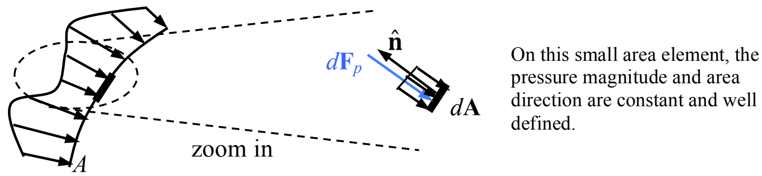
Pressure Forces on Submerged Surfaces and Center of Pressure

Recall from Chapter 01 that the small pressure force $d\mathbf{F}_p$ acting on a surface with a small area $d\mathbf{A}$ is,

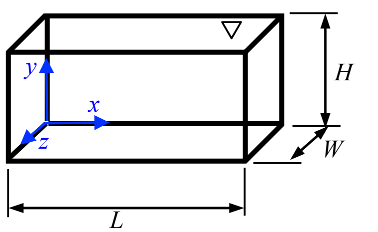
$$d\mathbf{F}_p = -p d\mathbf{A} . \tag{2.31}$$

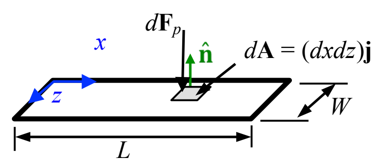


This force relationship was written specifically for a small area since it's possible that over a large area, the pressure and the direction of the area could vary over the area (shown in the figure above). Thus, to find the total pressure force on the whole area, the (small) force on a small area, where the area direction and pressure are well defined, is calculated first and then these are added, or integrated, over the whole area, i.e.,

$$\mathbf{F}_p = \int_A d\mathbf{F}_p = \int_A -p d\mathbf{A} . \tag{2.32}$$

Let's consider the example of a fish tank completely filled with water, as shown in the figure below. We wish to determine the net pressure force acting on bottom and right tank walls.





Start first with the pressure force on the tank bottom,

$$\mathbf{F}_{p,\text{bottom}} = \int_{z=0}^{z=W}\int_{x=0}^{x=L} -p\underbrace{\left[dxdz\left(\hat{\mathbf{j}}\right)\right]}_{=dA} = -\hat{\mathbf{j}}\int_{z=0}^{z=W}\int_{x=0}^{x=L}\underbrace{\rho g H}_{=p_\text{gage}} dx\,dz = -\hat{\mathbf{j}}\rho g H W L , \tag{2.33}$$

where, at the bottom of the tank, the gage pressure remains constant at,

$$p_{\substack{\text{bottom,}\\ \text{gage}}} = \rho g H . \tag{2.34}$$

Notes:

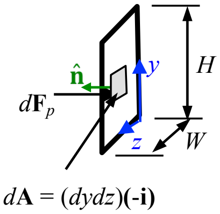
1. The magnitude of the pressure force on the bottom is equal to the weight of the water in the tank. This makes sense because if there are no shear stresses at the side walls, then the pressure force at the bottom of the tank must support all of the weight of the liquid sitting above it.
2. A gage pressure is used in Eq. (2.33) to simplify the pressure force calculation. Since there is atmosphere on the other side of the tank bottom, then the gage pressure due to the atmosphere is zero ($p_{\text{atm,gage}} = 0$) and the corresponding pressure force is zero. We get the same result as Eq. (2.33) if absolute pressures are used everywhere instead,

$$\mathbf{F}_{p,\text{bottom}} = \underbrace{\int_{z=0}^{z=W}\int_{x=0}^{x=L} -\left(p_\text{atm} + \rho g H\right)\left(dxdz\hat{\mathbf{j}}\right)}_{\substack{\text{pressure force due to water}\\ \text{using an absolute pressure}}} + \underbrace{\int_{z=0}^{z=W}\int_{x=0}^{x=L} -\left(p_\text{atm}\right)\left[dxdz\left(-\hat{\mathbf{j}}\right)\right]}_{\substack{\text{pressure force on bottom due to atmosphere}\\ \text{using an absolute pressure}}} = -\hat{\mathbf{j}}\rho g H W L . \tag{2.35}$$

Note that the unit normal vector for the atmospheric side (bottom side, second integral) is in the opposite direction of the unit normal vector for the water side (first integral) since we're on opposite sides of the wall.

