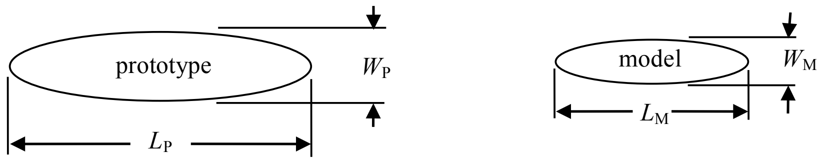
5. Modeling and Similarity

Models are often used in fluid mechanics to predict the kinematics and dynamics of full-scale (often referred to as prototype) flows. From previous discussions of dimensional analysis, we observe that we can write the governing equations and boundary conditions of our flow in dimensionless terms (Π terms). Thus, if we have two different flows (e.g., a large-scale, prototype flow and a small scale, model flow) that have identical dimensionless parameters, then the same solution, also in terms of dimensionless parameters, will hold for both. This is extremely helpful when modeling fluid systems.

When a model and the prototype have the same dimensionless parameters, we say that they are similar. We typically discuss similarity in three categories: geometric, dynamic, and kinematic.

Geometric similarity occurs when the model is an exact geometric replica of the prototype. In other words, all of the lengths in the model are scaled by exactly the same amount as in the prototype.

For example:



$L_P/L_M = W_P/W_M$

Note that surface roughness may even need to be scaled if it is a significant factor in the flow.

Dynamic similarity occurs when the ratio of forces in the model is the same as the ratio of forces in the prototype, i.e.,

(ratio of inertial to viscous forces)$_P$ = (ratio of inertial to viscous forces)$_M$ [$Re_P = Re_M$]
(ratio of pressure to inertial forces)$_P$ = (ratio of pressure to inertial forces)$_M$ [$Eu_P = Eu_M$]
(ratio of inertial to grav. forces)$_P$ = (ratio of inertial to grav. forces)$_M$ [$Fr_P = Fr_M$]
etc.

Kinematic similarity occurs when the prototype and model fluid velocity fields have identical streamlines (but scaled velocities). Since the forces affect the fluid motion, geometric similarity and dynamic similarity will automatically ensure kinematic similarity.

Note:

1. When modeling, we need to maintain similarity between all of the dimensionless parameters *that are important to the physics of the flow*. This means that we do not necessarily need to have similarity between all Π terms, just the ones that significantly affect the flow physics. Knowing a priori what dimensionless terms are important can be difficult but with experience the task is often easier.

2. It is not uncommon to have the important physics of a system change at different scales. For example, surface tension forces become more pronounced at smaller geometric scales. If one was scaling up a small system in which surface tension was an important effect, but didn't consider the dynamic similarity of the surface tension force at the larger scale, then the scaling experiments would result in incorrect results.

Partial Similarity

True similarity may be difficult to achieve in practice. In such cases, one must either: (a) acknowledge that model testing may not be possible, or (b) relax one or more similarity requirements and use a combination of experimentation and analysis to scale the measurements.

For example, in modeling the flow around ships both Reynolds number and Froude number similarity are important; however, both are difficult to achieve simultaneously. In such cases, one of the similarity requirements is relaxed (in boat modeling it's the Reynolds number similarity requirement) and a combination of experiments and analysis is utilized to scale the measurements.

The two primary components of drag on a ship's hull are viscous drag (i.e., the friction of the water against the hull's surface) and wave drag (i.e., the force required to create the waves generated by the hull). The two significant dimensionless parameters corresponding to these phenomena are:

Reynolds number: $\text{Re} = \dfrac{VL}{\nu}$ (ratio of inertial to viscous forces)

Froude number: $\text{Fr} = \dfrac{V}{\sqrt{gL}}$ (ratio of inertial to gravitational forces)

Maintaining both Reynolds number and Froude number similarity is difficult to achieve in practice.

$$\text{Fr}_M = \text{Fr}_P \Rightarrow \left(\frac{V}{\sqrt{gL}}\right)_M = \left(\frac{V}{\sqrt{gL}}\right)_P \Rightarrow V_M = V_P\sqrt{\frac{L_M}{L_P}}\sqrt{\frac{g_M}{g_P}} \overset{g_M = g_P}{\Rightarrow} V_M = V_P\sqrt{\frac{L_M}{L_P}} \qquad (10)$$

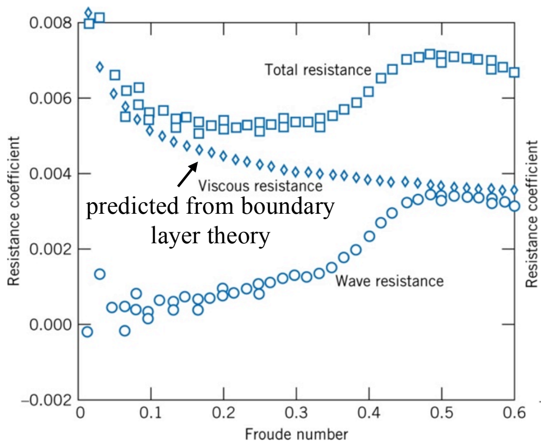
$$\text{Re}_M = \text{Re}_P \Rightarrow \left(\frac{VL}{\nu}\right)_M = \left(\frac{VL}{\nu}\right)_P \Rightarrow \nu_M = \nu_P \frac{V_M}{V_P}\frac{L_M}{L_P} \overset{\text{using previous result}}{\Rightarrow} \nu_M = \nu_P\left(\frac{L_M}{L_P}\right)^{3/2} \qquad (11)$$

As an example, consider a scale model which has $L_P = 100L_M$, $\nu_P = \nu_{H2O} = 1$ cSt $\Rightarrow$ $\nu_M = 0.001$ cSt. There is no such common model fluid available! Thus, we cannot easily maintain both Froude number and Reynolds number similarity.

