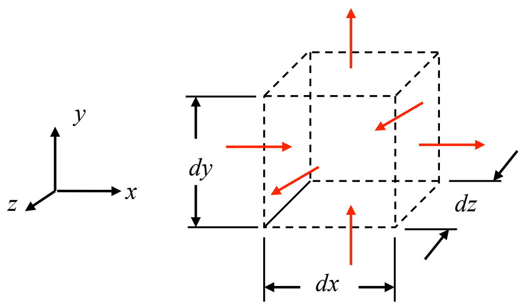
2. Continuity Equation (aka COM for a differential CV)

The continuity equation, which is simply conservation of mass for a differential fluid element or control volume, can be derived several different ways. Two of these methods are given below.

Method 1:

Apply the integral approach to a differential control volume:



Assume that the density and velocity are ρ and $\mathbf{u}$, respectively, at the control volume's center. The mass fluxes through each of the side of the control volume are given by:

$$\begin{aligned}\dot{m}_{\text{in through left}} &= \dot{m}_{x,\text{center}} + \frac{\partial \dot{m}_{x,\text{center}}}{\partial x}\left(-\tfrac{1}{2}dx\right)\\ &= \left(\rho u_x dydz\right) + \frac{\partial}{\partial x}\left(\rho u_x dydz\right)\left(-\tfrac{1}{2}dx\right)\\ &= \left[\rho u_x + \frac{\partial}{\partial x}\left(\rho u_x\right)\left(-\tfrac{1}{2}dx\right)\right]\left(dydz\right)\end{aligned}$$

(where $\dot{m}_{x,\text{center}}$ is the mass flux in the x-direction at the center of the control volume (Recall the Taylor Series approximation discussed in Chapter 01.))

$$\dot{m}_{\text{out through right}} = \left[\rho u_x + \frac{\partial}{\partial x}\left(\rho u_x\right)\left(\tfrac{1}{2}dx\right)\right]\left(dydz\right)$$

$$\dot{m}_{\text{in through bottom}} = \left[\rho u_y + \frac{\partial}{\partial y}\left(\rho u_y\right)\left(-\tfrac{1}{2}dy\right)\right]\left(dxdz\right)$$

$$\dot{m}_{\text{out through top}} = \left[\rho u_y + \frac{\partial}{\partial y}\left(\rho u_y\right)\left(\tfrac{1}{2}dy\right)\right]\left(dxdz\right)$$

$$\dot{m}_{\text{in through back}} = \left[\rho u_z + \frac{\partial}{\partial z}\left(\rho u_z\right)\left(-\tfrac{1}{2}dz\right)\right]\left(dxdy\right)$$

$$\dot{m}_{\text{out through front}} = \left[\rho u_z + \frac{\partial}{\partial z}\left(\rho u_z\right)\left(\tfrac{1}{2}dz\right)\right]\left(dxdy\right)$$

Thus, the net mass flow rate <u>into</u> the control volume is given by:

$$\dot{m}_{\text{net,into CV}} = -\left[\frac{\partial}{\partial x}\left(\rho u_x\right) + \frac{\partial}{\partial y}\left(\rho u_y\right) + \frac{\partial}{\partial z}\left(\rho u_z\right)\right]\left(dxdydz\right)$$

The rate at which mass is increasing within the control volume is given by:

$$\left.\frac{\partial m}{\partial t}\right|_{\text{within CV}} = \frac{\partial}{\partial t}\left(\rho dxdydz\right) \quad (4)$$

where ρ is the density at the center of the control volume. Note that since the density varies linearly within the CV (from the Taylor Series approximation), the average density in the CV is ρ.

Conservation of mass states that the rate of increase of mass within the control volume must equal the net rate at which mass enters the control volume:

$$\left.\frac{\partial m}{\partial t}\right|_{\text{within CV}} = \dot{m}_{\text{net,into CV}}$$

$$\frac{\partial \rho}{\partial t}(dxdydz) = -\left[\frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z)\right](dxdydz)$$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z) = 0} \qquad (5)$$

Written in a more compact form:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0} \qquad (6)$$

Or in index notation form:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0} \qquad (7)$$

Method 2:

Recall that the integral form of COM is given by:

$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A} = 0$$

Consider a fixed control volume so that:

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = \int_{\text{CV}} \frac{\partial \rho}{\partial t} dV \quad \text{and} \quad \mathbf{u}_{\text{rel}} = \mathbf{u}$$

By utilizing Gauss' Theorem (aka the Divergence Theorem), we can convert the area integral into a volume integral:

$$\int_{\text{CS}} \rho \mathbf{u} \cdot d\mathbf{A} = \int_{\text{CV}} \nabla \cdot (\rho \mathbf{u}) dV$$

Substitute these expressions back into COM to get:

$$\int_{\text{CV}} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u})\right] dV = 0$$

Since the choice of control volume is arbitrary, the kernel of the integral must be zero, i.e.:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \qquad \text{same result as before!}$$

Notes:

1. For a fluid in which the density remains uniform and constant (ρ = constant), the continuity equation simplifies to:

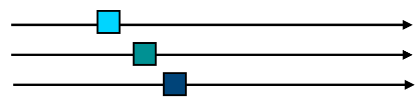
   $\boxed{\nabla \cdot \mathbf{u} = 0}$ **continuity equation for a constant density fluid**

2. An **incompressible fluid** is one in which the density <u>of a particular piece of fluid</u> remains constant, *i.e.*:

   $\boxed{\dfrac{D\rho}{Dt} = 0}$ **definition of an incompressible fluid**

   Note that an incompressible fluid does not necessarily imply that the density is the same everywhere in the flow (*i.e.* it's not uniform). An example of such a flow would be a stratified flow in the ocean where the density of various layers of ocean water varies due to salinity and temperature variations.

   The density of fluid particles varies from layer to layer but remains constant within a layer.

   A flow with a constant and uniform density fluid, however, is an incompressible flow.

   The continuity equation for an incompressible fluid can be found by using the definition of an incompressible flow:

   $$\frac{D\rho}{Dt} = 0 = \frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla)\rho \Rightarrow \frac{\partial \rho}{\partial t} = -(\mathbf{u} \cdot \nabla)\rho$$

   Substituting into the continuity equation:

   $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$
   $$-(\mathbf{u} \cdot \nabla)\rho + \nabla \cdot (\rho \mathbf{u}) = 0$$
   $$-(\mathbf{u} \cdot \nabla)\rho + \rho(\nabla \cdot \mathbf{u}) + (\mathbf{u} \cdot \nabla)\rho = 0$$

   $\boxed{\nabla \cdot \mathbf{u} = 0}$ **continuity equation for an incompressible flow** (8)

3. Another useful form of the continuity equation is as follows:

   $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 = \underbrace{\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i}}_{= D\rho/Dt} + \rho \frac{\partial u_i}{\partial x_i}$$

   $$\boxed{\frac{D\rho}{Dt} = -\rho \frac{\partial u_i}{\partial x_i}} \qquad (9)$$

4. The continuity equation (Eq. (7)) is valid for any continuous substance (e.g., a solid as well as a fluid).

5. Equation (7) is referred to as the <u>conservative</u> form of the continuity equation while Eq. (9) is the <u>non-conservative</u> form. The conservative form implies that the equation represents an Eulerian viewpoint of the continuity equation. The non-conservative form represents the Lagrangian viewpoint.