euler</strong> (relating forces to accelerations)</td>
<td>particle</td>
<td>$\sum \ddot{F} = m\ddot{a}$</td>
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<tr>
<td></td>
<td>rigid body (G = c.m. and A = any point on body)</td>
<td>$\sum \ddot{F} = m\ddot{a}_G$</td>
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<tr>
<td><strong>Work-energy</strong> (relating change in speed to change in position)</td>
<td>particle</td>
<td>$T_1 + V_1 + U_{1\rightarrow 2}^{(nc)} = T_2 + V_2$ where $T = \frac{1}{2}mv^2$</td>
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<td><strong>Linear impulse-momentum</strong> (relating change in velocity to change in time)</td>
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<td>$\int_{t_1}^{t_2} \sum \ddot{F} , dt = m\ddot{v}_2 - m\ddot{v}_1$</td>
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<td>rigid body (G = c.m.)</td>
<td>$\int_{t_1}^{t_2} \sum \ddot{F} , dt = m\ddot{v}<em>{G2} - m\ddot{v}</em>{G1}$</td>
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<td>particle (O = fixed point)</td>
<td>$\int_{t_1}^{t_2} \sum \ddot{M}<em>O , dt = \ddot{H}</em>{O2} - \ddot{H}_{O1}$ where $\ddot{H}<em>O = m\ddot{r}</em>{P/O} \times \ddot{v}_P$</td>
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<tr>
<td></td>
<td>rigid body (A = fixed point or c.m.)</td>
<td>$\int_{t_1}^{t_2} \sum \ddot{M}<em>A , dt = \ddot{H}</em>{A2} - \ddot{H}_{A1}$ where $\ddot{H}_A = I_A \dddot{\alpha}$</td>
</tr>
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</table>
**Newton’s 2nd law for particles**

**Cartesian Coordinates**
If we resolve the forces into their *Cartesian components* and balance the components on each side of this equation, we have:

\[
\sum \vec{F} = m \vec{a} \quad \Rightarrow \quad \sum F_x = m \ddot{x} \quad \sum F_y = m \ddot{y}
\]

**Path Coordinates**
If we resolve the forces into their *path components* and balance the components on each side of this equation, we have:

\[
\sum \vec{F} = m \vec{a} \quad \Rightarrow \quad \sum F_t = m \dot{v} \quad \sum F_n = m \frac{v^2}{\rho}
\]

**Polar Coordinates**
If we resolve the forces into their *polar components* and balance the components on each side of this equation, we have:

\[
\sum \vec{F} = m \vec{a} \quad \Rightarrow \quad \sum F_r = m \left( \ddot{r} - r \dot{\theta}^2 \right) \quad \sum F_\theta = m \left( r \ddot{\theta} + 2 \dot{r} \dot{\theta} \right)
\]
**Newton-Euler equations for rigid bodies**

\[
\sum \vec{F} = m \vec{a}_G \quad \text{(Newton)}
\]
\[
\sum \vec{M}_A = I_A \vec{\alpha} + \vec{r}_{G/A} \times (m \vec{a}_A) \quad \text{(Euler)}
\]

1. The above are known as the Newton-Euler equations for the dynamic analysis of planar motion of rigid bodies.

2. For the right-hand side of the Newton equation you MUST use the acceleration of the body’s center of mass G. There are not exceptions to this rule!

3. In the Euler Equation, you can use ANY point A on the rigid body. YOU are free to choose any point A on the body that you want. Regardless of the choice of point A, you must be consistent in the subscripts:

\[
\sum \vec{M}_A = I_A \vec{\alpha} + \vec{r}_{G/A} \times (m \vec{a}_A)
\]

Specifically:

- You must use the same point A for the mass moment of inertia \( I_A \) as the point that you use for summing the moments \( \vec{M}_A \).
- The acceleration \( \vec{a}_A \) must be of the same point A as the point that you use for summing moments \( \vec{M}_A \).
- The vector \( \vec{r}_{G/A} \) must extend FROM point A TO the mass center G.

4. Recall there are special forms of Euler’s Equation when A is either the center of mass G or a fixed point O on the rigid body. In these special cases the last term on the right-hand side of this equation vanishes. These are the preferred forms of Euler’s Equation because of their simplicity.

5. IMPORTANT POINT: You must use the same sign conventions on forces as you do for accelerations in the Newton Equation. Similarly, you must use the same sign convention on moments as you do for angular accelerations in the Euler Equation. For the Euler Equation, it is recommended that you use a right-hand rule convention for the moments/angular accelerations.
**Work-energy equation for particles**

\[ T_2 + V_2 = T_1 + V_1 + U_{1\rightarrow2}^{(nc)} \]

1. The above is the form of the *work-energy* equation that is commonly used.

2. Only the tangential component of the resultant force \( \vec{R} \) contributes to the work done by \( \vec{F} \):

\[ U_{1\rightarrow2} = \int_1^2 (\vec{R} \cdot \vec{t}) \, ds \]

If a force is perpendicular to the path of the particle on which it acts, then the force does NO work on the particle.

