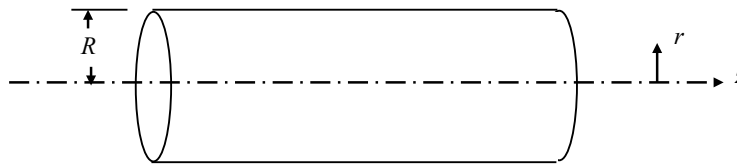


2. Fully Developed Laminar Circular Pipe Flow (Poiseuille Flow)

Consider the steady flow of an incompressible, constant viscosity, Newtonian fluid within an infinitely long, circular pipe of radius, R .



We'll make the following assumptions:

1. The flow is axi-symmetric and there is no "swirl" velocity. $\Rightarrow \frac{\partial}{\partial \theta}(\dots) = 0$ and $u_\theta = 0$
2. The flow is steady. $\Rightarrow \frac{\partial}{\partial t}(\dots) = 0$
3. The flow is fully-developed in the z -direction. $\Rightarrow \frac{\partial u_r}{\partial z} = \frac{\partial u_z}{\partial z} = 0$
4. There are no body forces. $\Rightarrow f_r = f_\theta = f_z = 0$

Let's first examine the continuity equation,

$$\frac{1}{r} \frac{\partial (ru_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0. \quad (4)$$

From assumptions #1 and #3 we see that,

$$ru_r = \text{constant} \quad (5)$$

Since there is no flow through the walls, the constant must be equal to zero and thus,

$$u_r = 0 \quad (\text{call this condition \#5}). \quad (6)$$

Now let's examine the Navier-Stokes equation in the z -direction,

$$\rho \left(\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} \right] + \rho f_z. \quad (7)$$

We can simplify this equation using our assumptions,

$$\rho \left(\underbrace{\frac{\partial u_z}{\partial t}}_{=0 \text{ (#2)}} + \underbrace{u_r}_{=0 \text{ (#5)}} \frac{\partial u_z}{\partial r} + \underbrace{\frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta}}_{=0 \text{ (#1)}} + \underbrace{u_z \frac{\partial u_z}{\partial z}}_{=0 \text{ (#3)}} \right) = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) + \underbrace{\frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2}}_{=0 \text{ (#1)}} + \underbrace{\frac{\partial^2 u_z}{\partial z^2}}_{=0 \text{ (#3)}} \right] + \rho \underbrace{f_z}_{=0 \text{ (#4)}} \quad (8)$$

$$\Rightarrow \frac{d}{dr} \left(r \frac{du_z}{dr} \right) = \frac{r}{\mu} \frac{dp}{dz}$$

$$\Rightarrow r \frac{du_z}{dr} = \frac{r^2}{2\mu} \frac{dp}{dz} + c_1$$

$$\Rightarrow u_z = \frac{r^2}{4\mu} \frac{dp}{dz} + c_1 \ln r + c_2 \quad (9)$$

Note that in the previous derivation the fact that u_z is a function only of r has been used to change the partial derivatives to ordinary derivatives. Furthermore, examining the Navier-Stokes equations in the r and θ directions demonstrates that the pressure, p , is a function only of z and thus ordinary derivatives can be used when differentiating the pressure with respect to z .

Now let's apply boundary conditions to determine the unknown constants c_1 and c_2 . First, note that the fluid velocity in a pipe must remain finite as $r \rightarrow 0$ so that the constant c_1 must be zero (this is a type of kinematic boundary condition). Also, the pipe wall is fixed so that we have $u_z(r=R)=0$ (no-slip condition). After applying boundary conditions we have,

$$u_z = \frac{R^2}{4\mu} \left(-\frac{dp}{dz} \right) \left(1 - \frac{r^2}{R^2} \right). \quad \text{Poiseuille Flow in a Circular Pipe} \quad (10)$$

Notes:

1. The velocity profile is a paraboloid with the maximum velocity occurring along the centerline. The average velocity in the pipe is found from,

$$\bar{u} = \frac{1}{\pi R^2} \int_{r=0}^{r=R} u_z (2\pi r dr) = \frac{R^2}{8\mu} \left(-\frac{dp}{dz} \right) = \frac{D^2}{32\mu} \left(-\frac{dp}{dz} \right) = \frac{1}{2} u_{\max}, \quad (11)$$

where u_{\max} is the maximum fluid velocity and D is the pipe diameter.

