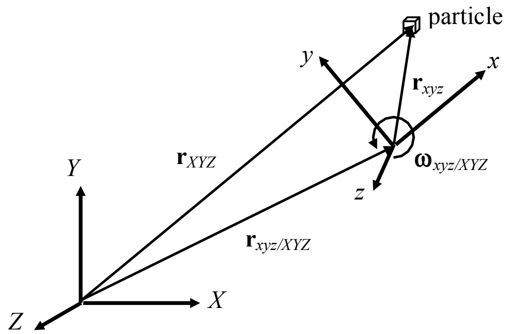
The LME for Non-Inertial Coordinate Systems

Recall that Newton's 2nd law holds strictly for inertial (non-accelerating) coordinate systems. Now let's consider coordinate systems that are non-inertial (accelerating). First let's examine how we can describe the motion of a particle in an accelerating coordinate system, call it frame *xyz*, in terms of a non-accelerating coordinate system, call it frame *XYZ*.



The position of a particle in *XYZ* is given by $\mathbf{r}_{XYZ}$ and in *xyz* the particle's position is given by $\mathbf{r}_{xyz}$. The two position vectors are related by the position vector of the origin of *xyz* in *XYZ*, $\mathbf{r}_{xyz/XYZ}$,

$$\mathbf{r}_{XYZ} = \mathbf{r}_{xyz/XYZ} + \mathbf{r}_{xyz}. \tag{34}$$

The velocity of the particle in *XYZ* can be found by taking the time derivative of the position vector, $\mathbf{r}_{XYZ}$, with respect to *XYZ* (as indicated by the subscript *XYZ* in the equation below),

$$\left.\frac{d\mathbf{r}_{XYZ}}{dt}\right|_{XYZ} = \left.\frac{d\mathbf{r}_{xyz/XYZ}}{dt}\right|_{XYZ} + \left.\frac{d\mathbf{r}_{xyz}}{dt}\right|_{XYZ}. \tag{35}$$

The time derivative of $\mathbf{r}_{xyz/XYZ}$ is simply the velocity of the origin of *xyz* with respect to *XYZ*, $\mathbf{u}_{xyz/XYZ}$, i.e.,

$$\left.\frac{d\mathbf{r}_{xyz/XYZ}}{dt}\right|_{XYZ} = \mathbf{u}_{xyz/XYZ}. \tag{36}$$

We must be careful, however, when evaluating the time derivative of $\mathbf{r}_{xyz}$ in *XYZ* since both the magnitude of $\mathbf{r}_{xyz}$ and the basis vectors of *xyz* can change with time (the basis vectors of *xyz* can change due to rotation of the *xyz* with respect to *XYZ*). To calculate the time derivative of $\mathbf{r}_{xyz}$ in *XYZ*, let's first write $\mathbf{r}_{xyz}$ as a magnitude, r_{xyz}, multiplied by the basis vectors of *xyz*, $\hat{\mathbf{e}}_{xyz}$, then use the product rule to evaluate the time derivative,

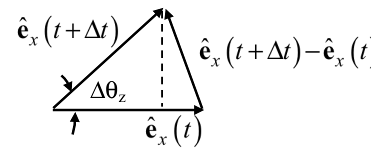
$$\left.\frac{d\mathbf{r}_{xyz}}{dt}\right|_{XYZ} = \left.\frac{d\left(r_{xyz}\hat{\mathbf{e}}_{xyz}\right)}{dt}\right|_{XYZ} = \left.\frac{dr_{xyz}}{dt}\right|_{XYZ}\hat{\mathbf{e}}_{xyz} + r_{xyz}\left.\frac{d\hat{\mathbf{e}}_{xyz}}{dt}\right|_{XYZ}. \tag{37}$$

Note that,

$$\left.\frac{dr_{xyz}}{dt}\right|_{XYZ}\hat{\mathbf{e}}_{xyz} = \mathbf{u}_{xyz}, \tag{38}$$

is the velocity of the particle in *xyz*.

