4. Experimental Uncertainty

In any experimental (or even computational) study, attention must be paid to the uncertainties involved in making measurements. Including the uncertainty allows one to judge the validity or accuracy of the measurements. Uncertainty analysis can also be useful when designing an experiment so that the propagation of uncertainties can be minimized (this will be covered in an example).

Consider a measurement of a flow rate through a pipe. Let's say that one measures a flow rate of 1.6 kg/s. Now consider a theoretical calculation that predicts a flow rate of 1.82 kg/s. Are the theory and measurement inconsistent? The answer depends upon the uncertainty in the measurement. If the experimental uncertainty is ±0.3 kg/s, then the true measured value could very well be equal to the theoretical value. However, if the experimental uncertainty is ±0.1 kg/s, then the two results are likely to be inconsistent.

There are two parts to uncertainty analysis. These include:
1. estimating the uncertainty associated with a measurement and
2. analyzing the propagation of uncertainty in subsequent analyses.

Both of these parts will be reviewed in the following sections. There are many texts (such as Holman, J.P., *Experimental Methods for Engineers*, McGraw-Hill) that can be referred to for additional information concerning experimental uncertainty.

Estimation of Uncertainty

There are three common types of error. These include "blunders," systematic (or fixed) errors, and random errors.

1. "Blunders" are errors caused by mistakes occurring due to inattention or an incorrectly configured experimental apparatus.

   Examples:
   Blatant blunder: An experimenter looks at the wrong gauge or misreads a scale and, as a result, records the wrong quantity.

   Less blatant blunder: A measurement device has the wrong resolution (spatial or temporal) to measure the parameter of interest. For example, an experimenter who uses a manometer to measure the pressure fluctuations occurring in an automobile piston cylinder will not be able to capture the rapid changes in pressure due to the manometer's slow response time.

   Subtle blunder: A measurement might affect the phenomenon that is being measured. For example, an experimenter using an ordinary thermometer to make a very precise measurement of a hot cavity's temperature might inadvertently affect the measurement by conducting heat out of the cavity through the thermometer's stem.

2. Systematic (or fixed) errors occur when repeated measurements are in error by the same amount. These errors can be removed via calibration or correction.

   Example:
   The error in length caused by a blunt ruler. This error could be corrected by calibrating the ruler against a known length.

3. Random errors occur due to unknown factors. These errors are not correctable, in general.

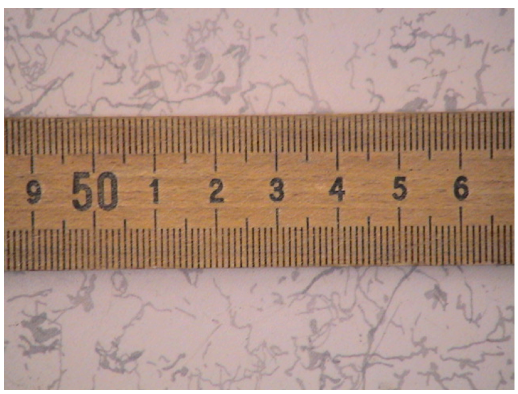
Blunders and systematic errors can be avoided or corrected. It is the random errors that we must account for in uncertainty analyses. How we quantify random errors depends on whether we conduct a single experiment or multiple experiments. Each case is examined in the following sections.

1. Single Sample Experiments

   A single sample experiment is one in which a measurement is made only once. This approach is common when the cost or duration of an experiment makes it prohibitive to perform multiple experiments.

   The measure of uncertainty in a single sample experiment is $\pm{}^{1}/_{2}$ the smallest scale division (or least count) of the measurement device. For example, given a thermometer where the smallest discernable scale division is 1 °C, the uncertainty in a temperature measurement will be ±0.5 °C. If your eyesight is poor and you can only see 5 °C divisions, then the uncertainty will be ±2.5 °C. One should use an uncertainty within which they are 95% certain that the result lies.

   Example:



   The least count for the ruler to the left is 1 mm. Hence, the uncertainty in the length measurement will be ±0.5 mm.

   Example:
   You use a manual electronic stop watch to measure the speed of a person running the 100 m dash. The stop watch gives the elapsed time to $1/1000^{th}$ of a second. What is the least count for the measurement?

   SOLUTION:

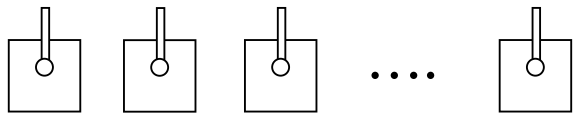
   Although the stop watch has a precision of $1/1000^{th}$ of a second, you cannot respond quickly enough to make this the limiting uncertainty. Most people have a reaction time of $1/10^{th}$ of a second. (Test yourself by having a friend drop a ruler between your fingers. You can determine your reaction time by where you catch the ruler.) Hence, to be 95% certain of your time measurement, you should use an uncertainty of $\pm{}^{1}/_{2}(0.1 \text{ sec}) = \pm 0.05$ sec.