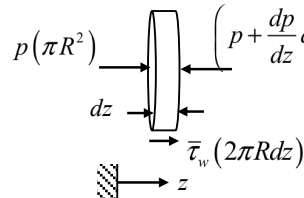
2. The volumetric flow rate through the pipe is,

$$Q = \bar{u} \left(\frac{\pi}{4} D^2 \right) = \frac{\pi D^4}{128\mu} \left(-\frac{dp}{dz} \right). \quad (12)$$

3. We can determine stresses using the constitutive relations for a Newtonian fluid. The shear stress that the pipe walls apply to the fluid, τ_w , is,

$$\tau_w = \frac{R}{2} \left(\frac{dp}{dz} \right) = \frac{-4\mu\bar{u}}{R}, \quad (13)$$

where \bar{u} is the average velocity in the pipe. Note that an alternate method for determining the average wall shear stress, which in this case is equal to the exact wall shear stress, is to balance shear forces and pressure forces on a small slice of the flow as shown below.



$$\sum F_z = 0 = p\pi R^2 - \left(p + \frac{dp}{dz} dz \right) \pi R^2 + \bar{\tau}_w 2\pi R dz \quad (14)$$

$$\bar{\tau}_w = \frac{R}{2} \frac{dp}{dz} \quad (\text{The same answer as before!}) \quad (15)$$

In engineering applications, it is common to express the average shear stress in terms of a dimensionless **(Darcy) friction factor, f_D** , which is defined as,

$$f_D \equiv \left| \frac{4\bar{\tau}_w}{\frac{1}{2}\rho\bar{u}^2} \right| = 64 \left(\frac{\mu}{\rho\bar{u}D} \right) = \frac{64}{\text{Re}} \quad (16)$$

where $D=2R$ is the pipe diameter and Re is the Reynolds number. The Darcy friction factor commonly appears in the Moody chart for incompressible, viscous pipe flow. Note again that this solution is only valid only for a laminar flow. The condition for the flow to remain laminar is found experimentally to be:

$$\text{Re} \equiv \frac{\rho\bar{u}D}{\mu} < 2300 \quad (17)$$

4. Let's re-write Eq. (11),

$$\bar{u} = \frac{D^2}{32\mu} \left(-\frac{dp}{dz} \right) \Rightarrow |\bar{u}| = \frac{D^2}{32\mu} \left(\frac{\Delta p}{L} \right), \quad p+\Delta p \quad \text{---} \quad p \quad L \quad (18)$$

where, in the fully developed region, the pressure gradient remains constant and so we may write dp/dz as $\Delta p/L$ where Δp is the pressure drop over a length L of the pipe. Re-arranging Eq. (18) and dropping the absolute value symbol for convenience,

$$\Delta p = \frac{32\mu\bar{u}L}{D^2}. \quad (19)$$

Make the previous equation dimensionless by dividing through by the dynamic pressure based on the average flow speed,

$$\frac{\Delta p}{\frac{1}{2}\rho\bar{u}^2} = \frac{64\mu}{\rho\bar{u}D} \left(\frac{L}{D} \right) = \left(\frac{64}{\text{Re}_D} \right) \left(\frac{L}{D} \right). \quad (20)$$

The dimensionless pressure drop is also referred to as the **loss coefficient, k** . Hence, for a laminar flow, the loss coefficient corresponding to the viscous stresses due to the pipe walls is,

$$k_{\text{laminar, wall stresses}} = \left(\frac{64}{\text{Re}_D} \right) \left(\frac{L}{D} \right) = f_D \left(\frac{L}{D} \right). \quad (21)$$

4. Moody Chart

The previous friction factor relations have been summarized into a single plot known as the Moody chart, which is shown below.

