1. Lagrangian and Eulerian Perspectives

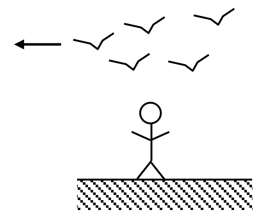
There are two common ways to study a moving fluid:

1. Look at a particular location and observe how all the fluid passing that location behaves. This is called the **Eulerian** point of view.
2. Look at a particular piece of fluid and observe how it behaves as it moves from location to location. This is called the **Lagrangian** point of view.

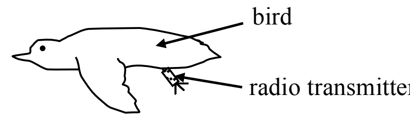
Example:

Let's say we want to study migrating birds. We could either:

1. stand in a fixed spot and make measurements as birds fly by (Eulerian point of view), or



2. tag some of the birds and make measurements as they fly along (Lagrangian point of view).



2. Reynolds' Transport Theorem (RTT)

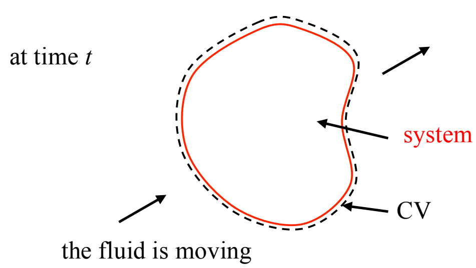
Recall that we can look at the behavior of small pieces of fluid in two ways: the Eulerian perspective or the Lagrangian perspective. Often we're interested in the behavior of an entire *system* of fluid (many pieces of fluid) rather than just an individual piece. How do we analyze this situation? We can use Eulerian and Lagrangian approaches for analyzing a macroscopic amount of fluid but we need to first develop an important tool called the Reynolds Transport Theorem.

Why do we want to do this? It turns out that the behavior of fluids (most substances in fact) can be described in terms of a few fundamental laws. These laws include:

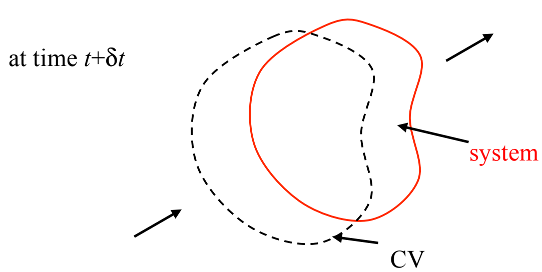
- conservation of mass (COM),
- Newton's 2nd Law,
- the angular momentum principle,
- conservation of energy (COE, aka the First Law of Thermodynamics), and
- the second law of thermodynamics.

These laws are typically easiest to apply to a particular system of fluid particles (Lagrangian perspective). However, the Lagrangian forms of the laws are typically difficult to use in practical applications since we can't easily keep track of many individual bits of fluid. It's much easier to apply the laws to a particular volume in space instead (referred to as a control volume, an Eulerian perspective). For example, tracking the behavior of individual bits of gas flowing through a rocket nozzle would be difficult. It's much easier to just look at the behavior of the gas flowing into, out of, and within the volume enclosed by the rocket nozzle. The **Reynolds Transport Theorem** is a tool that will allow us to convert from a system point of view (Lagrangian) to a control volume point of view (Eulerian).

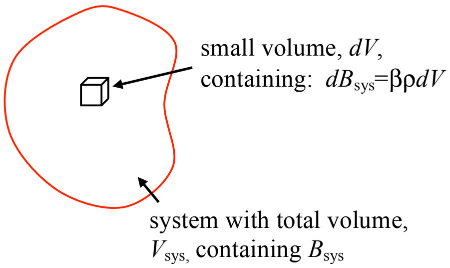
Let's consider a system of fluid particles that is *coincident* (occupying the same region in space) as our control volume (CV) at some time, t:



At some later time, $t+\delta t$, the system may have moved relative to the CV.



Let B be some transportable property (i.e., some property that can be transported from one location to another, e.g., mass, momentum, energy) and β be the corresponding amount of B per unit mass, i.e.:



$$B_{\text{sys}} = \int_{V_{\text{sys}}} \beta \rho dV$$
$$B_{\text{CV}} = \int_{\text{CV}} \beta \rho dV \tag{11}$$

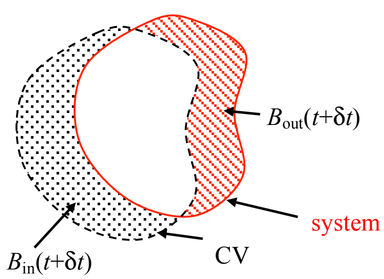
where B_{sys} and B_{CV} refer to the total amount of B in the system and control volume, respectively.

Note that at time, t, the total amounts of B in the system and control volume are equal since the system and CV are coincident,

$$B_{\text{sys}}(t) = B_{\text{CV}}(t) \tag{12}$$

However, at time, $t+\delta t$, the system and CV no longer occupy the same region in space so that, in general, $B_{\text{sys}}(t+\delta t) \neq B_{\text{CV}}(t+\delta t)$.

at time $t+\delta t$



Note that B may be changing with time so that, in general, $B_{\text{sys}}(t+\delta t) \neq B_{\text{sys}}(t)$.

