2. Buckingham Pi Theorem

The key component to dimensional analysis is the:

Buckingham Pi Theorem: If an equation involving *k* variables is dimensionally homogeneous, it can be reduced to a relationship among *k*-*r* independent dimensionless products (referred to as Π terms), where *r* is the minimum number of reference dimensions required to describe the variables, i.e.,

$$\left(\#\text{ of }\Pi\text{ terms}\right) = \underbrace{\left(\#\text{ of variables}\right)}_{=k} - \underbrace{\left(\#\text{ of reference dimensions}\right)}_{=r}$$

(The proof to this theorem will not be presented here.)

Notes:

1. Dimensionally homogeneous means that each term in the equation has the same units. For example, the following form of Bernoulli's equation:
$$\frac{p}{\rho g} + \frac{V^2}{2g} + z = \text{constant}$$
is dimensionally homogeneous since each term has units of length (L).
2. A dimensionless product, also commonly referred to as a Pi (Π) term, is a term that has no dimensions. For example,
$$\frac{p}{\rho V^2}$$
is a dimensionless product since both the numerator and denominator have the same dimensions.
3. Reference dimensions are *usually* basic dimensions such as mass (M), length (L), and time (T) or force (F), length (L), and time (T). We'll discuss the "usually" modifier a little later when discussing the method of repeating variables.

3. Method of Repeating Variables

The Buckingham Pi Theorem merely states that a relationship among dimensional variables may be written, perhaps in a more compact form, in terms of dimensionless variables (Π terms). The Pi Theorem does not, however, tell us what these dimensionless variables are. The method of repeating variables is an algorithm that can be used to determine these dimensionless variables.

The method of repeating variables algorithm is as follows:

1. *List all variables involved in the problem.*
   a. This is the most difficult step since it requires experience and insight.
   b. Variables are things like pressure, velocity, gravitational acceleration, viscosity, etc.
   c. List only independent variables. For example, you can list:
      ρ (density) and g (gravitational acceleration), or
      ρ and γ (specific weight), or
      g and γ
      but you should not list ρ, g, and γ since one of the variables is dependent on the others.
   d. If you include variables that are unimportant to the system, then you'll form Π terms that won't have an impact in practice. This situation is the same one you'd have if dimensional variables were used.
   e. If you leave out an important variable, then you'll find that in practice your relationship between dimensionless terms can't fully describe the system behavior. Again, this situation is the same one you'd have if you used dimensional terms.

2. *Express each variable in terms of basic dimensions.*
   a. For fluid mechanics problems we typically will use mass (M), length (L), and time (T) or force (F), length (L), and time (T) as basic dimensions. We may occasionally need other basic dimensions such as temperature (Θ).
   b. For example, the dimensions of density can be written as:
      $$[\rho] = \frac{M}{L^3} = \frac{FT^2}{L^4}$$ (*Note*: The square brackets are used to indicate "dimensions of.")

3. *Determine the number of Π terms using the Buckingham-Pi Theorem.*
   a. (# of Π terms) = (# of variables) – (# of reference dimensions)
   b. Usually the # of reference dimensions will be the same as the # of basic dimensions found in step 2. There are (rare) cases where some of the basic dimensions always appear in particular combinations so that the # of reference dimensions is less than the # of basic dimensions. For example, say that the variables in the problem are A, B, and C, and their corresponding basic dimensions, are:
      $$[A] = \frac{M}{L^3} \quad [B] = \frac{M}{L^3T^2} \quad [C] = \frac{MT}{L^3}$$
      The basic dimensions are M, L, and T (there are 3 basic dimensions). Notice, however, that the dimensions M and L always appear in the combination M/L^3. Thus, we really only need two reference dimensions, M/L^3 and T, to describe all of the variables' dimensions.

4. *Select repeating variables where the number of repeating variables is equal to the number of reference dimensions.*
   a. The repeating variables should come from the list of variables. In our previous example, the list of variables is $\Delta p/L$, V, D, ρ, and μ.
   b. Each repeating variable must have units independent of the other repeating variables.
   c. Don't make the dependent variable one of the repeating variables. In our previous example, $\Delta p/L$ is the dependent variable.
   d. All of the reference dimensions must be included in the group of repeating variables.

5. *Form a* Π *term by multiplying one of the non-repeating variables by the product of the repeating variables, each raised to an exponent that will make the combination dimensionless*.
   a. This step is most clearly illustrated in an example and will not be discussed here.
   b. Repeat this step for all non-repeating variables.

6. *Check that all* Π *terms are dimensionless.*
   a. This is an important, and often overlooked, step to verify that your Π terms are, in fact, dimensionless.

7. *Express the final form of the dimensional analysis as a relationship among the* Π *terms.*
   a. For example, $\Pi_1 = \text{fcn}(\Pi_2, \Pi_3, \ldots, \Pi_{k-r})$.

Example:

Let's use our pipe flow experiment as an example.

