5. Conservation of Energy for a Control Volume

The reader is should review Chapter 03: Basic Thermodynamics before continuing with this section.

To write COE for a control volume, we utilize the Reynolds Transport Theorem (RTT) to convert our system expression to a control volume expression. Let's first rewrite the 1st Law of Thermodynamics for a system using the Lagrangian derivative notation (we're interested in how things change with respect to time as we follow a particular system of fluid) and write the total energy of a system in terms of an integral,

$$\frac{D}{Dt}\underbrace{\int_{V_{\text{sys}}} e\rho\, dV}_{E_{\text{system}}} = \dot{Q}_{\text{into system}} + \dot{W}_{\text{on system}} \tag{64}$$

Applying the Reynolds Transport Theorem and noting that the system and control volume are coincident at the time we apply the RTT gives,

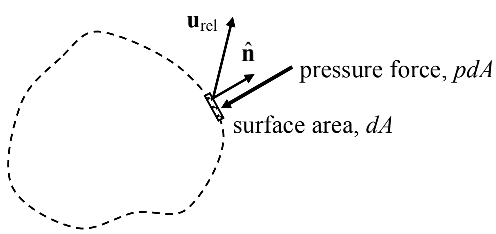
$$\boxed{\frac{d}{dt}\int_{\text{CV}} e\rho\, dV + \int_{\text{CS}} e\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \dot{Q}_{\text{into CV}} + \dot{W}_{\text{on CV}}} \qquad \textbf{COE for a CV} \tag{65}$$

Note: $e=u+{}^1/_2V^2+G$ where G is a conservative potential energy function with the specific gravitational force given by $f_{\text{gravity}} = -\nabla G$. For the remainder of these notes, G will be assumed to be $G = gz$ ($\Rightarrow f_{\text{gravity}} = -g$) where g is the acceleration due to gravity which points in the negative z-direction.

Now let's expand the rate of work (power) term into rate of pressure work (pdV power), shaft power, and the power due to other effects such as viscous forces, electromagnetic forces, etc.,

$$\dot{W}_{\text{on CV}} = \dot{W}_{\text{pressure,on CV}} + \dot{W}_{\text{shaft,on CV}} + \dot{W}_{\text{other,on CV}} \tag{66}$$

In particular, we can write the rate of pressure work term in the following way,



$$\begin{aligned} d\dot{W}_{\text{pressure,on CV}} &= d\mathbf{F}_{\text{on CV}} \cdot \mathbf{u}_{\text{rel}} \\ &= -pdA\hat{\mathbf{n}} \cdot \mathbf{u}_{\text{rel}} \\ &= -p\left(\mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) \end{aligned}$$

The rate of pressure work over the entire CS is,

$$\boxed{\dot{W}_{\text{pressure, on CV}} = \int_{\text{CS}} -p\left(\mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right)} \tag{67}$$

Equation (67) is the rate at which pressure work is performed on the fluid flux through the control surface. The rate at which pressure work is done on a moving solid boundary is included in the $\dot{W}_{\text{other}}$ term.

Substituting Eqs. (67) and (66) into Eq. (65), expanding the specific total energy term in the surface integral, and bringing the rate of pressure work term to the LHS gives,

$$\begin{aligned} \frac{d}{dt}\int_{\text{CV}} e\rho\, dV + \int_{\text{CS}} \left(u + \tfrac{p}{\rho} + \tfrac{1}{2}V^2 + gz\right)\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) \\ = \dot{Q}_{\text{into CV}} + \dot{W}_{\text{shaft, on CV}} + \dot{W}_{\text{other, on CV}} \end{aligned} \tag{68}$$

The quantity (u+p/ρ) shows up often in thermal-fluid systems and is given the special name of **specific enthalpy, *h*,**

$$\boxed{h \equiv u + \tfrac{p}{\rho} = u + pv} \qquad \textbf{specific enthalpy} \tag{69}$$

where $v=V/m$ is the specific volume

Note that just as with internal energy, tables of thermodynamic properties typically list the value of the specific enthalpy for various substances under various conditions.

Substituting Eq. (69) into Eq. (68) gives,

$$\boxed{\frac{d}{dt}\int_{CV} e\rho\, dV + \int_{CS}\left(h + \tfrac{1}{2}V^2 + gz\right)\left(\rho \mathbf{u}_{rel}\cdot d\mathbf{A}\right) = \dot{Q}_{\text{into CV}} + \dot{W}_{\text{shaft, on CV}} + \dot{W}_{\text{other, on CV}}} \quad (70)$$

Notes:

1. For a flow where the total energy within the CV does not change with time (a steady flow), we have,

$$\int_{CS}\left(h + \tfrac{1}{2}V^2 + gz\right)\left(\rho \mathbf{u}_{rel}\cdot d\mathbf{A}\right) = \dot{Q}_{\text{into CV}} + \dot{W}_{\text{shaft, on CV}} + \dot{W}_{\text{other, on CV}} \quad (71)$$

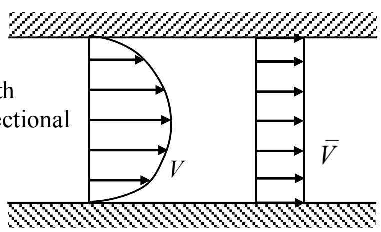
Note that flows may be unsteady at the local level (e.g., detailed flow within a pump), but may be steady on a larger scale (e.g., the average conditions within the pump housing).