Now let's calculate the pressure force acting on the right side wall.

$$\mathbf{F}_{p,\text{right}} = \int_{z=0}^{z=W}\int_{y=0}^{y=H} -p\underbrace{\left[dydz\left(-\hat{\mathbf{i}}\right)\right]}_{=d\mathbf{A}} = \hat{\mathbf{i}}\int_{z=0}^{z=W}\int_{y=0}^{y=H} \underbrace{\rho g\left(H-y\right)}_{=p_{\text{gage}}}dy\,dz = \hat{\mathbf{i}}\rho g\frac{1}{2}H^2W \ . \tag{2.36}$$

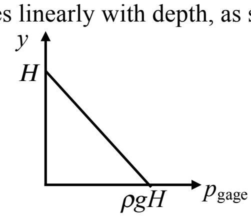


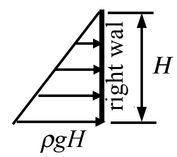
Notes:

1. Recall from the diagram that the coordinate system is located at the bottom of the tank. Thus, the (gage) pressure varies as,

   $$p_{\text{gage}} = \rho g\left(H-y\right) . \tag{2.37}$$

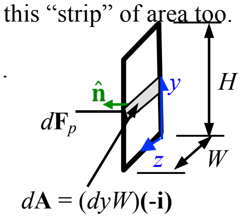
   This pressure still varies linearly with depth, as shown in the following figure.

2. The small area element $dA = dydz$ (in the picture at the top of the page) is used since the pressure has a well-defined value on this area. Since the pressure only varies in the y direction, we could have also used the area element $dA = Wdy$. The pressure is well defined on this "strip" of area too.

   $$\mathbf{F}_{p,\text{right}} = \int_{y=0}^{y=H} -p\underbrace{\left[dyW\left(-\hat{\mathbf{i}}\right)\right]}_{=d\mathbf{A}} = \hat{\mathbf{i}}W\int_{y=0}^{y=H} \underbrace{\rho g\left(H-y\right)}_{=p_{\text{gage}}}dy = \hat{\mathbf{i}}\rho g\frac{1}{2}H^2W \ . \tag{2.38}$$

   A vertical strip of area, i.e., $dA = Hdz$, can't be used to determine the pressure force since the pressure isn't well defined on this surface. The pressure varies in the y direction so over this vertical strip, the pressure doesn't remain constant.

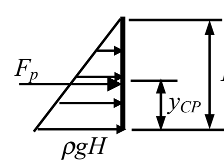
3. The pressure force is equal in magnitude to the area under the pressure curve shown in Note #1,

   $$\left|d\mathbf{F}_p\right| = \frac{1}{2}\underbrace{\left(\rho gH\right)}_{\text{base}}\underbrace{\left(H\right)}_{\text{height}}\underbrace{\left(W\right)}_{\text{depth}} = \frac{1}{2}\rho gH^2W \ . \tag{2.39}$$

   This same behavior is true for the pressure force on the base.

Now that we've determined the resultant pressure forces on the bottom and right surfaces, let's determine where these resultant forces act. This location is known as the **center of pressure (CP)**. The center of pressure is found by ensuring that the moment generated by the resultant pressure force will equal the moment generated by the actual, distributed pressure forces. Consider the right side of the tank. Balancing moments about the z axis,

$$F_{p,\text{right}}y_{CP} = \int_{y=0}^{y=H} y\underbrace{\left(pdA\right)}_{=dF_p} = \int_{y=0}^{y=H} y\left[\rho g\left(H-y\right)\cdot dyW\right] = \frac{1}{6}\rho gH^3W \ , \tag{2.40}$$

