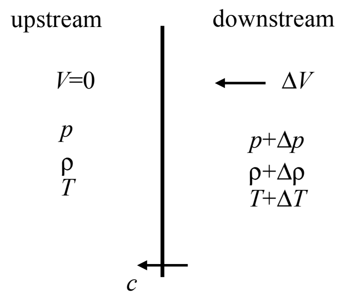
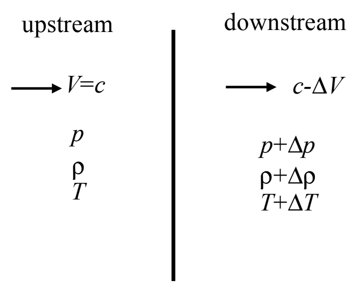
4. Speed of Sound

The speed of sound, c, in a substance is the speed at which infinitesimal pressure disturbances propagate through the surrounding substance. To understand how the speed of sound depends on the substance properties, let's examine the following model.

Consider a wave moving at velocity, c, through a stagnant fluid. Across the wave, the fluid properties such as pressure, p, density, ρ, temperature, T, and the velocity, V, can all change as shown in the figure below.

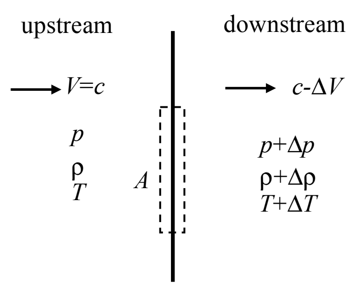


Now let's change our frame of reference such that it moves with the wave:



stationary wave

Now apply COM and the LME on a very thin CV surrounding the wave:



COM:

$$\frac{d}{dt}\int_{\text{CV}} \rho dV + \int_{\text{CS}} \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = 0$$

where

$$\frac{d}{dt}\int_{\text{CV}} \rho dV = 0$$

$$\int_{\text{CS}} \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -\rho c A + \left(\rho + \Delta\rho\right)\left(c - \Delta V\right) A$$

Substitute and simplify.

$$\rho c A = \left(\rho + \Delta\rho\right)\left(c - \Delta V\right) A$$

$$\rho c = \rho c - \rho \Delta V + \Delta\rho c - \Delta\rho \Delta V$$

$$\Delta V = \frac{c\Delta\rho}{\rho + \Delta\rho} \tag{13}$$

LME:

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{\substack{x,\text{ body}\\ \text{on CV}}} + F_{\substack{x,\text{ surface}\\ \text{on CV}}}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV = 0$$

$$\int_{\text{CS}} u_x \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = -\dot{m}c + \dot{m}\left(c - \Delta V\right) = -\dot{m}\Delta V$$

$$= -\rho c A \Delta V$$

$$F_{\substack{x,\text{ body}\\ \text{on CV}}} = 0$$

$$F_{\substack{x,\text{ surface}\\ \text{on CV}}} = pA - \left(p + \Delta p\right) A = -\Delta p A$$

Substituting and simplifying:

$$-\rho c A \Delta V = -\Delta p A$$

$$c = \frac{\Delta p}{\rho \Delta V}$$

Substituting the result from Eq. (13):

$$c = \frac{\Delta p\left(\rho + \Delta\rho\right)}{\rho c \Delta\rho}$$

$$\boxed{c^2 = \frac{\Delta p}{\Delta\rho}\left(1 + \frac{\Delta\rho}{\rho}\right)} \tag{14}$$

For a sound wave, the changes across the wave are infinitesimal so that:

$$c^2 = \lim_{\substack{\Delta p \to dp \\ \Delta\rho \to d\rho}} \frac{\Delta p}{\Delta \rho}\left(1+\frac{\Delta\rho}{\rho}\right) = \frac{\partial p}{\partial \rho} \tag{15}$$

We also need to specify the process by which these changes occur. Since the changes across the wave are infinitesimal, we can regard the wave as a reversible process. Additionally, the temperature gradient on either side of the wave is small so there is negligible heat transfer into the control volume so the process is adiabatic. Thus, the changes across a sound wave occur isentropically (an adiabatic, reversible process is also isentropic):

$$\boxed{c^2 = \left.\frac{\partial p}{\partial \rho}\right|_s} \qquad \textbf{speed of sound in a continuous substance} \tag{16}$$

Notes:

1. Note that Eq. (16) is the speed of sound in any continuous substance. It's not limited to fluids.

2. If the wave is not "weak" (i.e., the changes in the flow properties across the wave are not infinitesimal), then viscous effects and temperature gradients within the wave will be significant and the process can no longer be considered isentropic. We will discuss this situation later when examining shock waves.