2. For a steady flow with a single inlet (call it state 1) and outlet (call it state 2) we can write COE as,

$$\left(h + \alpha\tfrac{1}{2}\bar{V}^2 + gz\right)_2 - \left(h + \alpha\tfrac{1}{2}\bar{V}^2 + gz\right)_1 = q_{\text{into CV}} + w_{\text{shaft, on CV}} + w_{\text{other, on CV}} \quad (72)$$

where $q = \dot{Q}/\dot{m}$ and $w = \dot{W}/\dot{m}$ (note that from COM the mass flow rate is a constant).

   i. The quantity, α, is known as the **kinetic energy correction factor**. It is a correction factor accounting for the fact that the average velocity, $\bar{V}$, does not contain the same kinetic energy as a non-uniform distribution. For example, consider the kinetic energy contained in the two flow profiles shown below:



The average velocity is found by,

$$\bar{V} = \frac{1}{A}\int_A V dA$$

The average specific kinetic energy flux is,

$$\overline{ke} = \int_A \tfrac{1}{2}V^2\left(\rho\mathbf{V}\cdot d\mathbf{A}\right) \neq \tfrac{1}{2}\dot{m}\bar{V}^2$$

We define the kinetic energy correction factor, α,

$$\boxed{\alpha \equiv \frac{\int_A \tfrac{1}{2}V^2\rho\left(\mathbf{V}\cdot d\mathbf{A}\right)}{\tfrac{1}{2}\dot{m}\bar{V}^2}} \quad (73)$$

so that,

$$\overline{ke} = \alpha\, \tfrac{1}{2}\dot{m}\bar{V}^2$$

For a laminar flow in a circular pipe, the velocity profile is parabolic resulting in α=2. For a turbulent flow, α→1 as the Reynolds number, defined as $\text{Re}_D \equiv \rho\bar{V}D/\mu$, increases (suggesting that the velocity profile becomes more uniform as the Reynolds number increases).

   ii. The quantity,

$$h_T = h_0 \equiv h + \alpha\tfrac{1}{2}\bar{V}^2 + gz \quad (74)$$

is referred to as the **total specific enthalpy** or the **stagnation specific enthalpy**. Note that for gases, the gz term is often much smaller than the other terms and thus is often neglected.

   iii. If the flow is adiabatic ($q = 0$) and the rate of work by forces other than pressure can be neglected, then,

$$\boxed{h_T = \text{constant}} \quad \textbf{adiabatic flow with no shaft or other work} \quad (75)$$

3. Now let's re-write Eq. (72) but expand the specific enthalpy terms,

$$\left(u + \frac{p}{\rho} + \alpha\tfrac{1}{2}\bar{V}^2 + gz\right)_2 - \left(u + \frac{p}{\rho} + \alpha\tfrac{1}{2}\bar{V}^2 + gz\right)_1 = q_{\text{into CV}} + w_{\text{shaft, on CV}} + w_{\text{other, on CV}} \quad (76)$$

Re-arranging terms and dividing through by the gravitational acceleration gives,

$$\left(\frac{p}{\rho g}+\alpha\frac{\overline{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\overline{V}^2}{2g}+z\right)_1 -\frac{\left(u_2-u_1-q_{\text{into CV}}\right)}{g}+\frac{\dot{W}_{\text{shaft, on CV}}}{\dot{m}g}+\frac{\dot{W}_{\text{other, on CV}}}{\dot{m}g} \tag{77}$$

Note that each term in this equation has the dimensions of length. The terms are also referred to as **head** quantities as defined below:

$\frac{p}{\rho g}\equiv$ **pressure head** $\frac{\overline{V}^2}{2g}\equiv$ **velocity head** $z\equiv$ **elevation head**

$\frac{\left(u_2-u_1-q_{\text{into CV}}\right)}{g}\equiv$ **head loss, H_L** (head lost due to mechanical energy being converted to thermal energy, and heat lost from the CV)

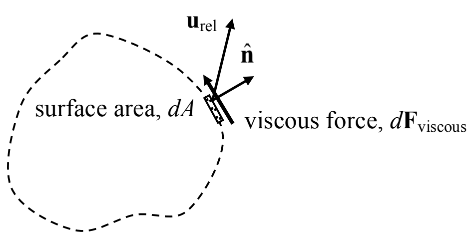
$\frac{\dot{W}_{\text{shaft on CV}}}{\dot{m}g}\equiv$ **shaft head, H_S** (head added due to shaft work; recall that $\dot{W}=T\omega$)

$\frac{\dot{W}_{\text{other on CV}}}{\dot{m}g}\equiv$ **other head, H_O** (head added due to other work on the fluid)