The time derivative of the *xyz* basis vectors is found from geometric considerations. Consider the drawing shown below of the change in the *x*-basis vector as a function of time. For simplicity, we'll assume that the rotation only occurs in the *xy* plane (i.e., $\Delta\theta_x = \Delta\theta_y = 0$):



The time derivative of the basis vector is given by,

$$\frac{d\hat{\mathbf{e}}_x}{dt} = \lim_{\Delta t\to 0}\frac{\hat{\mathbf{e}}_x(t+\Delta t)-\hat{\mathbf{e}}_x(t)}{\Delta t}.$$

Note from the diagram that,

$$\begin{aligned}\hat{\mathbf{e}}_x(t+\Delta t)-\hat{\mathbf{e}}_x(t) &= \hat{\mathbf{e}}_x(t)\cos\Delta\theta_z+\hat{\mathbf{e}}_y(t)\sin\Delta\theta_z-\hat{\mathbf{e}}_x(t)\\ &= \hat{\mathbf{e}}_x(t)\left(\cos\Delta\theta_z-1\right)+\hat{\mathbf{e}}_y(t)\sin\Delta\theta_z\end{aligned}$$

In addition, as $\Delta t \to 0$, $\Delta\theta_z \to 0$ and,

$$\left(\cos\Delta\theta_z-1\right)\approx 1-\left(\Delta\theta_z\right)^2/2-1=-\tfrac{1}{2}\left(\Delta\theta_z\right)^2 \text{ and } \sin\Delta\theta_z\approx\Delta\theta_z,$$

so that,

$$\begin{aligned}\frac{d\hat{\mathbf{e}}_x}{dt} &= \lim_{\Delta t\to 0}\frac{\hat{\mathbf{e}}_x(t+\Delta t)-\hat{\mathbf{e}}_x(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{-\tfrac{1}{2}\left(\Delta\theta_z\right)^2\hat{\mathbf{e}}_x(t)+\Delta\theta_z\hat{\mathbf{e}}_y}{\Delta t}\\ &= \frac{d\theta_z}{dt}\hat{\mathbf{e}}_y\end{aligned}$$

$$\therefore \frac{d\hat{\mathbf{e}}_x}{dt} = \omega_z\hat{\mathbf{e}}_y \quad \text{(where } \omega_z = d\theta_z/dt\text{).} \tag{39}$$

In general, it can be shown that,

$$\left.\frac{d\hat{\mathbf{e}}_{xyz}}{dt}\right|_{XYZ} = \boldsymbol{\omega}_{xyz/XYZ}\times\hat{\mathbf{e}}_{xyz} \tag{40}$$

so that,

$$r_{xyz}\left.\frac{d\hat{\mathbf{e}}_{xyz}}{dt}\right|_{XYZ} = r_{xyz}\left(\boldsymbol{\omega}_{xyz/XYZ}\times\hat{\mathbf{e}}_{xyz}\right)=\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}. \tag{41}$$

Combining Eqs. (35) – (38) and (41) we find that the velocity of a fluid particle in the inertial coordinate system *XYZ* is,

$$\boxed{\underbrace{\mathbf{u}_{XYZ}}_{\substack{\text{velocity of particle}\\ \text{in } XYZ}} = \underbrace{\mathbf{u}_{xyz/XYZ}}_{\substack{\text{velocity of } xyz\\ \text{w/r/t } XYZ}} + \underbrace{\mathbf{u}_{xyz}}_{\substack{\text{velocity of particle}\\ \text{in } xyz}} + \underbrace{\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}}_{\substack{\text{velocity of particle in } XYZ\\ \text{due to rotation of } xyz\\ \text{w/r/t } XYZ}}} \tag{42}$$

where $\mathbf{u}_{xyz}$ is the particle velocity in non-inertial coordinate system *xyz*, $\boldsymbol{\omega}_{xyz/XYZ}$ is the angular velocity of *xyz* with respect to *XYZ*, and $\mathbf{r}_{xyz}$ is the position vector of the particle from the origin of *xyz*.