3. Note that according to Eq. (15) the stronger the wave, i.e., the greater $\Delta\rho$, the faster the wave will propagate. This will also be examined when discussing shock waves.

4. For an ideal gas undergoing an isentropic process ($ds = 0$):

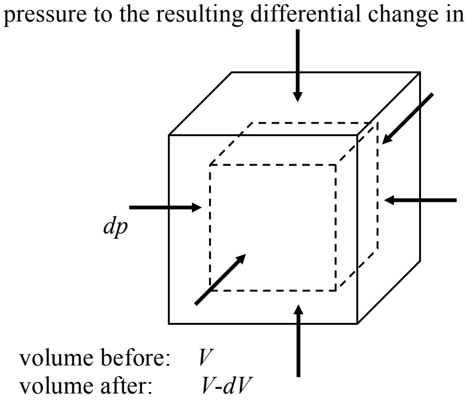
   $$\frac{R}{c_v}\frac{d\rho}{\rho} = \frac{R}{c_p}\frac{dp}{p} \quad \Rightarrow \quad \left.\frac{\partial p}{\partial \rho}\right|_s = \gamma\frac{p}{\rho} = \gamma RT \tag{17}$$

   Substituting into Eq. (16), the sound speed for an ideal gas is:

   $$\boxed{c = \sqrt{\gamma RT}} \qquad \textbf{speed of sound in an ideal gas} \tag{18}$$

   Notes:
   a. The absolute temperature must be used when calculating the speed of sound.
   b. The speed of sound in air (γ=1.4, R=287 J/(kg·K)) at standard conditions (T=288 K) is 340 m/s (=1115 ft/s ≈ 1/5 mile/s). This value helps explain the rule of thumb whereby the distance to a thunderstorm in miles is roughly equal to the number of seconds between a lightening flash and the corresponding thunder divided by five.
   c. It is not unexpected that the speed of sound is proportional to the square root of the temperature. Since disturbances travel through the gas as a result of molecular impacts, we should expect the speed of the disturbance to be proportional to the speed of the molecules. The temperature is equal to the random kinetic energy of the molecules and so the molecular speed is proportional to the square root of the temperature. Thus, the speed of sound is proportional to the square root of the temperature.

5. Equation (16) can also be written in terms of the **bulk modulus**. The bulk modulus, E_v, of a substance is a measure of the compressibility of the substance. It is defined as the ratio of a differential applied pressure to the resulting differential change in volume of a substance at a given volume:



$$E_v \equiv \frac{\partial p}{\left(-\partial V/V\right)} = \rho \frac{\partial p}{\partial \rho} \tag{19}$$

Notes:
1. $dp>0 \Rightarrow dV<0 \Rightarrow E_v>0$
2. From COM: $dV/V = -d\rho/\rho$
3. $E_v \uparrow \Rightarrow$ compressibility $\downarrow$

The isentropic bulk modulus, $E_v|_s$ is defined as:

$$E_v\Big|_s \equiv \frac{\partial p}{\left(-\partial V/V\right)} = \rho \frac{\partial p}{\partial \rho}\Big|_s \tag{20}$$

Thus, the speed of sound can also be written as:

$$\boxed{c^2 = \frac{E_v|_s}{\rho}} \qquad \textbf{alternate sound speed expression} \tag{21}$$

Notes:

a. The isentropic bulk modulus for air is $E_v|_s = \gamma\rho RT$.