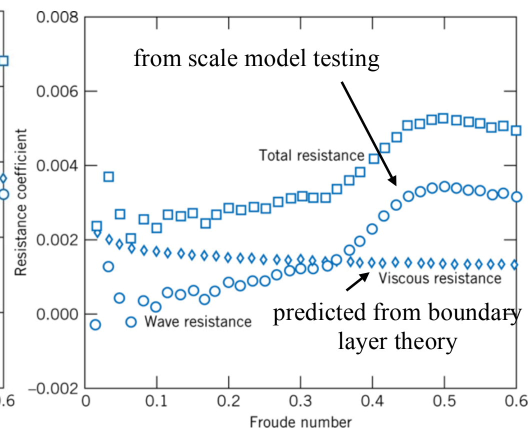
How do we resolve this difficulty? In practice, Froude number similarity is maintained with water as both the prototype and model fluid (i.e., Eq. (10) holds). Reynolds number similarity is neglected in the experiment and instead analysis or computation is used to estimate the viscous drag contribution. The procedure is as follows:

1. The total drag acting on the model is measured in the experiment. This drag force is usually expressed in terms of a dimensionless resistance coefficient.
2. The viscous drag contribution to the total drag is calculated using analysis (e.g., boundary layer analysis) or computation (e.g., computational fluid dynamics).
3. The difference between the total drag and the viscous drag is the wave drag.
4. The wave drag is a function of the Froude number, which is held constant during scaling. So, the experimental wave drag data can be scaled between the model and prototype.
5. Estimate the viscous drag contribution for the prototype using analysis or computation.
6. Sum the predicted viscous drag force (step 5) with the scaled wave drag force (step 4) to get the total prototype drag force.

As a demonstration of how well the procedure works, consider the resistance coefficient data from a 1:80 scale model test of the U.S. Navy guided missile frigate *Oliver Hazard Perry* (FFG-7) as shown in the plots below. (Note that these plots are from Figs. 7.2 and 7.3 in Fox, R.W., Pritchard, P.J., and McDonald, A.T., 2008, *Introduction to Fluid Mechanics*, 7th ed., Wiley.) The error between the scaled and actual total drag force measurements is approximately ±5%.



Scale Model



Prototype

Experimental observations have shown that in many (but not all!) cases, Reynolds number similarity may be neglected for sufficiently large Reynolds numbers.

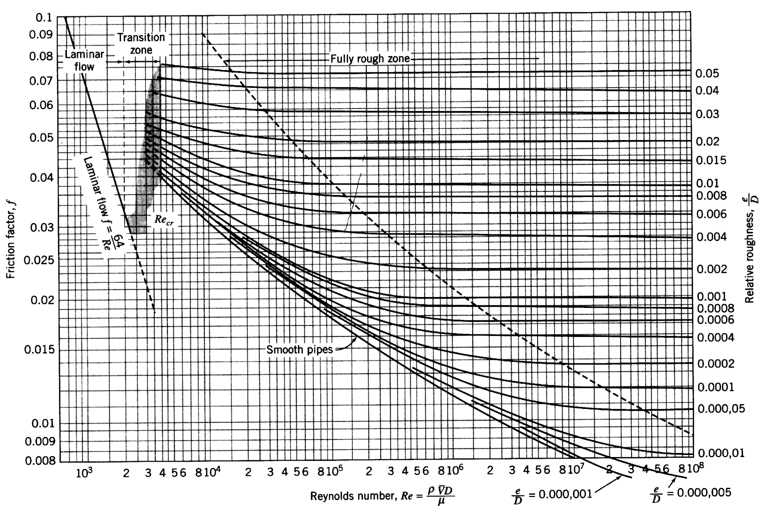
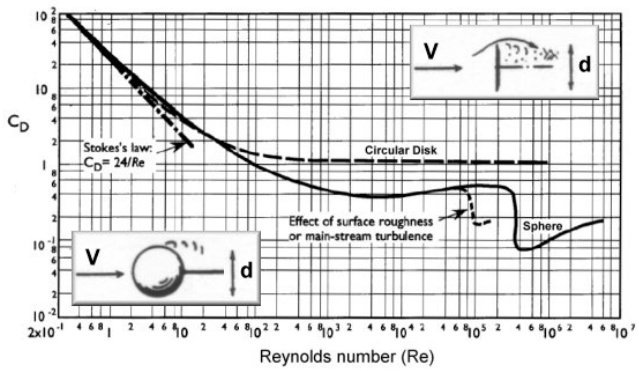


Fig. 8.13 Friction factor for fully developed flow in circular pipes. (Data from [8], used by permission.)

(Figure from Fox, R.W., Pritchard, P.J., and McDonald, A.T., 2008, *Introduction to Fluid Mechanics*, 7th ed., Wiley.)

Moody diagram – Friction factor as a function of Reynolds number and pipe relative roughness. Note that the friction factor becomes independent of Reynolds number for sufficiently large Reynolds numbers.



(Figure from Fox, R.W., Pritchard, P.J., and McDonald, A.T., 2008, *Introduction to Fluid Mechanics*, 7th ed., Wiley.)

Drag coefficients for a sphere and circular disk as a function of Reynolds number. Note that the drag coefficient is insensitive to Reynolds number over a wide range of Reynolds numbers.