Be Sure To:
1. Always indicate the uncertainty of any experimental measurement.
2. Carefully design your experiments to minimize sources of error.
3. Carefully evaluate your least count. The least count is not always $\pm{}^{1}/_{2}$ of the smallest scale division.

2. Multiple Sample Experiment

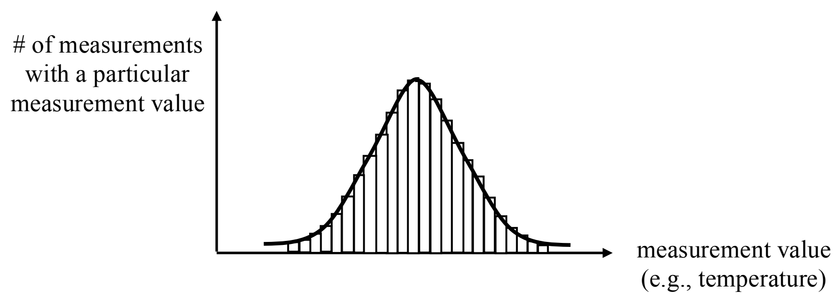
A multiple sample experiment is one where many different trials are conducted in which the same measurement is made.

Example:
- making temperature measurements in many "identical" hot cavities (shown below) or making temperature measurements in the same cavity many different times



many identical cavities and thermometers

We can use statistics to estimate the random error associated with a multiple sample experiment. For truly random errors, the distribution of errors will follow a Gaussian (or normal) distribution which has the following qualitative histogram:



To quantify the set of measurement data, we commonly use the mean of the data set and its standard deviation or variance. For example, consider N measurements of some parameter x: $x_1, x_2, \ldots, x_N$.

sample mean, $\bar{x}$ (a type of average)

$$\bar{x} = \frac{1}{N}\sum_{n=1}^{N} x_n \tag{18}$$

sample standard deviation, σ (a measure of how precise the measurements are: as $\sigma \downarrow$, the precision $\uparrow$)

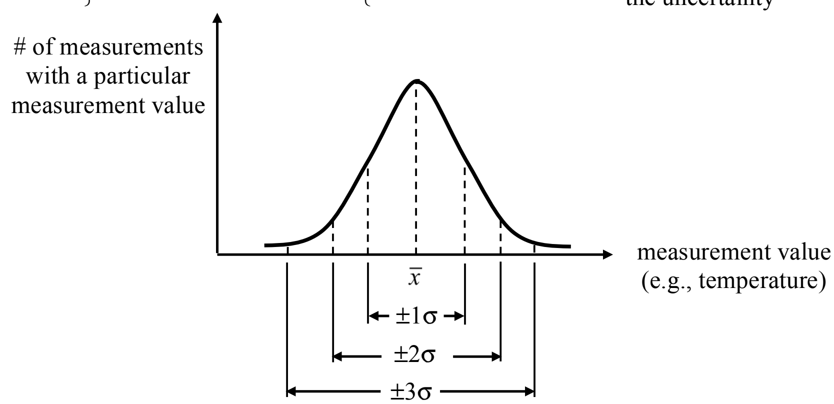
$$\sigma = \left[\frac{1}{N-1}\sum_{n=1}^{N}\left(x_n - \bar{x}\right)^2\right]^{1/2} \quad \text{(Note: The variance is } \sigma^2\text{.)} \tag{19}$$

Notes:

1. It is not reasonable to comprehensively discuss statistical analyses of data within the scope of these notes. The reader is encouraged to look through an introductory text on statistics for additional information (see, for example, Vardeman, S.B., *Statistics for Engineering Problem Solving*, PWS Publishing, Boston).
2. The square of the standard deviation (σ^2) is known as the <u>variance</u>.
3. The coefficient of variation, CoV or CV (also rsd = relative standard deviation), is defined as the ratio of the standard deviation to the mean, i.e., $CoV \equiv \sigma/\bar{x}$. A small CoV means that the scatter in your measurements is small compared to the mean.
4. For random data (a Gaussian or normal distribution) and a very large number of measurements:

$$\left.\begin{matrix}68\% \\ 95\% \\ 99\%\end{matrix}\right\} \text{of measurements fall between} \left\{\begin{matrix}\bar{x} \pm 1\sigma \\ \bar{x} \pm 2\sigma \\ \bar{x} \pm 3\sigma\end{matrix}\right.$$

← used in most engineering situations as a measure of the uncertainty (for $\bar{x} \pm 2\sigma$)



4. If the number of measurements is not very large (N < 30, for example), it is better to use the Student's t-distribution for estimating the uncertainty (refer to an introductory text on statistics such as Vardeman, S.B., *Statistics for Engineering Problem Solving*, PWS Publishing, Boston):

$$\bar{x} \pm t\sigma \tag{20}$$

where t is a factor related to the degree of confidence desired (again, a 95% uncertainty is typically desired in engineering applications), σ is the standard deviation given in Eq. (19), and N is the number of measurements made. The following table gives the value of t for various values of N and a 95% confidence level. Note that as $N \rightarrow \infty$ the t factor approaches the large sample size value of 1.96.