The acceleration of a particle in *XYZ* in terms of *xyz* quantities can be found in a similar manner,

$$\underbrace{\left.\frac{d\mathbf{u}_{XYZ}}{dt}\right|_{XYZ}}_{=\mathbf{a}_{XYZ}} = \underbrace{\left.\frac{d\mathbf{u}_{xyz/XYZ}}{dt}\right|_{XYZ}}_{=\mathbf{a}_{xyz/XYZ}} + \underbrace{\left.\frac{d\mathbf{u}_{xyz}}{dt}\right|_{XYZ}}_{=\left.\frac{d}{dt}\left(u_{xyz}\hat{\mathbf{e}}_{xyz}\right)\right|_{XYZ}} + \underbrace{\left.\frac{d}{dt}\left(\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)\right|_{XYZ}}_{=\dot{\boldsymbol{\omega}}_{xyz/XYZ}\times\mathbf{r}_{xyz}+\boldsymbol{\omega}_{xyz/XYZ}\times\left.\frac{d\left(r_{xyz}\hat{\mathbf{e}}_{xyz}\right)}{dt}\right|_{XYZ}} \tag{43}$$

where the results from Eqs. (37), (38), (40), and (41) are used to simplify the last two expressions in Eq. (43),

$$\begin{aligned}\left.\frac{d}{dt}\left(u_{xyz}\hat{\mathbf{e}}_{xyz}\right)\right|_{XYZ} &= \frac{du_{xyz}}{dt}\hat{\mathbf{e}}_{xyz}+u_{xyz}\frac{d\hat{\mathbf{e}}_{xyz}}{dt}\\ &= \mathbf{a}_{xyz}+\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{u}_{xyz}\end{aligned} \tag{44}$$

and,

$$\dot{\boldsymbol{\omega}}_{xyz/XYZ} \times \left.\frac{d\left(r_{xyz}\hat{\mathbf{e}}_{xyz}\right)}{dt}\right|_{XYZ} = \boldsymbol{\omega}_{xyz/XYZ} \times \left(\mathbf{u}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz}\right)$$
$$= \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{u}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ} \times \left(\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz}\right) \quad (45)$$

Substituting Eqs. (44) and (45) into Eq. (43) and simplifying gives,

$$\underbrace{\mathbf{a}_{XYZ}}_{\substack{\text{rectilinear acceleration}\\ \text{of particle in } XYZ}} = \underbrace{\mathbf{a}_{xyz/XYZ}}_{\substack{\text{rectilinear acceleration}\\ \text{of } xyz \text{ w/r/t } XYZ}} + \underbrace{\mathbf{a}_{xyz}}_{\substack{\text{rectilinear acceleration}\\ \text{of particle in } xyz}} + \underbrace{\left(\dot{\boldsymbol{\omega}}_{xyz/XYZ} \times \mathbf{r}_{xyz}\right)}_{\substack{\text{tangential acceleration of}\\ \text{particle in } XYZ \text{ due to}\\ \text{rotational acceleration of } xyz}}$$
$$+ \underbrace{\left(2\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{u}_{xyz}\right)}_{\substack{\text{Coriolis acceleration of}\\ \text{particle in } XYZ \text{ due to}\\ \text{rectilinear motion of particle}\\ \text{in } xyz}} + \underbrace{\left[\boldsymbol{\omega}_{xyz/XYZ} \times \left(\boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz}\right)\right]}_{\substack{\text{centripital acceleration of particle}\\ \text{in } XYZ \text{ due to rotation of } xyz}} \quad (46)$$

Now let's use these relations to determine an expression for the LME using a non-inertial coordinate system. Recall that the Lagrangian statement for the LME is (refer to Eq. (26)),

$$\frac{D}{Dt}\left(\int_{V_{\text{system}}} \mathbf{u}_{XYZ}\,\rho dV\right) = \mathbf{F}_{\text{on system}}. \quad (47)$$

Substitute Eq. (42) into Eq. (47) and re-arrange,

$$\mathbf{F}_{\text{on system}} = \frac{D}{Dt}\left[\int_{V_{\text{system}}} \left(\mathbf{u}_{xyz/XYZ} + \mathbf{u}_{xyz} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz}\right)\rho dV\right]$$
$$= \frac{D}{Dt}\left[\int_{V_{\text{system}}} \mathbf{u}_{xyz}\,\rho dV\right] + \frac{D}{Dt}\left[\int_{V_{\text{system}}} \left(\mathbf{u}_{xyz/XYZ} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz}\right)\rho dV\right] \quad (48)$$

Now use the Reynolds Transport Theorem to convert the first term on the right hand side to a control volume perspective and re-arrange,

$$\mathbf{F}_{\text{body on CV}} + \mathbf{F}_{\text{surface on CV}} - \frac{D}{Dt}\left[\int_{V_{\text{system}}} \left(\mathbf{u}_{xyz/XYZ} + \boldsymbol{\omega}_{xyz/XYZ} \times \mathbf{r}_{xyz}\right)\rho dV\right]$$
$$= \frac{d}{dt}\int_{\text{CV}} \mathbf{u}_{xyz}\,\rho dV + \int_{\text{CS}} \mathbf{u}_{xyz}\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) \quad (49)$$

