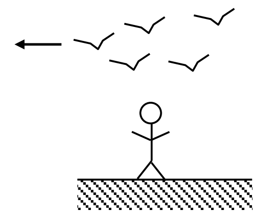
1. Lagrangian and Eulerian Perspectives

There are two common ways to study a moving fluid:

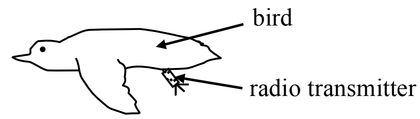
1. Look at a particular location and observe how all the fluid passing that location behaves. This is called the **Eulerian** point of view.
2. Look at a particular piece of fluid and observe how it behaves as it moves from location to location. This is called the **Lagrangian** point of view.

Example:



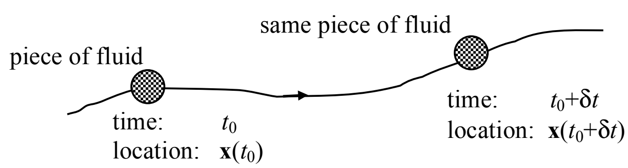
Let's say we want to study migrating birds. We could either:

1. stand in a fixed spot and make measurements as birds fly by (Eulerian point of view), or
2. tag some of the birds and make measurements as they fly along (Lagrangian point of view).



Lagrangian (aka Material, Particle, Substantial) Derivative
(Go to *https://engineering.purdue.edu/~wassgren/teaching* for an interactive Java applet on this topic.)

If we follow a piece of fluid (Lagrangian viewpoint), how will some property of that particular piece of fluid change with respect to time?



Let's say we're interested in looking at the time rate of change of temperature, T, that the particle observes as it moves from location to location. The particle may experience a temperature change because the temperature of the entire field of fluid may be changing with respect to time (i.e., the temperature field may be unsteady). In addition, the temperature field may have spatial gradients (different temperatures at different locations, i.e., non-uniform) so that as the particle moves from point to point it will experience a change in temperature. Thus, there are two effects that can cause a time rate of change of temperature that the particle experiences: unsteady effects, also known as **local or Eulerian** effects, and spatial gradient effects, also known as **convective** effects. We can describe this in mathematical terms by writing the temperature of the entire field as a function of time, t, and location, $\mathbf{x}$,

$$T = T(t, \mathbf{x}) \tag{1}$$

Note that the location of the fluid particle is a function of time: $\mathbf{x}=\mathbf{x}(t)$ so that,

$$T = T\big(t, \mathbf{x}(t)\big) \tag{2}$$

Taking the time derivative of the temperature, expanding the location vector into its x, y, and z components, and using the chain rule gives,

$$\left.\frac{dT}{dt}\right|_{\substack{\text{following a}\\ \text{fluid particle}}} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\underbrace{\frac{dx}{dt}}_{=u_x} + \frac{\partial T}{\partial y}\underbrace{\frac{dy}{dt}}_{=u_y} + \frac{\partial T}{\partial z}\underbrace{\frac{dz}{dt}}_{=u_z} \tag{3}$$

Note that dx/dt, dy/dt, and dz/dt are the particle velocities u_x, u_y, and u_z respectively. Writing this in a more compact form,

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u_x \frac{\partial T}{\partial x} + u_y \frac{\partial T}{\partial y} + u_z \frac{\partial T}{\partial z} = \frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T \tag{4}$$

The notation, D/Dt, indicating a **Lagrangian** (also sometimes referred to as the **material**, **particle**, or **substantial**) **derivative,** has been used in Eq. (4) to indicate that we're following a particular piece of fluid. More generally, we have,

$$\boxed{\begin{aligned} \underbrace{\frac{D}{Dt}(\cdots)}_{\substack{\text{Lagrangian rate of}\\ \text{change (changes as we}\\ \text{follow a fluid particle)}}} &= \underbrace{\frac{\partial}{\partial t}(\cdots)}_{\substack{\text{local or Eulerian}\\ \text{rate of change (changes}\\ \text{due to unsteady effects)}}} + \underbrace{(\mathbf{u}\cdot\nabla)(\cdots)}_{\substack{\text{convective rate}\\ \text{of change (changes due to}\\ \text{a change in particle position)}}} \\ &= \frac{\partial}{\partial t}(\cdots) + u_x \frac{\partial}{\partial x}(\cdots) + u_y \frac{\partial}{\partial y}(\cdots) + u_z \frac{\partial}{\partial z}(\cdots) \end{aligned}} \tag{5}$$

where $(\cdots)$ represents any field quantity of interest.

Notes:

1. The Lagrangian derivatives in cylindrical and spherical coordinates are,

cylindrical: $$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z} \tag{6}$$

spherical: $$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{7}$$

2. The acceleration experienced by a fluid particle is given by,

Cartesian: $$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \frac{\partial \mathbf{u}}{\partial t} + u_x \frac{\partial \mathbf{u}}{\partial x} + u_y \frac{\partial \mathbf{u}}{\partial y} + u_z \frac{\partial \mathbf{u}}{\partial z} \tag{8}$$

cylindrical: $$\begin{aligned} a_r &= \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} \\ a_\theta &= \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} \\ a_z &= \frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} \end{aligned} \tag{9}$$

spherical: $$\begin{aligned} a_r &= \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{1}{r}\left(u_\theta^2 + u_\phi^2\right) \\ a_\theta &= \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{1}{r}\left(u_r u_\theta - u_\phi^2 \cot\theta\right) \\ a_\phi &= \frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{1}{r}\left(u_r u_\phi + u_\theta u_\phi \cot\theta\right) \end{aligned} \tag{10}$$