3. The work-energy equation is a scalar equation. That is, this equation does not have vector components. In particular, you can not separate the work-energy equation into, say, \( x \)- and \( y \)-components of the motion.

4. Note that if there are no non-conservative forces doing work on the system, \( U_{1\rightarrow2}^{(nc)} = 0 \), then the mechanical energy (kinetic + potential energy) is constant: \( T_1 + V_1 = T_2 + V_2 \). In this case, we say that energy is *conserved*.

5. In using this equation, be careful with signs on the potential energy terms:

   (a) The potential of a spring is ALWAYS non-negative, regardless of whether the spring is stretched or compressed. That is, a spring that is compressed has exactly the same potential energy as a spring that is stretched by the same amount.

   (b) The potential of the weight force depends on whether the particle is above or below the *datum line* (the line from which the height \( h \) is measured). YOU are free to choose whatever datum you want. However, once you have chosen the datum line, the sign of the potential depends entirely on whether the particle is above or below that line.

6. The work-energy equation also applies to systems of particles. For systems of particles, simply sum up the kinetic energy for all particles in the system to find the total kinetic energy. Likewise, the potential energy for all springs and weight forces are added together. Many *internal* forces acting within the system (such as the reaction forces due to rigid connections) do no net work on the system and are not included in the work term.
**Work-energy equation for rigid bodies**

\[ T_2 + V_2 = T_1 + V_1 + U^{(nc)}_{1 \rightarrow 2} \]

where

\[ T = \frac{1}{2} m v_P^2 + \frac{1}{2} I_P \omega^2 + m \vec{v}_P \cdot (\vec{\omega} \times \vec{r}_{G/P}) \]

1. The point P in the above equation for kinetic energy can be ANY point on the rigid body. Typically, we choose P to be EITHER
   - the center of mass G for the body \((\vec{r}_{G/G} = \vec{0})\), in which case the kinetic energy expression reduces to:
     \[ T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2 \]
   - OR, a fixed point O (possibly instant center) for which \(\vec{v}_P = \vec{0}\), in which case the kinetic energy reduces to:
     \[ T = \frac{1}{2} I_O \omega^2 \]
Linear impulse-momentum equation for particles

\[ m \vec{v}_2 = m \vec{v}_1 + \int_{1}^{2} \vec{R} \, dt \]

where \( m \vec{v} \) is the linear momentum of the particle and \( \int_{1}^{2} \vec{R} \, dt \) is the impulse of the net force acting on the particle.

1. The above is the impulse-momentum equation for a particle since the equation relates the change in linear momentum to the impulse acting on the particle.

2. Note that this equation relates the change in velocity of the particle to a change in time (from \( t = t_1 \) to \( t = t_2 \)).

3. In contrast with the work-energy equation, the impulse-momentum is a vector equation. That is, the equation can be resolved into components. For example, if the forces acting on the particle are resolved into a set of \( x-y \) Cartesian components, then the above vector equation represents the following set of two scalar equations:

\[ mv_{x2} = mv_{x1} + \int_{1}^{2} R_x \, dt \]

\[ mv_{y2} = mv_{y1} + \int_{1}^{2} R_y \, dt \]

4. If the resultant force in a given direction is zero, then linear momentum is conserved in that direction. For example, suppose that \( R_x = 0 \); then from above we see that:

\[ mv_{x2} = mv_{x1} \quad \Rightarrow \quad v_{x2} = v_{x1} \]

Conservation of momentum in one direction does NOT imply conservation of momentum in any other direction.

5. Conservation of momentum does NOT imply or result from conservation of energy. Momentum and energy are two independent quantities. We will see many examples in the course for which momentum is conserved but energy is not conserved, and vice versa.
Linear impulse-momentum equation for particles (continued)

1. The above set of four equations can be used to analyze the central impact of a pair of particles A and B.

2. An impulsive force is a force with a large magnitude and very short duration in time. During impact, we will neglect the contributions of non-impulsive forces such as those due to weight and springs during the short time of impact \( \Delta t \).
   - The impact force acting between particles A and B above is an impulsive force.
   - Reaction forces can be impulsive.
   - Gravitational and spring forces are not impulsive.

3. The application of the above four equations for A and B assumes that the impact force between A and B is the only impulsive forces acting on A and B during the impact.

4. In working problems using these four equations, be sure to clearly define the \( n-t \) coordinate axes prior to starting the problem.

5. The coefficient of restitution equation (the fourth equation above) is valid for only for velocity components along the line of impact (the normal direction, \( n \)). This equation is NOT valid for relating the total magnitudes of velocities or for the components of velocity along the plane of contact, \( t \).