The remaining Lagrangian term can be simplified by changing the volume integral to a mass integral and noting that the mass of the system doesn't change with time,

$$\begin{aligned}
&\frac{D}{Dt}\left[\int_{V_{\text{system}}}\left(\mathbf{u}_{xyz/XYZ}+\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)\rho dV\right]\\
&\qquad=\frac{D}{Dt}\left[\int_{M_{\text{system}}}\left(\mathbf{u}_{xyz/XYZ}+\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)dm\right]\\
&\qquad=\int_{M_{\text{system}}}\frac{D}{Dt}\left(\mathbf{u}_{xyz/XYZ}+\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)dm\\
&\qquad=\int_{V_{\text{system}}}\frac{D}{Dt}\left(\mathbf{u}_{xyz/XYZ}+\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)\rho dV
\end{aligned} \tag{50}$$

Since $\mathbf{u}_{xyz/XYZ}$ and $\boldsymbol{\omega}_{xyz/XYZ}$ are functions only of time (these variables describe the motion of the coordinate system xyz and not the fluid field), and because $D\mathbf{r}_{xyz}/Dt=\mathbf{u}_{xyz}$[1], we can replace the Lagrangian time derivative with an Eulerian time derivative and substitute in our result from Eq. (46),

$$\begin{aligned}
&\int_{V_{\text{system}}}\frac{D}{Dt}\left(\mathbf{u}_{xyz/XYZ}+\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)\rho\, dV\\
&\qquad=\int_{V_{\text{system}}}\frac{d}{dt}\left(\mathbf{u}_{xyz/XYZ}+\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)\rho\, dV\\
&\qquad=\int_{V_{\text{system}}}\left[\mathbf{a}_{xyz/XYZ}+\dot{\boldsymbol{\omega}}_{xyz/XYZ}\times\mathbf{r}_{xyz}+2\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{u}_{xyz}+\boldsymbol{\omega}_{xyz/XYZ}\times\left(\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)\right]\rho\, dV
\end{aligned} \tag{51}$$

Substituting Eqs. (50) and (51) back into Eq. (49) and noting that when we apply the Reynolds Transport Theorem the control volume and system volume are coincident (so that the system volume integral in Eq. (51) can be replaced by a control volume integral), we find that the LME can be applied using a non-inertial coordinate, xyz, if the following form is used,

$$\boxed{\begin{aligned}
&\mathbf{F}_{\text{body on CV}}+\mathbf{F}_{\text{surface on CV}}\\
&-\int_{\text{CV}}\left\{\mathbf{a}_{xyz/XYZ}+\left(\dot{\boldsymbol{\omega}}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)+\left(2\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{u}_{xyz}\right)+\left[\boldsymbol{\omega}_{xyz/XYZ}\times\left(\boldsymbol{\omega}_{xyz/XYZ}\times\mathbf{r}_{xyz}\right)\right]\right\}\rho\, dV\\
&\qquad=\frac{d}{dt}\int_{\text{CV}}\mathbf{u}_{xyz}\rho\, dV+\int_{\text{CS}}\mathbf{u}_{xyz}\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)
\end{aligned}} \tag{52}$$

Let's consider a few examples to see how this form of the LME is applied.

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1 $$\frac{D\mathbf{r}_{xyz}}{Dt}=\underbrace{\frac{\partial\mathbf{r}_{xyz}}{\partial t}}_{=0}+u_x\underbrace{\frac{\partial\mathbf{r}_{xyz}}{\partial x}}_{=\hat{\mathbf{e}}_x}+u_y\underbrace{\frac{\partial\mathbf{r}_{xyz}}{\partial y}}_{=\hat{\mathbf{e}}_y}+u_z\underbrace{\frac{\partial\mathbf{r}_{xyz}}{\partial z}}_{=\hat{\mathbf{e}}_z}=\mathbf{u}_{xyz}$$

where $\mathbf{r}_{xyz}=x\hat{\mathbf{e}}_x+y\hat{\mathbf{e}}_y+z\hat{\mathbf{e}}_z$