Step 1:

The variables that are important in this problem are:

$\Delta p/L \equiv$ average pressure gradient over length L

$V \equiv$ average flow velocity

$D \equiv$ pipe diameter

$\rho \equiv$ fluid density

$\mu \equiv$ fluid dynamic viscosity

$\Delta p/L = \text{fcn}_1(V, D, \rho, \mu)$

Step 2:

The basic dimensions of each variable are:

$$\left[\Delta p/L\right] = \frac{F}{L^3} = \frac{M}{L^2T^2}$$

$$[V] = \frac{L}{T}$$

$$[D] = L$$

$$[\rho] = \frac{M}{L^3}$$

$$[\mu] = \frac{FT}{L^2} = \frac{M}{LT}$$

Step 3:

(# of variables) = 5 ($\Delta p/L$, V, D, ρ, μ)

(# of reference dimensions) = 3 (F, L, T or M, L, T)

(# of Π terms) = (# of variables) – (# of reference dimensions) = 2

Thus, instead of having a relation involving 5 terms, we actually have a relationship involving just 2 terms!

Step 4:

Select the following 3 repeating variables (We require three since the # of reference dimensions is three.):

ρ, V, D

Notes:

1. These three repeating variables have independent dimensions.
2. The dependent variable ($\Delta p/L$) is not one of the repeating variables.
3. We could have also selected (ρ, μ, V) or (ρ, μ, D) or (D, μ, V) as repeating variables. The choice of repeating variables is somewhat arbitrary (as long as they have independent reference dimensions and do not include the dependent variable).

Step 5:

Form Π terms from the remaining, non-repeating variables:

$$\Pi_1 = \left(\Delta p/L\right)\rho^a V^b D^c$$

$$\Rightarrow [MLT]^0 = \left[\frac{M}{L^2T^2}\right]\left[\frac{M}{L^3}\right]^a\left[\frac{L}{T}\right]^b\left[\frac{L}{1}\right]^c$$

$$M^0 = M^1M^a \qquad 0 = 1 + a \qquad a = -1$$

$$\Rightarrow L^0 = L^{-2}L^{-3a}L^bL^c \Rightarrow 0 = -2 - 3a + b + c \Rightarrow b = -2$$

$$T^0 = T^{-2}T^{-b} \qquad 0 = -2 - b \qquad c = 1$$

$$\therefore \Pi_1 = \frac{\left(\Delta p/L\right)D}{\rho V^2}$$

$$\Pi_2 = \mu\rho^a V^b D^c$$

$$\Rightarrow [MLT]^0 = \left[\frac{M}{LT}\right]\left[\frac{M}{L^3}\right]^a\left[\frac{L}{T}\right]^b\left[\frac{L}{1}\right]^c$$

$$M^0 = M^1M^a \qquad 0 = 1 + a \qquad a = -1$$

$$\Rightarrow L^0 = L^{-1}L^{-3a}L^bL^c \Rightarrow 0 = -1 - 3a + b + c \Rightarrow b = -1$$

$$T^0 = T^{-1}T^{-b} \qquad 0 = -1 - b \qquad c = -1$$

$$\therefore \Pi_2 = \frac{\mu}{\rho VD}$$

Step 6:

Verify that each Π term is, in fact, dimensionless.

$$[\Pi_1] = \left[\frac{\left(\Delta p/L\right)D}{\rho V^2}\right] = \frac{M}{L^2T^2}\frac{L}{1}\frac{L^3}{M}\frac{T^2}{L^2} = M^0L^0T^0 \qquad \text{OK!}$$

$$[\Pi_2] = \left[\frac{\mu}{\rho VD}\right] = \frac{M}{LT}\frac{L^3}{M}\frac{T}{L}\frac{1}{L} = M^0L^0T^0 \qquad \text{OK!}$$

Step 7:

Re-write the original relationship in dimensionless terms.

$$\boxed{\frac{\left(\Delta p/L\right)D}{\rho V^2} = fcn_2\left(\frac{\mu}{\rho VD}\right)}$$

Notes:

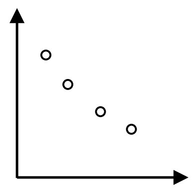
1. Instead of having to run four different sets of experiments as was discussed at the beginning of this chapter, we only really need to run one set of experiments where we vary:

   $$\frac{\mu}{\rho V D}$$

   and measure:

   $$\frac{\left(\Delta p / L\right) D}{\rho V^2}$$

   $$\frac{\left(\Delta p / L\right) D}{\rho V^2}$$

   $$\frac{\rho V D}{\mu}$$

   Note: We can use 1/Π in place of Π since we haven't actually specified what the function, fcn_2, looks like.

   This is a Reynolds number!

   All of the information contained in the previous four plots is contained within this single plot! This reduces the complexity, cost, and time required to determine the relationship between the average pressure gradient and the other variables.

2. Dimensional analysis is a very powerful tool because it tells us what terms really are important in an equation. For example, we started with the relation:

   $$\Delta p / L = fcn_1\left(V, D, \rho, \mu\right)$$

   leading us to believe that V, D, ρ, and μ are all important terms by themselves. However, dimensional analysis shows us that instead of the terms by themselves, it is the following <u>grouping</u> of terms:

   $$\frac{\left(\Delta p / L\right) D}{\rho V^2} = fcn_2\left(\frac{\mu}{\rho V D}\right)$$

   that is important in the relationship. This is a subtle but very important point.

3. Dimensional analysis tells us how many dimensionless terms are important in a relation. It does not tell us what the functional relationship is.

4. The dimensionless Π terms found via dimensionless analysis are not necessarily unique. Had we chosen different repeating variables in the previous example, we would have ended up with different Π terms. One can multiply, divide, or raise their set of Π terms to form the Π terms found by another. The *number* of Π terms, however, is unique.

5. After a bit of practice, one can quickly form Π terms by inspection rather than having to go through the method of repeating variables.