b. The isentropic bulk modulus for water is 2.19 GPa so that the speed of sound in water (ρ=1000 kg/m^3) is 1480 m/s (=4900 ft/s ≈ 1 mile/s ≈ 5X faster than the speed of sound in air at sea level).

c. For solids, the bulk modulus, E_v, is related to the modulus of elasticity, E, and Poisson's ratio, μ, by:

$$\frac{E_v}{E} = 3(1-2\mu)$$

For many metals (e.g., steel and aluminum), Poisson's ratio is approximately $\mu \approx {}^1/_3$ so that $E_v/E \approx 1$. The speed of sound in stainless steel (E = 163 GPa; ρ = 7800 kg/m^3) is 4570 m/s (= 15000 ft/s ≈ 3 mile/s ≈ 3X faster than the speed of sound in water).

6. The **Mach number, Ma**, is a dimensionless parameter that is commonly used when discussing compressible flows. The Mach number is defined as:

$$\boxed{\text{Ma} \equiv \frac{V}{c}} \tag{22}$$

where V is the flow velocity and c is the speed of sound in the flow.

Notes:

a. Compressible flows are often classified by their Mach number:

Ma < 1	subsonic
Ma = 1	sonic
Ma > 1	supersonic

Additional sub-classifications include:

Ma < 0.3	incompressible
Ma ≈ 1	transonic
Ma > 5	hypersonic

b. The square of the Mach number, Ma^2, is a measure of a flow's macroscopic kinetic energy to its microscopic kinetic energy.

7. The change in the properties across the sound wave can be found from the following:

COM (note that A = constant)

$$-\rho c A + (\rho + d\rho)(c - dV)A = 0 \quad \Rightarrow \quad -\rho dV + c d\rho = 0 \quad \Rightarrow \quad \frac{dV}{c} = \frac{d\rho}{\rho} \tag{23}$$

Speed of Sound (isentropic process):

$$c^2 = \frac{dp}{d\rho} \tag{24}$$

Ideal Gas:

$$p = \rho R T \quad \Rightarrow \quad \frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \tag{25}$$

Combine Eqs. (23) and (24) and simplify.

$$\frac{\rho}{p}c^2 = \frac{dp}{p}\frac{\rho}{d\rho} \Rightarrow \frac{\gamma RT}{RT} = \frac{dp}{p}\frac{c}{dV}$$

$$\boxed{\frac{dV}{c} = \frac{d\rho}{\rho} = \frac{1}{\gamma}\frac{dp}{p}} \tag{26}$$

Now combine Eqs. (26) and (25).

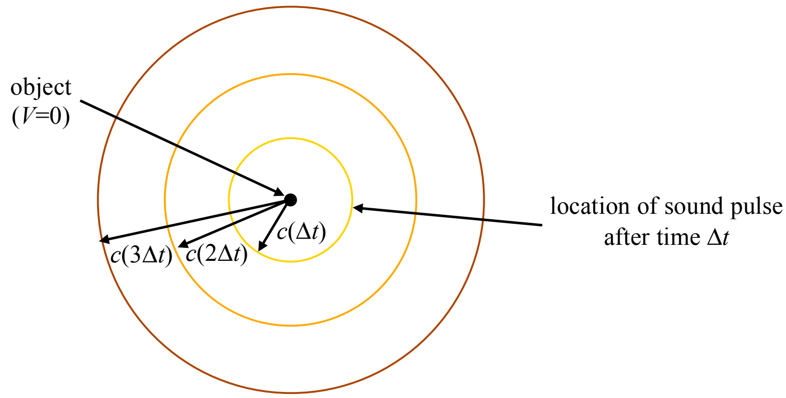
$$\frac{dp}{p} = \frac{1}{\gamma}\frac{dp}{p} + \frac{dT}{T}$$