The equation in this form is also known as the **Extended Bernoulli Equation**,

$$\boxed{\left(\frac{p}{\rho g}+\alpha\frac{\overline{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\overline{V}^2}{2g}+z\right)_1-H_L+H_S+H_O} \tag{78}$$

4. Now let's examine the "other" work term more closely. Specifically, let's concern ourselves with the work done by viscous effects. Consider the rate of viscous work done on the CV shown below,



$$d\dot{W}_{\text{viscous,on CV}}=d\mathbf{F}_{\text{viscous on CV}}\cdot\mathbf{u}_{\text{rel}}$$

so that the total rate of viscous work acting on the CS is,

$$\dot{W}_{\text{viscous,on CV}}=\int_{CS}d\mathbf{F}_{\text{viscous on CV}}\cdot\mathbf{u}_{\text{rel}}$$

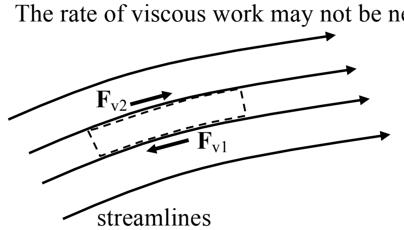
i. Note that at a solid boundary, $\mathbf{u}_{rel}=\mathbf{0}$ (due to the no-slip condition) so that the rate of viscous work is zero at solid surfaces. If the flow is inviscid then $\mathbf{u}_{rel}\neq\mathbf{0}$ but $d\mathbf{F}_{viscous}=\mathbf{0}$ and so the rate of viscous work is also zero.

ii. If the control volume is oriented so that the velocity vectors are perpendicular to the normal vectors of the CS, then the rate of viscous work done on the CV will be zero,

$$d\mathbf{F}_{\text{viscous on CV}}\cdot\mathbf{u}_{\text{rel}}=0$$

since the viscous force will be perpendicular to the velocity vector.

iii. The rate of viscous work may not be negligible if the control volume is chosen as shown below:



Viscous forces along streamline surfaces may be significant if the shear stress, τ, is large,

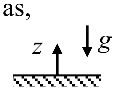
$$\tau=\mu\frac{\partial u}{\partial n}$$

where n is the direction normal to the streamlines.

6. Extended Bernoulli Equation

Recall from previous notes that conservation of energy may be written as,

$$\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_2=\left(\frac{p}{\rho g}+\alpha\frac{\bar{V}^2}{2g}+z\right)_1-H_{L,1\to2}+H_{S,1\to2}\ , \qquad (50)$$



where each of the terms in the equation has dimensions of length. The "1" and "2" subscripts refer to the inlet and outlet conditions, respectively. The individual terms are referred to as:

$\frac{p}{\rho g}$ ≡ pressure head

$\alpha\frac{\bar{V}^2}{2g}$ ≡ velocity or dynamic head

z ≡ elevation head

$H_{L,1\to2}$ ≡ head loss

$H_{S,1\to2}$ ≡ shaft head

Recall that the α in the velocity head term is the kinetic energy correction factor which accounts for the fact that an average speed is used in the Extended Bernoulli Equation rather than the real velocity profile (refer back to earlier notes concerning conservation of energy). A value of $\alpha = 2$ is used for laminar flows while $\alpha = 1$ is typically assumed for turbulent flows (actually, $\alpha \to 1$ as $\text{Re}_D \to \infty$).

The head loss term, H_L, accounts for both major and minor pressure losses, and may be written as,

$$H_{L,1\to2}=\sum_i k_i\frac{\bar{V}_i^2}{2g}, \qquad (51)$$

where the subscript "i" accounts for every loss in the pipe system. Recall that the major loss coefficient may be written as,

$$k_{\text{major}}=f_D\left(\frac{L}{D}\right). \qquad (52)$$

The shaft head term, H_S, accounts for the pressure addition (reduction) resulting from the inclusion of devices such as pumps, compressors, fans, turbines, and windmills. Those devices that add head to the flow are positive (e.g., pumps), while those that extract head are negative (e.g., turbines). The shaft head term may be written in terms of shaft power, $\dot{W}_S$, as,

$$H_{S,1\to2}=\frac{\dot{W}_{S,1\to2}}{\dot{m}g}, \qquad (53)$$

where $\dot{m}$ is the mass flow rate through the device.

Notes:

1. One often must make various assumptions at the beginning of a pipe flow solution, e.g., the flow is laminar, the flow is turbulent, or the flow is in the fully rough zone. For example, for flow through a hypodermic needle, it's reasonable to assume that the flow will be laminar since the needle diameter is so small. Having experience with pipe flow systems helps one to make good assumptions. Regardless of what assumptions are made, it is important that one verifies that the calculated flow conditions are consistent with the assumptions that were made. For example, if one assumes laminar flow in the hypodermic needle then solves for the flow velocity, then the Reynolds number should be checked to verify that a laminar flow assumption was correct. If so, then the solution procedure is consistent. If not, then the laminar flow assumption was incorrect and a turbulent flow assumption should be made and the problem re-solved.