N	2	3	4	5	6	7	8	9	10	15	20	30	∞
$t_{95\%}$	12.71	4.30	3.18	2.78	2.57	2.45	2.36	2.31	2.26	2.14	2.09	2.04	1.96

5. Often one presents data in terms of its true (rather than sample) mean, μ, and a confidence interval. For random data, the true mean lies within the interval,

$$\mu = \bar{x} \pm \frac{t\sigma}{\sqrt{N}} \tag{21}$$

where $\bar{x}$ and σ are the sample mean (Eq. (18)) and sample standard deviation (Eq. (19)), respectively, t is the confidence interval factor (found from the Student t-distribution as in the table above), and N is the number of data points.
6. When reporting the variability in multiple sample experiments, remember that there is still uncertainty in individual measurements. Thus, when reporting results from multiple sample experiments, be sure to report the mean and (95%) confidence interval (using Eq. (21)) as well as the uncertainty in an individual measurement.

Be Sure To:

1. Report the uncertainty in an individual measurement as well as the mean and 95% confidence interval for multiple sample experiments.

Propagation of Uncertainty

Let R be a result that depends on several measurements: $x_1, \ldots, x_N$, or, in mathematical terms:

$$R = R\left(x_1, \ldots, x_N\right). \tag{22}$$

For example, the volume of a cylinder is:

$$V = \pi r^2 h \quad \Rightarrow \quad V = V\left(r, h\right). \tag{23}$$

How do we determine the uncertainty in the result R due to the uncertainties in the measurements $x_1, \ldots, x_N$? In the example above, what is the uncertainty in the volume V given the uncertainties in the radius, r, and height, h?

To address this issue, consider how a small variation in parameter, x_n, call it δx_n, causes a variation in R, call this variation δR_{xn}:

$$\delta R_{x_n} = R\left(x_{1,} \ldots, x_n + \delta x_n, \ldots x_N\right) - R\left(x_{1,} \ldots, x_n, \ldots x_N\right)$$

$$\delta R_{x_n} = \frac{R\left(x_{1,} \ldots, x_n + \delta x_n, \ldots x_N\right) - R\left(x_{1,} \ldots, x_n, \ldots x_N\right)}{\delta x_n} \delta x_n$$

$$\boxed{\underbrace{\delta R_{x_n}}_{\substack{\text{uncertainty} \\ \text{in } R \text{ due to} \\ \text{uncertainty in } x_n}} \approx \underbrace{\frac{\partial R}{\partial x_n}}_{\substack{\text{partial derivative} \\ \text{of } R \text{ w/r/t } x_n}} \underbrace{\delta x_n}_{\substack{\text{uncertainty} \\ \text{in measurement } x_n}}} \quad \text{(Note that an “=” is only strictly true as } \delta x_\text{n} \to dx_\text{n}.) \tag{24}$$

The total uncertainty in R, δR, due to uncertainties in all measurements $x_1, \ldots, x_N$, assuming that the x_n are independent so that the variations in one parameter do not affect the variations in the others, is estimated as,

$$\boxed{\delta R = \left[\sum_{n=1}^{N} \left(\delta R_{x_n}\right)^2\right]^{½} = \left[\sum_{n=1}^{N} \left(\frac{\partial R}{\partial x_n} \delta x_n\right)^2\right]^{½}} \tag{25}$$

The <u>relative uncertainty</u> in R, u_R, is given by,

$$\boxed{u_R = \frac{\delta R}{R}} \tag{26}$$

For example, the uncertainty in the cylinder volume, $V=\pi r^2 h$, due to uncertainties in the radius, r, and height, h, is,

$$\delta V = \left[\left(\frac{\partial V}{\partial r} \delta r\right)^2 + \left(\frac{\partial V}{\partial h} \delta h\right)^2\right]^{½}$$

$$= \left[\left(2\pi r h \delta r\right)^2 + \left(\pi r^2 \delta h\right)^2\right]^{½}$$

and the relative uncertainty is,

$$u_V = \frac{\delta V}{V} = \frac{1}{\pi r^2 h}\left[\left(2\pi r h \delta r\right)^2 + \left(\pi r^2 \delta h\right)^2\right]^{½}$$

$$= \left[\left(2\frac{\delta r}{r}\right)^2 + \left(\frac{\delta h}{h}\right)^2\right]^{½}$$

$$= \left[\left(2u_r\right)^2 + \left(u_h\right)^2\right]^{½}$$

where $u_r = \delta r/r$ and $u_h = \delta h/h$ are the relative uncertainties in r and h, respectively.

Note:

1. Use absolute quantities when calculating the uncertainty. For example, use °R or K as opposed to °F or °C for temperature, and use absolute pressures rather than gage pressures.
2. In an uncertainty analysis the uncertainty of some quantities may be so small compared to the uncertainties in the remaining quantities that they can be considered “exactly” known. This is

generally the case for well-characterized constants and material parameters, e.g., the acceleration due to gravity.

Be Sure To:

1. Use absolute quantities when evaluating uncertainties, e.g., absolute temperature and pressure.
2. Review your uncertainty analyses to determine which measurements result in the greatest error in a derived quantity. Design your experiments to reduce these uncertainties.