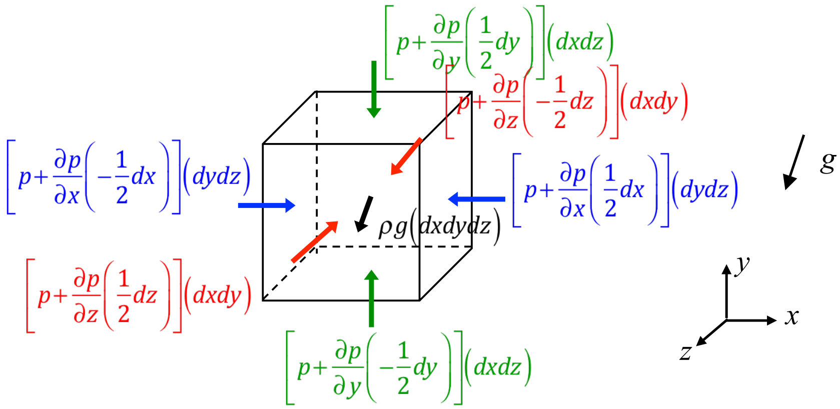
Pressure Distribution in a Fluid with No Shear Stresses
Recall from Chapter 01 that for a flow in which there are no velocity gradients, e.g., $du/dy = 0$, such as a static fluid, the shear stresses are zero. Draw a free body diagram of a (differentially) small piece of fluid, with width, height, and depth of dx, dy, and dz, respectively, under static conditions. Note that only pressure forces and weight will act on the fluid element. If we assume that the pressure, density, and gravitational acceleration at the center of the element are p, ρ, and g, respectively, then the free body diagram looks as follows (making use of the Taylor Series approximation discussed in Chapter 01).



Summing forces in each of the three directions and noting that the fluid element is static,

$$\sum F_x = 0 = \left[p+\frac{\partial p}{\partial x}\left(-\frac{1}{2}dx\right)\right](dydz) - \left[p+\frac{\partial p}{\partial x}\left(\frac{1}{2}dx\right)\right](dydz) + \underbrace{\rho dxdydz}_{=dm} g_x\,, \tag{2.1}$$

$$0 = -\frac{\partial p}{\partial x}dxdydz + \rho dxdydz g_x\,, \tag{2.2}$$

$$0 = -\frac{\partial p}{\partial x} + \rho g_x\,. \tag{2.3}$$

Similar approaches in the y and z directions give,

$$0 = -\frac{\partial p}{\partial y} + \rho g_y\,, \tag{2.4}$$

$$0 = -\frac{\partial p}{\partial z} + \rho g_z\,. \tag{2.5}$$

The last three equations may be written more compactly in vector form as,

$$\boxed{\nabla p = \rho \mathbf{g}\,. \quad \text{Force balance for a static fluid particle.}} \tag{2.6}$$

What this equation tells us is that for a static piece of fluid, a difference in pressure is required to balance the weight of the fluid particle.

Now consider the case where the gravitational acceleration points in the positive y direction, $\mathbf{g} = g\hat{\mathbf{j}}$, so that the y-component of Eq. (2.6) is,

$$\frac{dp}{dy} = \rho g\,. \qquad y\downarrow \;\; \downarrow g \tag{2.7}$$

Note that the x and z components indicate that there is no change in pressure in those directions since there is no component of weight to balance, i.e., the pressure only changes in the y direction (hence the use of an ordinary derivative in Eq. (2.7) as opposed to a partial derivative since $p = p(y)$).