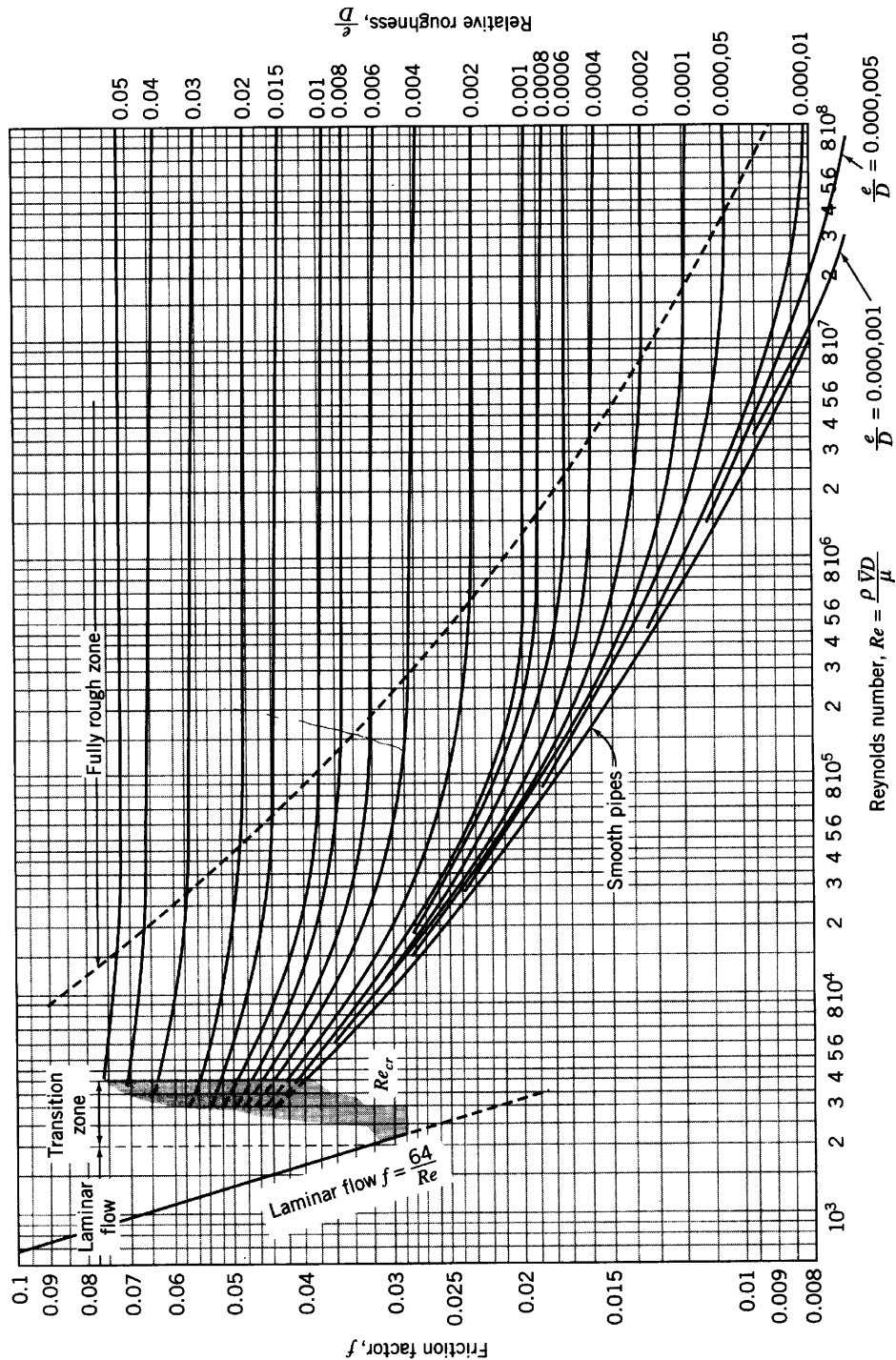


Fig. 8.13 Friction factor for fully developed flow in circular pipes. (Data from [8], used by permission.)

(Figure from Fox, *et al.*, *Introduction to Fluid Mechanics*, Wiley.)

Notes:

1. For Reynolds numbers less than 2300, one may use either the analytical expression for the friction factor,

$$f_D = \frac{64}{\text{Re}_D}, \quad (41)$$

or the Moody chart.