In the figure at the left, B_{out} is the amount of B that has left the CV and B_{in} is the amount of B that has entered the CV.

Utilizing the figure shown above, we see that,

$$B_{\text{CV}}(t+\delta t) = B_{\text{sys}}(t+\delta t) - B_{\text{out}}(t+\delta t) + B_{\text{in}}(t+\delta t) \tag{13}$$

Subtracting $B_{\text{sys}}(t)$ from both sides and dividing through by δt gives,

$$\frac{B_{\text{CV}}(t+\delta t) - B_{\text{sys}}(t)}{\delta t} = \frac{B_{\text{sys}}(t+\delta t) - B_{\text{sys}}(t)}{\delta t} - \frac{B_{\text{out}}(t+\delta t)}{\delta t} + \frac{B_{\text{in}}(t+\delta t)}{\delta t} \tag{14}$$

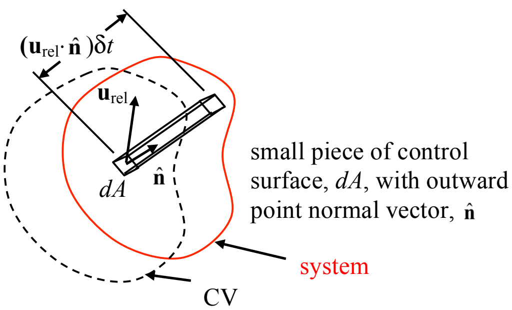
Now let's substitute $B_{\text{CV}}(t) = B_{\text{sys}}(t)$ on the left hand side, subtract $B_{\text{out}}(t)/\delta t$ and $B_{\text{in}}(t)/\delta t$ on the right-hand side (note that $B_{\text{out}}(t)=B_{\text{in}}(t)=0$), and then take the limit of the entire equation as $\delta t \to 0$,

$$\lim_{\delta t \to 0} \frac{B_{\text{CV}}(t+\delta t) - B_{\text{CV}}(t)}{\delta t} =$$
$$\lim_{\delta t \to 0} \frac{B_{\text{sys}}(t+\delta t) - B_{\text{sys}}(t)}{\delta t} - \lim_{\delta t \to 0} \frac{B_{\text{out}}(t+\delta t) - B_{\text{out}}(t)}{\delta t} + \lim_{\delta t \to 0} \frac{B_{\text{in}}(t+\delta t) - B_{\text{in}}(t)}{\delta t} \tag{15}$$
$$\Rightarrow \frac{dB_{\text{CV}}}{dt} = \frac{DB_{\text{sys}}}{Dt} - \frac{dB_{\text{out}}}{dt} + \frac{dB_{\text{in}}}{dt}$$

Note that the D/Dt notation has been used to signify that the first term on the right hand side of Eq. (15) represents the time rate of change as we follow a particular system of fluid (Lagrangian perspective). Re-arranging the equation and substituting in for B_{CV} and B_{sys} using Eq. (11),

$$\frac{D}{Dt}\left[\int_{V_{\text{sys}}} \beta\rho dV\right] = \frac{d}{dt}\left[\int_{\text{CV}} \beta\rho dV\right] + \frac{d\left(B_{\text{out}} - B_{\text{in}}\right)}{dt} \tag{16}$$

The last term on the right hand side of Eq. (16) represents the net rate at which B is leaving the control volume through the control surface (CS). Let's examine this term more closely by zooming in on a small piece of the control surface and observing how much B leaves through this surface in time δt.



The component of the fluid velocity out of the control volume through surface, dA, is given by,

$$\mathbf{u}_{\text{rel}} \cdot \hat{\mathbf{n}}$$

where $\mathbf{u}_{\text{rel}}=\mathbf{u}_{\text{sys}}-\mathbf{u}_{\text{CS}}$ is the velocity of the fluid relative to the control surface. The volume of fluid leaving through surface dA in time δt is then,

$$dV = \left(\mathbf{u}_{\text{rel}} \cdot \hat{\mathbf{n}}\right)\delta t dA = \left(\mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right)\delta t$$

Thus, the **volumetric flowrate,** dQ, (volume per unit time) through surface dA is given by,

$$dQ = \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} \tag{17}$$

Now use Eq. (17) to write the net rate at which B leaves the control volume,

$$\frac{d\left(B_{\text{out}} - B_{\text{in}}\right)}{dt} = \int_{\text{CS}} \beta\rho dQ = \int_{\text{CS}} \beta\left(\rho\mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) \tag{18}$$

Combining Eq. (18) with Eq. (16) gives,

$$\boxed{\underbrace{\frac{D}{Dt}\left[\int_{V_{\text{sys}}} \beta\rho\, dV\right]}_{\substack{\text{rate of increase of}\\ B \text{ within the system}}} = \underbrace{\frac{d}{dt}\left[\int_{\text{CV}} \beta\rho\, dV\right]}_{\substack{\text{rate of increase of}\\ B \text{ within the CV}}} + \underbrace{\int_{\text{CS}} \beta\left(\rho\mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right)}_{\substack{\text{net rate at which } B \text{ leaves}\\ \text{the CV through the CS}}}} \tag{19}$$

The **Reynolds Transport Theorem**!