6. Mechanical energy in System AB is NOT conserved during impact, except for the special case of \( e = 1 \). (How do we know this?) You should NEVER attempt to use the work-energy equation during impact.
4. Vibrations

We have seen that the general form of the differential EOM for our single degree-of-freedom systems can be written as:

\[ M\ddot{x} + C\dot{x} + Kx = f(t) \]

where the parameters \( M, C \) and \( K \) depend on the particular problem at hand.

We have also seen that the homogeneous solution of this EOM (and the total free response when \( f(t) = 0 \)) explicitly depends on two important parameters: (i) the undamped natural frequency \( \omega_n \), and (ii) the damping ratio \( \zeta \). We will find it convenient to rewrite our EOM explicitly in terms of these two parameters. To do so, divide the above EOM by the coefficient \( M \) of the \( \ddot{x} \):

\[ \ddot{x} + \frac{C}{M}\dot{x} + \frac{K}{M}x = \frac{1}{M}f(t) \]

Using our previous definition \( \omega_n = \sqrt{K/M} \), we see that the coefficient of the \( x \) term in the EOM becomes \( \omega_n^2 \). Using our previous definition \( \zeta = C/2\sqrt{KM} \), the coefficient of the \( \dot{x} \) term in the EOM can be written as:

\[ \frac{C}{M} = \frac{2\zeta\sqrt{KM}}{M} \]
\[ = 2\zeta\sqrt{\frac{K}{M}} \]
\[ = 2\zeta\omega_n \]

Therefore, a standard form of our EOM in terms of damping ratio and undamped natural frequency as naturally appears as:

\[ \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{M}f(t) \]

Why do we care about such a standard form for the EOM? We know that the free response depends only on the damping ratio \( \zeta \), the undamped natural frequency \( \omega_n \) and the initial conditions \( x(0) \) and \( \dot{x}(0) \). This standard form allows us to explicitly observe the two parameters \( \zeta \) and \( \omega_n \) directly from the EOM. In particular, from this standard form we can immediately see if the free response is underdamped \( (\zeta < 1) \), critically damped \( (\zeta = 1) \) or overdamped \( (\zeta > 1) \). Furthermore, for the underdamped case, we can immediately see the frequency of the oscillations \( \omega_d = \omega_n\sqrt{1 - \zeta^2} \).
For underdamped ($0 < \zeta < 1$) free response we have:

$$x(t) = e^{-\zeta \omega_n t} \left[ C \cos \omega_d t + S \sin \omega_d t \right]$$

where $\omega_d = \omega_n \sqrt{1 - \zeta^2} = \text{damped natural frequency}$.

1. The coefficients $C$ and $S$ for the free response are found by enforcing the initial conditions $x(0)$ and $\dot{x}(0)$.

2. For the special case of undamped response ($\zeta = 0$), the above solution reduces to: $x(t) = C \cos \omega_n t + S \sin \omega_n t$. This response is purely harmonic (non-decaying in time). The frequency of free response is given by $\omega_n$, the undamped natural frequency.

3. For the general damped (underdamped) case, $0 < \zeta < 1$, the response has a harmonic component $C \cos \omega_d t + S \sin \omega_d t$ (of the damped natural frequency $\omega_d$) that is amplitude modulated by a decaying exponential term $e^{-\zeta \omega_n t}$. Note that the damped frequency is less than the undamped frequency since $\omega_d = \omega_n \sqrt{1 - \zeta^2}$. 

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Particular solution for harmonic excitation $F(t) = F_0 \sin \omega t$:

$$x_p(t) = \left[ \frac{F_0/K}{1 - (\omega/\omega_n)^2} \right] \sin \omega t = X \sin \omega t \quad \text{(undamped response)}$$

$$x_p(t) = \frac{F_0/K}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta \omega/\omega_n]^2}} \sin (\omega t - \varphi) \quad \text{(damped response)}$$

$$= X \sin (\omega t - \varphi)$$

where

$$\varphi = \tan^{-1}\left[ \frac{2\zeta \omega/\omega_n}{1 - (\omega/\omega_n)^2} \right]$$

The following plots show the response amplitude $X(\omega)$ and phase $\phi(\omega)$ for a damping ratio of $\zeta = 0.1$. We can make the following observations about these plots as related to the expressions for $X(\omega)$ and $\phi(\omega)$ provided above.

- For low excitation frequencies $\omega$, the response amplitude $X$ approaches a value of $F_0/K$. This is expected since for low excitation frequencies we are observing quasistatic behavior of the system. The response phase $\phi$ approaches a zero value for small $\omega$.

- For large excitation frequencies $\omega$, the response amplitude $X$ approaches a value of zero. The response phase $\phi$ approaches a value of $\pi$ for large $\omega$.

- For excitation frequencies near the undamped natural frequency, $\omega \approx \omega_n$, the response amplitude is large, particularly for small damping ratios. The maximum response occurs for excitation frequencies less than $\omega_n$. The response phase $\phi$ is exactly equal to $\pi/2$ for $\omega = \omega_n$. 