To determine how the pressure varies in the y direction, we must solve the differential equation given by Eq. (2.7),

$$dp = \rho g dy \quad => \quad \int_{p=p_{y=0}}^{p=p} dp = \int_{y=0}^{y=y} \rho g dy \quad => \quad p - p_{y=0} = \int_{y=0}^{y=y} \rho g dy \ . \tag{2.8}$$

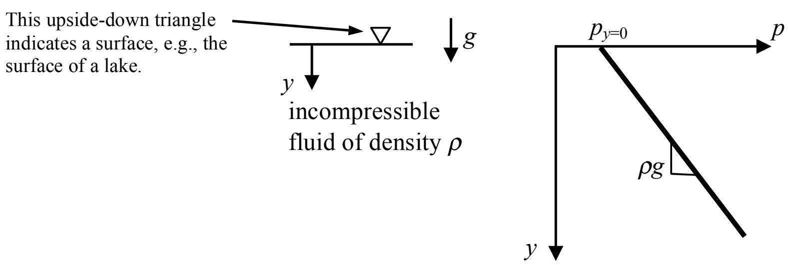
In order to solve the integral on the right hand side of the previous equation, we must know how the density and gravitational acceleration vary with y. It's reasonable in most applications to assume that the gravitational acceleration is constant, so it can be pulled outside the integral. If we further assume that we're dealing with an incompressible fluid, then the density can also be pulled outside the integral and we're left with,

$$p - p_{y=0} = \rho g \int_{y=0}^{y=y} dy = \rho g y \ , \tag{2.9}$$

$$\boxed{p = p_{y=0} + \rho g y} \text{ or, alternately, } \Delta p = \rho g \Delta y \ . \tag{2.10}$$

The previous equation is the **hydrostatic pressure variation in an incompressible fluid in which gravity points in the positive *y* direction**.

Notes:
1. The pressure in Eq. (2.10) only changes when there are variations in elevation in the direction of gravity (the y direction). Moving perpendicular to the direction of gravity does not change the pressure.
2. The pressure increases linearly in the direction of the gravitational acceleration. A plot of this variation is shown in the following figure.



3. As mentioned previously, the reason the pressure increases with depth is because the pressure must balance the weight of all the fluid sitting above it.
4. Since changes in pressure correspond to changes in elevation (refer to Eq. (2.10)), pressure differences are often expressed in terms of lengths, or depths of fluid. For example, the standard atmospheric pressure of 101 kPa (abs) corresponds to 760 mm of mercury,

$$\underbrace{101 \text{ kPa}}_{=\Delta p} = \underbrace{\left(13600 \ \tfrac{\text{kg}}{\text{m}^3}\right)}_{=\rho_{\text{Hg}}} \underbrace{\left(9.81 \ \tfrac{\text{m}}{\text{s}^2}\right)}_{=g} \underbrace{\left(760 * 10^{-3} \text{ m}\right)}_{=\Delta y} . \tag{2.11}$$

5. When measuring pressure differences in Eq. (2.10), either both pressures must be absolute pressures or both must be gage. Do not mix gage and absolute pressures.

Now consider the **pressure variation in a compressible fluid.** This case would be of particular interest for airplanes, rockets, and mountain climbers where large changes of elevations in the atmosphere are common. Recall from Eq. (2.7),

$$\frac{dp}{dy} = \rho g \text{ . } (y \text{ and } g \text{ point in the same direction}) \tag{2.12}$$

If we're dealing with an ideal gas (like air), then the pressure and density are related via,

$$p = \rho RT \Rightarrow \rho = \frac{p}{RT}, \tag{2.13}$$

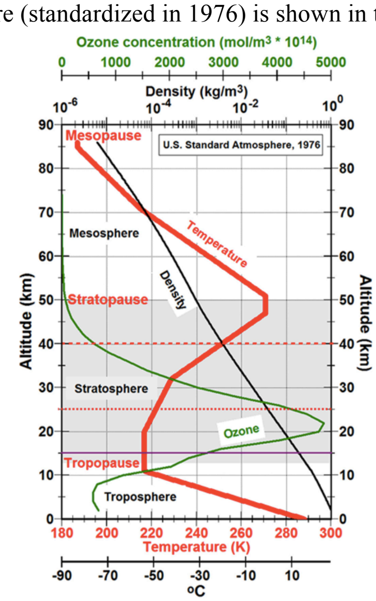
where R is the gas constant for air. Substituting Eq. (2.13) into Eq. (2.12) and re-arranging gives,

$$\frac{dp}{p} = \frac{g}{RT}dy \text{ . } \quad y\downarrow \quad g\downarrow \tag{2.14}$$