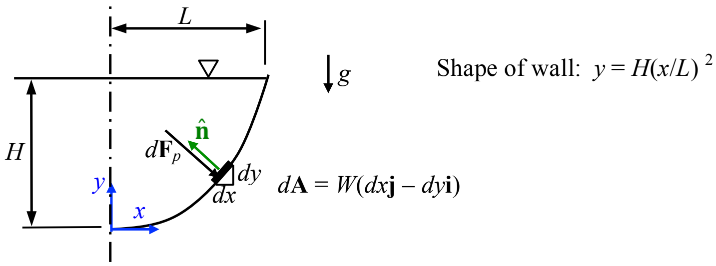


$$\left(\frac{1}{2}\rho gH^2W\right)y_{CP} = \frac{1}{6}\rho gH^3W \quad \text{(making use of Eq. (2.38))}, \tag{2.41}$$

$$y_{CP} = \frac{1}{3}H \ . \tag{2.42}$$

Notes:
1. The center of pressure is also equal to the center of area under the pressure distribution curve.
2. We can take moments about any location and get the same result.
3. The center of pressure for the right wall in the z direction is $z_{CP} = W/2$. This result may be found using a similar method as shown above, or can be determined from symmetry.

The **resultant pressure force and center of pressure location for curved surfaces** may be found in the same way as for straight surfaces. The only significant difference is that the unit normal vectors for the differentially small area elements may change with position. For example, let's determine the net pressure force and center of pressure on the parabolically-shaped wall shown in the figure below. Assume the wall is planar and has a depth W into the page.



$$\mathbf{F}_p = \int_A -p\,d\mathbf{A} = \int_A -\underbrace{\rho g\left(H-y\right)}_{=p_{\text{gage}}}\underbrace{\left(Wdx\hat{\mathbf{j}}-Wdy\hat{\mathbf{i}}\right)}_{=d\mathbf{A}} = -\rho g W\int_A \left(H-y\right)\left(dx\hat{\mathbf{j}}-dy\hat{\mathbf{i}}\right). \tag{2.43}$$

Before setting the limits on the integral, note that y is a function of x on the wall surface, which also means that a small displacement in the y direction is related to a small displacement in the x direction,

$$y = H\left(\frac{x}{L}\right)^2 \Rightarrow dy = \frac{2H}{L^2}x\,dx\,. \tag{2.44}$$

We can use this information to express the integral in terms of a single variable (we'll use x, but we could use y instead too). Substituting Eq. (2.44) into Eq. (2.43) gives,

$$\mathbf{F}_p = -\rho g W\int_{x=0}^{x=L}\left[H - H\left(\frac{x}{L}\right)^2\right]\left(dx\hat{\mathbf{j}} - \frac{2H}{L^2}x\,dx\,\hat{\mathbf{i}}\right) = -\rho g W H\left[\hat{\mathbf{j}}\int_{x=0}^{x=L}\left(1-\frac{x^2}{L^2}\right)dx - \hat{\mathbf{i}}\frac{2H}{L^2}\int_{x=0}^{x=L}\left(x-\frac{x^3}{L^2}\right)dx\right], \tag{2.45}$$

$$\mathbf{F}_p = -\rho g W H\left[\hat{\mathbf{j}}\left(L - \frac{1}{3}\frac{L^3}{L^2}\right) - \hat{\mathbf{i}}\frac{2H}{L^2}\left(\frac{1}{2}L^2 - \frac{1}{4}\frac{L^4}{L^2}\right)\right], \tag{2.46}$$

$$\mathbf{F}_p = \rho g W H\left(\frac{1}{2}H\hat{\mathbf{i}} - \frac{2}{3}L\hat{\mathbf{j}}\right) = \frac{1}{2}\rho g W H^2\hat{\mathbf{i}} - \frac{2}{3}\rho g W H L\hat{\mathbf{j}}\,. \tag{2.47}$$

This result is the pressure force the fluid exerts on the wall.

The center of pressure is found by balancing moments, identical to what was used for planar surfaces. Balance moments about the origin,

$$\mathbf{r}_{CP}\times\mathbf{F}_p = \int_A \underbrace{\left(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}\right)}_{\text{moment arm}}\times\left[-\underbrace{\rho g\left(H-y\right)}_{=p_{\text{gage}}}\underbrace{\left(Wdx\hat{\mathbf{j}}-Wdy\hat{\mathbf{i}}\right)}_{=d\mathbf{A}}\right] = -\rho g W\int_A \left(H-y\right)\left[x\,dx + y\,dy\right]\hat{\mathbf{k}}\,, \tag{2.48}$$