2. Reynolds numbers between approximately 2300 and 4000 correspond to the transitional regime between laminar and turbulent flow. The gray region in the Moody chart reflects the fact that the friction factor can vary significantly in this region.
3. The **fully rough zone** (aka wholly turbulent zone, fully turbulent zone) in the Moody chart is a region where the friction factor is a weak function of the Reynolds number, but a strong function of the relative roughness (refer to the second half of Section 3 of these notes). If the Reynolds number of a flow is unknown, but is expected to be large, it is often helpful to assume that the flow is in the fully rough zone as an initial first guess.
4. The roughnesses of various types of pipe materials have been compiled into tables such as the following one.

Material (new)	ϵ [ft]	ϵ [mm]
riveted steel	0.003 – 0.03	0.9 – 9.0
concrete	0.001 – 0.01	0.3 – 3.0
wood stave	0.0006 – 0.003	0.18 – 0.9
cast iron	0.0085	0.26
galvanized iron	0.0005	0.15
asphalted cast iron	0.0004	0.12
commercial steel or wrought iron	0.00015	0.046
drawn tubing	0.000005	0.0015
glass	smooth	smooth

Example:

1. Using the Moody chart, determine the friction factor for a Reynolds number of 10^5 and a relative roughness of 0.001.
2. What is the friction factor for a Reynolds number of 1000?
3. What is the friction factor for a Reynolds number of 10^6 in a smooth pipe?

SOLUTION:

1. The friction factor is $f_D \approx 0.0225$ (Follow the red lines in the following figure.)
2. Since the Reynolds number is less than 2300, we can use the exact laminar flow relation:

$$f_D = \frac{64}{Re_D} \Rightarrow f_D = 0.064 \quad (42)$$

Alternately, we could use the Moody chart by following the blue lines in the following figure.

3. The friction factor is $f_D \approx 0.012$ (Follow the green lines in the following figure.)

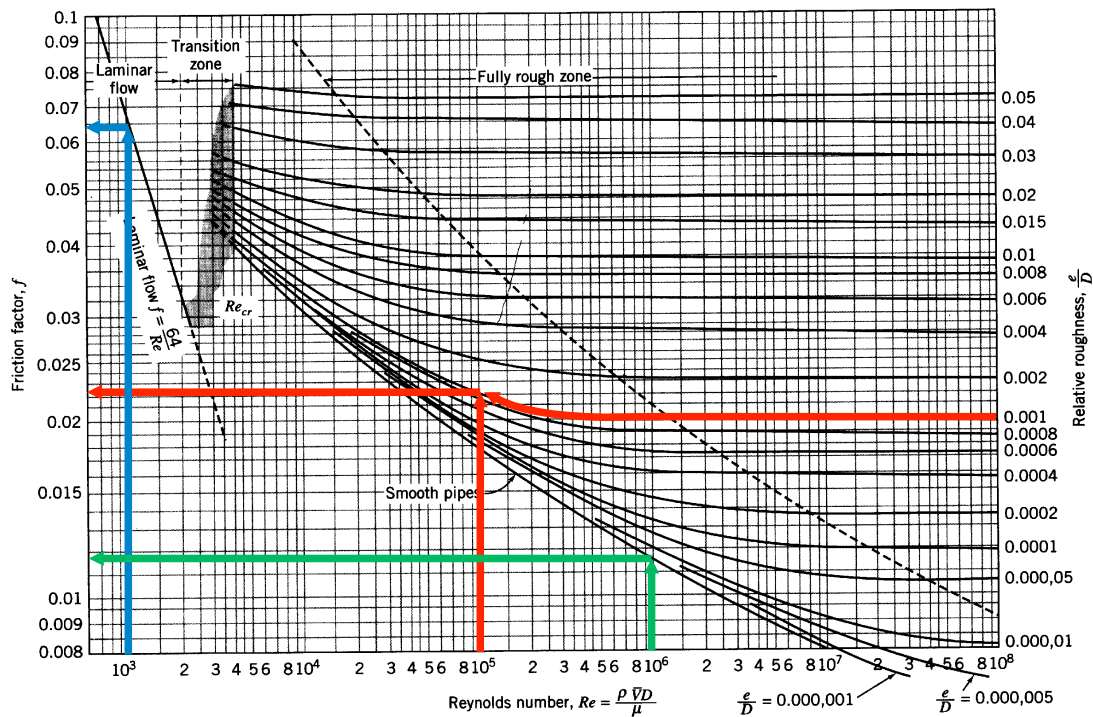
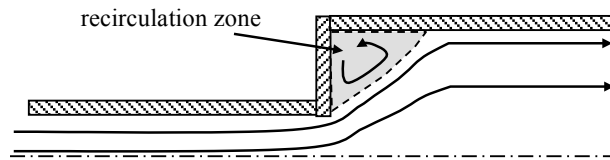


Fig. 8.13 Friction factor for fully developed flow in circular pipes. (Data from [8], used by permission.)