$$\boxed{\frac{dT}{T} = \frac{\gamma - 1}{\gamma}\frac{dp}{p}} \tag{27}$$

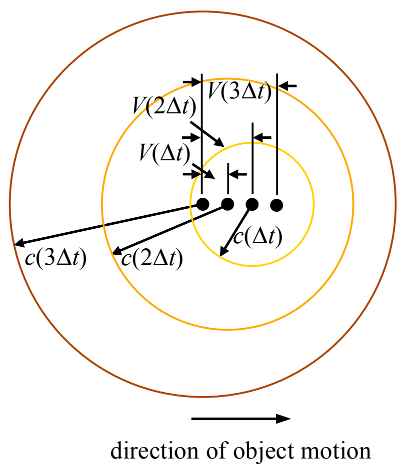
Thus, across a compression wave ($dp > 0$): $dV > 0$, $d\rho > 0$, and $dT > 0$.
Across a rarefaction, or expansion wave ($dp < 0$): $dV < 0$, $d\rho < 0$, and $dT < 0$.

The Mach Cone

Consider the propagation of infinitesimal pressure waves, i.e., sound waves, emanating from an object at rest. The waves will travel at the speed of sound, *c*.

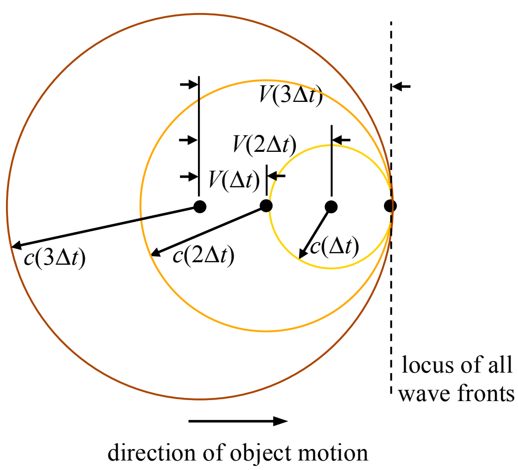


Now consider an object moving at subsonic velocity, *V*<*c*:



The pressure pulses are more closely spaced in the direction of the object's motion and more widely spaced behind the object. Thus, the frequency of the sound in front of the object increases, while the frequency behind the object decreases. This is known as the **Doppler Shift**.

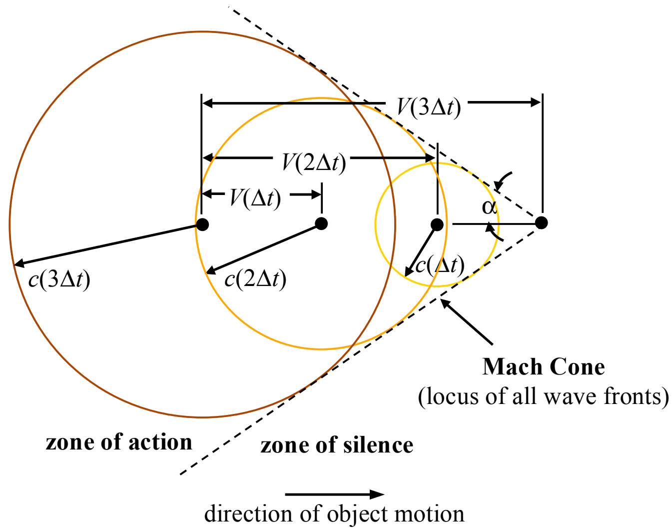
Now consider an object traveling at the sonic speed, *V*=*c*:



Since no wave fronts propagate ahead of the object, an observer standing in front of the object will not hear it approaching until the object reaches the observer.

Note that the infinitesimal pressure changes in front of the object "pile up" on one another producing a sudden, finite pressure change (a shock wave!)

Lastly, consider an object travelling at supersonic speeds, $V > c$:



The object out runs the pressure pulses it generates.

The locus of wave fronts forms a cone which is known as the **Mach Cone**. The object cannot be heard outside the Mach Cone. This region is termed the **zone of silence**. Inside the cone, the region known as the **zone of action**, the object can be heard.

The angle of the cone, known as the **Mach angle, α,** is given by:

$$\sin\alpha = \frac{c\Delta t}{V\Delta t} = \frac{c}{V} = \frac{1}{\text{Ma}}$$

$$\boxed{\alpha = \sin^{-1}\left(1/\text{Ma}\right)} \tag{28}$$