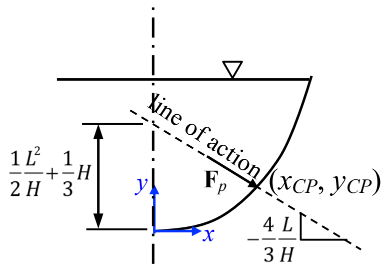
$$\left(x_{CP}\hat{\mathbf{i}}+y_{CP}\hat{\mathbf{j}}\right)\times\left(\frac{1}{2}\rho g W H^2\hat{\mathbf{i}} - \frac{2}{3}\rho g W H L\hat{\mathbf{j}}\right) = -\rho g W\hat{\mathbf{k}}\int_{x=0}^{x=L}\left[H\left(1-\frac{x^2}{L^2}\right)x\,dx + H\left(1-\frac{x^2}{L^2}\right)H\left(\frac{x^2}{L^2}\right)\frac{2H}{L^2}x\,dx\right], \tag{2.49}$$

$$-\rho g W H\left(x_{CP}\frac{2}{3}L + y_{CP}\frac{1}{2}H\right)\hat{\mathbf{k}} = -\rho g W H\hat{\mathbf{k}}\int_{x=0}^{x=L}\left[\left(x-\frac{x^3}{L^2}\right) + \frac{2H^2}{L^4}\left(x^3 - \frac{x^5}{L^2}\right)\right]dx\,, \tag{2.50}$$

$$x_{CP}\frac{2}{3}L + y_{CP}\frac{1}{2}H = \frac{1}{2}L^2 - \frac{1}{4}\frac{L^4}{L^2} + \frac{2H^2}{L^4}\left(\frac{1}{4}L^4 - \frac{1}{6}\frac{L^6}{L^2}\right) = \frac{1}{4}L^2 + \frac{1}{6}H^2\,, \tag{2.51}$$

$$y_{CP}=\left(-\frac{4}{3}\frac{L}{H}\right)x_{CP}+\left(\frac{1}{2}\frac{L^2}{H}+\frac{1}{3}H\right). \tag{2.52}$$

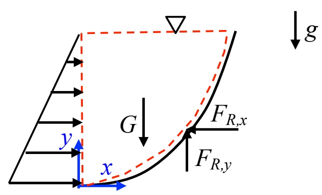
This previous equation, which is the equation of a line, is known as the **line of action**. It is the line along which the resultant forces act. This line of action is shown graphically in the following figure.



In order to find (x_{CP}, y_{CP}), we would need to find the intersection of the line of action with the equation of the wall (Eq. (2.44)). This calculation is tedious for the current example and will not be performed here.

Notes:

1. An alternate method for determining the resultant force and center of pressure is to balance forces on a region of fluid bordered by the wall. For example, balance forces on the region of fluid identified by the dotted line in the figure below.



G is the weight of the liquid within the red dotted line.

$F_{R,x}$ and $F_{R,y}$ are the force components the wall exerts on the fluid element.

$$\sum F_x=0=\frac{1}{2}\rho gH^2W-F_{R,x}\Rightarrow F_{R,x}=\frac{1}{2}\rho gH^2W\,, \tag{2.53}$$

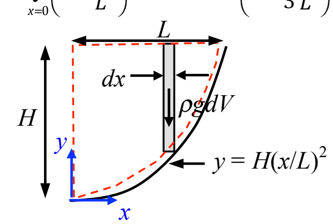
$$\sum F_y=0=-G+F_{R,y}\Rightarrow F_{R,y}=G\,, \tag{2.54}$$

where,

$$G=\int_{x=0}^{x=L}\rho g\underbrace{(H-y)dxW}_{=dV}=\int_{x=0}^{x=L}\rho g\left[H-H\left(\frac{x}{L}\right)^2\right]dxW=\rho gHW\int_{x=0}^{x=L}\left(1-\frac{x^2}{L^2}\right)dx=\rho gHW\left(L-\frac{1}{3}\frac{L^3}{L^2}\right), \tag{2.55}$$

$$G=\frac{2}{3}\rho gHWL\,, \tag{2.56}$$

(dV is a small amount of volume),



so that,

$$F_{R,y}=\frac{2}{3}\rho gHWL\,. \tag{2.57}$$

These magnitudes for $F_{R,x}$ and $F_{R,y}$ are exactly the same as what was found in Eq. (2.47). Note that here $F_{R,x}$ and $F_{R,y}$ are the force components the wall exerts on the fluid so, from Newton's 3rd Law, the fluid exerts equal and opposite force components on the wall.