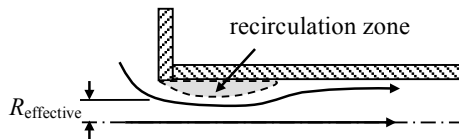
5. Other Losses

The loss due to the viscous resistance caused by the pipe walls is referred to as a **major loss**. Pressure losses may occur due to viscous dissipation resulting from fluid interactions with other parts of a pipe system such as valves, bends, contractions/expansions, inlets, and connectors. These losses are known as **minor losses**. The names can be misleading since it's not uncommon in pipe systems to have most of the pressure loss resulting from the minor losses (e.g., a pipe system with a large number of bends and valves, but short sections of straight pipe).

What causes these minor losses? The pressure loss results primarily from viscous dissipation in regions with large velocity gradients, such as in a recirculation zone as shown in the figure below.



A closely related phenomenon known as the **vena contracta** acts to effectively reduce the diameter at entrances and bends. The recirculation zone also results in a pressure loss.



Although minor loss coefficients can be determined analytically for certain situations, most frequently the loss coefficient for a particular device is found experimentally. Essentially, one measures the pressure drop across the device, Δp , and forms the loss coefficient, k , using,

$$k = \frac{\Delta p}{\frac{1}{2} \rho \bar{V}^2}, \quad (43)$$

where ρ is the fluid density and \bar{V} is the average speed through the device. Many tables with experimentally determined loss coefficients have been generated.

Notes:

1. When using a loss coefficient, it is important to know what velocity has been used to form the coefficient. For example, the loss coefficient for a contraction is typically based on the speed downstream of the contraction, while the loss coefficient for an expansion is based on the speed upstream of the expansion.
2. Minor losses are sometimes given in terms of **equivalent lengths** of pipe. An equivalent minor loss of 10 pipe diameters worth of a particular type of pipe means that the major loss caused by a pipe of that type, 10 diameters in length will give the same pressure loss as the minor loss. Thus, a loss coefficient and equivalent pipe length, L_e , can be related by,

$$k = f_D \left(\frac{L_e}{D} \right). \quad (44)$$

3. For non-circular pipes or pipes that are not completely filled, the same method of determining the friction factor and loss coefficients are used, except that a **hydraulic diameter**, D_h , is used in place of the diameter. The hydraulic diameter is defined as,

$$D_h \equiv \frac{4A}{P_w}, \quad (45)$$

where A and P_w are the cross-sectional flow area (not necessarily the pipe cross-sectional area) and wetted perimeter of the pipe (the part of the pipe that is in contact with the fluid). For example, consider a completely filled square pipe of side length L . The hydraulic diameter for such a pipe is,

$$D_h \equiv \frac{4(L^2)}{(4L)} = L. \quad \text{Diagram: A blue square with side length } L. \quad (46)$$

Now consider a completely filled annular pipe with outer diameter D_o and inner diameter D_i . The hydraulic diameter for this case is,

$$D_h \equiv \frac{4\left(\frac{\pi}{4}D_o^2 - \frac{\pi}{4}D_i^2\right)}{(\pi D_o + \pi D_i)} = \frac{D_o^2 - D_i^2}{D_o + D_i} = D_o - D_i. \quad \text{Diagram: A blue annular pipe with outer diameter } D_o \text{ and inner diameter } D_i. \quad (47)$$

Note:

1. Often a **hydraulic radius, R_h** , is used instead of a hydraulic diameter for flows in conduits with a free surface. The hydraulic radius is defined as,

$$R_h \equiv \frac{A}{P_w}. \quad (49)$$

Using this definition, $D_h \neq 2R_h$, but is instead, $D_h = 4R_h$, which can be confusing. The *Manning Formula* (not covered in these notes) is frequently used in the analysis of free surface conduit flows.