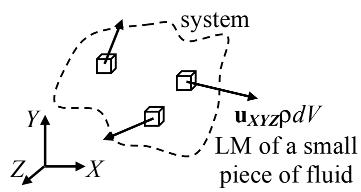
Linear Momentum Equation (LME)

In this section we'll consider **Newton's 2nd law** applied to a control volume of fluid. Recall that linear momentum is a vector quantity (it has both magnitude and direction) and is given by the mass*velocity. In words and in mathematical terms, Newton's 2nd Law for a system is:

The rate of change of a system's linear momentum is equal to the net force acting on the system.



$$\frac{D}{Dt}\Bigg(\underbrace{\int_{V_{\text{system}}} \mathbf{u}_{XYZ}\rho\, dV}_{\text{LM of the system}}\Bigg) = \mathbf{F}_{\text{on system}} \tag{26}$$

where D/Dt is the Lagrangian derivative (implying that we're using the rate of change as we follow the system), V is the volume, and ρ is the density. The quantity, $\mathbf{u}_{XYZ}$, represents the velocity of a small piece of fluid in the system with respect to an **inertial (aka non-accelerating) coordinate system** *XYZ* (recall that Newton's 2nd law holds strictly for inertial coordinate systems). Note that a coordinate system moving at a <u>constant</u> velocity in a straight line is non-accelerating and thus is inertial.

The term, $\mathbf{F}_{\text{on system}}$, represents the net forces acting on the system. These forces can be of two different types. The first are **body forces**, $\mathbf{F}_{\text{body}}$, and the second are **surface forces**, $\mathbf{F}_{\text{surface}}$. Body forces are those forces that act on each piece of fluid in the system. Examples include gravitational and electro-magnetic forces. Surface forces are those forces acting only at the surface of the system. Examples of surface forces include pressure and shear forces. Thus,

$$\mathbf{F}_{\text{on system}} = \mathbf{F}_{\text{body on system}} + \mathbf{F}_{\text{surface on system}} \tag{27}$$

Using the Reynolds Transport Theorem to convert the left hand side of Eq. (20) from a system point of view to an expression for a control volume gives:

$$\frac{D}{Dt}\left(\int_{V_{\text{system}}} \mathbf{u}_{XYZ}\rho dV\right) = \frac{d}{dt}\int_{\text{CV}} \mathbf{u}_{XYZ}\rho dV + \int_{\text{CS}} \mathbf{u}_{XYZ}\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right) \tag{28}$$

Since the Reynolds Transport Theorem is applied to a coincident system and control volume, the forces acting on the system will also act on the control volume. Thus, the LME for a CV becomes:

$$\underbrace{\mathbf{F}_{\text{body on CV}}}_{\substack{\text{net body force}\\ \text{acting on the CV}}} + \underbrace{\mathbf{F}_{\text{surface on CV}}}_{\substack{\text{net surface force}\\ \text{acting on the CV}}} = \underbrace{\frac{d}{dt}\int_{\text{CV}} \mathbf{u}_{XYZ}\rho\, dV}_{\substack{\text{rate of increase of}\\ \text{LM inside the CV}}} + \underbrace{\int_{\text{CS}} \mathbf{u}_{XYZ}\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)}_{\substack{\text{net rate at which LM leaves}\\ \text{the CV through the CS}}} \tag{29}$$

LME for a CV

Notes:

1. Recall that the LME is a vector expression. There are actually three equations built into Eq. (29). For example, in a rectangular coordinate system (Cartesian coordinates) we have:

$$\begin{aligned} F_{X,\text{body on CV}} + F_{X,\text{surface on CV}} &= \frac{d}{dt}\int_{\text{CV}} u_X \rho dV + \int_{\text{CS}} u_X \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) \\ F_{Y,\text{body on CV}} + F_{Y,\text{surface on CV}} &= \frac{d}{dt}\int_{\text{CV}} u_Y \rho dV + \int_{\text{CS}} u_Y \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) \\ F_{Z,\text{body on CV}} + F_{Z,\text{surface on CV}} &= \frac{d}{dt}\int_{\text{CV}} u_Z \rho dV + \int_{\text{CS}} u_Z \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) \end{aligned} \tag{30}$$