The center of pressure about the z axis is found by balancing moments about the origin, the same as what was done for planar walls,

$$\mathbf{r}_{CP}\times\mathbf{F}_p=\left(\underbrace{\frac{1}{3}H\hat{\mathbf{j}}}_{\substack{CP\text{ for}\\ \text{pressure}\\ \text{on left side}}}\times\underbrace{\frac{1}{2}\rho gH^2W\hat{\mathbf{i}}}_{\substack{\text{resultant pressure}\\ \text{force on left side}}}\right)+\left[\underbrace{\left(x_{CM}\hat{\mathbf{i}}+y_{CM}\hat{\mathbf{j}}\right)}_{\text{center of mass}}\times\underbrace{-\frac{2}{3}\rho gHWL\hat{\mathbf{j}}}_{\substack{\text{weight of}\\ \text{fluid region}}}\right]. \quad (2.58)$$

Since the weight has no x component, we need not worry about calculating y_{CM}. However, we do need the x component of the center of mass, which we can find via integration (refer to the previous figure),

$$x_{CM}G=\int_{x=0}^{x=L}x\rho g\underbrace{\left(H-y\right)Wdx}_{=dV}=\int_{x=0}^{x=L}x\rho g\left[H-H\left(\frac{x}{L}\right)^2\right]W\,dx=\rho gHW\int_{x=0}^{x=L}\left(x-\frac{x^3}{L^2}\right)dx\,, \quad (2.59)$$

$$x_{CM}\frac{2}{3}\rho gHWL=\rho gHW\left(\frac{1}{2}L^2-\frac{1}{4}\frac{L^4}{L^2}\right)=\frac{1}{4}\rho gHWL^2\,, \quad (2.60)$$

$$x_{CM}=\frac{3}{8}L\,. \quad (2.61)$$

Substituting this value back into Eq. (2.58), making use of the resultant pressure force, and simplifying,

$$\left(x_{CP}\hat{\mathbf{i}}+y_{CP}\hat{\mathbf{j}}\right)\times\left(\frac{1}{2}\rho gH^2W\hat{\mathbf{i}}-\frac{2}{3}\rho gHWL\hat{\mathbf{j}}\right)=\left(\frac{1}{3}H\hat{\mathbf{j}}\times\frac{1}{2}\rho gH^2W\hat{\mathbf{i}}\right)+\left(\frac{3}{8}L\hat{\mathbf{i}}\times-\frac{2}{3}\rho gHWL\hat{\mathbf{j}}\right), \quad (2.62)$$

$$-x_{CP}\frac{2}{3}\rho gHWL\hat{\mathbf{k}}-y_{CP}\frac{1}{2}\rho gH^2W\hat{\mathbf{k}}=-\frac{1}{6}\rho gH^3W\hat{\mathbf{k}}-\frac{1}{4}\rho gHWL^2\hat{\mathbf{k}}\,, \quad (2.63)$$

$$y_{CP}=\left(-\frac{4}{3}\frac{L}{H}\right)x_{CP}+\left(\frac{1}{3}H+\frac{1}{2}\frac{L^2}{H}\right), \quad (2.64)$$

which is the same result found previously.

2. Either approach to finding the resultant force and center of pressure (integration or balancing forces on a wisely chosen region of fluid) is fine. One method may be easier than the other, depending on the geometry of the problem.
3. Yet another method to finding the resultant pressure force and center of pressure relies on calculating the center of area of the wall surface and calculating moments of inertia. This approach isn't described in these notes since it's a more "formulaic" approach to the solving the problem and is less connected to the actual physics of the problem. Moreover, this moment-of-inertia approach often requires access to moment of inertia tables, which may be inconvenient. A number of texts that discuss fluid statics present this "moments-of-inertia" approach.