In order to be more convenient for use in the atmosphere where altitude (y) points in the direction opposite to the gravitational acceleration, we can change the sign on y in Eq. (2.14) to give,

$$\frac{dp}{p} = -\frac{g}{RT}dy \text{ . } \quad y\uparrow \quad g\downarrow \tag{2.15}$$

Numerous measurements have been made of the average atmospheric temperature as a function of altitude, i.e., $T = T(y)$, and can be substituted into Eq. (2.15). For example, the temperature variation with altitude from the U.S. Standard Atmosphere (standardized in 1976) is shown in the following figure.



Source: http://ozonedepletiontheory.info/Images/atmosphere-temp-ozone-density.jpg

In each region of the atmosphere, the temperature varies linearly and can be expressed as,

$$T = T_a - \beta y, \tag{2.16}$$

where T_a and β are constants and y is the altitude measured from sea level. For example, in the troposphere (the part of the atmosphere closest to the ground), T_a = 288 K (= 15 °C = 59 degF) and β = 6.50 K/km (≈ 4 °F/1000 ft). If we substitute Eq. (2.16) into Eq. (2.15) and solve the differential equation, we get,

$$\frac{dp}{p} = -\frac{g}{R(T_a - \beta y)}dy \Rightarrow \int_{p=p_a}^{p} \frac{dp}{p} = -\frac{g}{R}\int_{y=0}^{y=y} \frac{dy}{T_a - \beta y} \Rightarrow \ln\left(\frac{p}{p_a}\right) = \frac{g}{\beta R}\ln\left(\frac{T_a - \beta y}{T_a}\right), \tag{2.17}$$

$$\boxed{p = p_a\left(1 - \frac{\beta y}{T_a}\right)^{\frac{g}{\beta R}}} \text{. } \textbf{(Pressure variation in the U.S. Standard Atmosphere.)} \tag{2.18}$$

Notes:

1. For an **isothermal change** (T = constant) in elevation in an ideal gas, Eq. (2.15) becomes,

$$\frac{dp}{p}=-\frac{g}{RT}dy \Rightarrow \int_{p=p_{y=0}}^{p}\frac{dp}{p}=-\frac{g}{RT}\int_{y=0}^{y=y}dy \Rightarrow \ln\left(\frac{p}{p_{y=0}}\right)=-\frac{g}{RT}y \Rightarrow p=p_{y=0}\exp\left(-\frac{g}{RT}y\right). \qquad (2.19)$$

This type of variation would be more applicable for determining pressure changes in a controlled environment, such as inside a tall building.

2. Large changes in elevation are required to make appreciable changes in pressure in a gas, such as air. For example, estimate the altitude change required to drop the pressure by 1% in the U.S. Standard atmosphere in the troposphere,

$$\frac{p}{p_a}=0.99=\left(1-\frac{\beta y}{T_a}\right)^{\frac{g}{\beta R}} \Rightarrow (0.99)^{\frac{\beta R}{g}}=1-\frac{\beta y}{T_a} \Rightarrow y=\frac{T_a}{\beta}\left[1-(0.99)^{\frac{\beta R}{g}}\right], \qquad (2.20)$$

which gives y = 85 m. An isothermal variation gives approximately the same value. Thus**, it's reasonable to assume that unless large elevation differences occur, the pressure does not vary with elevation in the atmosphere (or any gas, for that matter).** However, even small elevation differences in liquids do result in appreciable pressure changes (and can be found using Eq. (2.10)).