2. It is important to distinguish between the two velocities $\mathbf{u}_{XYZ}$ and $\mathbf{u}_{\text{rel}}$. The velocity, $\mathbf{u}_{XYZ}$, represents the fluid velocity with respect to an inertial coordinate system *XYZ* (e.g., a coordinate system fixed with respect to the stars or moving at a constant velocity in a straight line). The velocity, $\mathbf{u}_{\text{rel}}$, is the velocity of the fluid with respect to the control surface, e.g., $\mathbf{u}_{\text{rel}} = \mathbf{u}_{\text{fluid},\,XYZ} - \mathbf{u}_{\text{CS},\,XYZ}$. The velocities $\mathbf{u}_{XYZ}$ and $\mathbf{u}_{\text{rel}}$ will be equal if the control surface velocity in *XYZ* is zero, i.e., $\mathbf{u}_{\text{CS},XYZ} = \mathbf{0}$.

3. So far we've only discussed the LME for inertial (aka non-accelerating) coordinate systems. We can also apply the LME to non-inertial (aka accelerating) coordinate systems but we need to add additional acceleration terms. We'll consider accelerating coordinate systems a little later.

4. In order to avoid mistakes, be sure to do the following when applying the LME:
   a. draw the CV that the LME is being applied to,
   b. indicate the coordinate system that is being used,
   c. state your significant assumptions,
   d. draw a free body diagram (FBD) of the pertinent forces, and
   e. write down the LME and then indicate the value of each term in the LME

   While these things may seem trivial and unnecessary, writing them down in a clear and concise manner can greatly reduce the likelihood of mistakes.
5. Note that the first term on the right hand side of Eq. (29) is the rate of increase of linear momentum in the CV which can be re-written as:

$$\frac{d}{dt}\int_{\text{CV}} \mathbf{u}_{XYZ} \rho dV = \frac{d\mathbf{L}_{CV}}{dt} \tag{31}$$

   where $\mathbf{L}_{CV}$ is the linear momentum (= $m\mathbf{u}$) contained within the CV with respect to the inertial coordinate system *XYZ*. Similarly, the second term on the right hand side of Eq. (29), which is the net rate at which linear momentum leaves the CV through the CS, may be written as,

$$\int_{\text{CS}} \mathbf{u}_{XYZ}\left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \sum_{\text{all outlets}} \left(\dot{m}\mathbf{L}\right)_{\text{out}} - \sum_{\text{all inlets}} \left(\dot{m}\mathbf{L}\right)_{\text{in}} \tag{32}$$

   where $\dot{m}$ is a mass flow rate and $\mathbf{L}$ is the linear momentum of the outlet/inlet stream using the inertial coordinate system *XYZ*. Combining Eqs. (29), (31), and (32) gives,

$$\boxed{\mathbf{F}_B + \mathbf{F}_S = \frac{d\mathbf{L}_{CV}}{dt} + \sum_{\text{all outlets}} \left(\dot{m}\mathbf{L}\right)_{\text{out}} - \sum_{\text{all inlets}} \left(\dot{m}\mathbf{L}\right)_{\text{in}}} \quad \text{Alternate form of LME} \tag{33}$$

Helpful Hints:

1. Carefully draw your control volume. Don't neglect to draw a control volume or draw a control volume and then use a different one.
2. Clearly indicate your coordinate system. Don't neglect to indicate a coordinate system or draw a coordinate system and then use a different one.
3. Carefully draw free body diagrams for your control volume. This will facilitate your evaluation of forces in the linear momentum equation.
4. Make sure you understand what each term in the linear momentum equation represents.
5. Carefully evaluate the dot product in the mass flow rate term.
6. Be sure to use the correct velocity components in the CV and CS integral terms.
7. You must integrate the terms in the linear momentum equation when the density or velocity are not uniform.
8. Don't forget to include pressure and shear forces in the surface force term.
9. Don't forget to include the weight of everything inside the control volume when gravitational body forces are significant.

Let's consider a few examples to see how LME is applied.