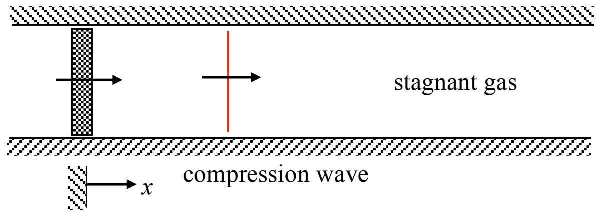
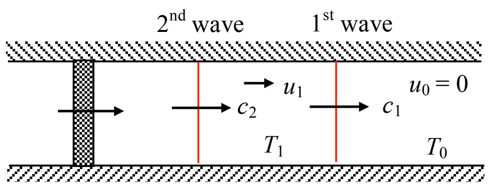
9. Normal Shock Waves

Consider the movement of a piston in a cylinder:

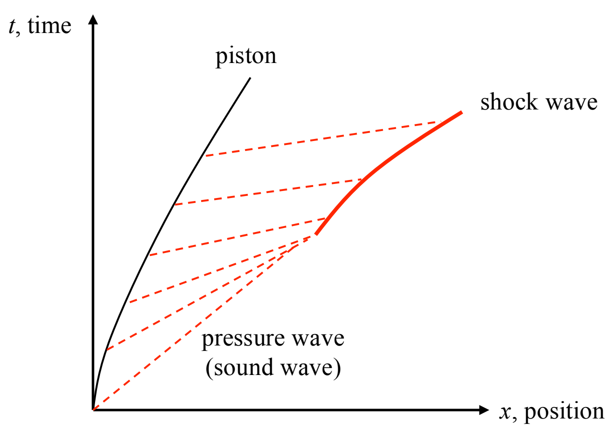


When we first move the piston, an infinitesimal (compression) pressure wave travels down the cylinder at the sonic speed. Behind the wave, the pressure, temperature, density, and increase slightly and the fluid has a small velocity following the wave.

If we continue to increase the piston velocity, additional pressure waves will propagate down the cylinder. However, these waves travel at a slightly increased speed relative to a fixed observer due to the increased fluid temperature and fluid movement. The result is that the waves formed later catch up to the previous waves. When the waves catch up to the first wave, their effects add together so that the small changes across the individual waves now become a sudden and finite change called a **shock wave**.



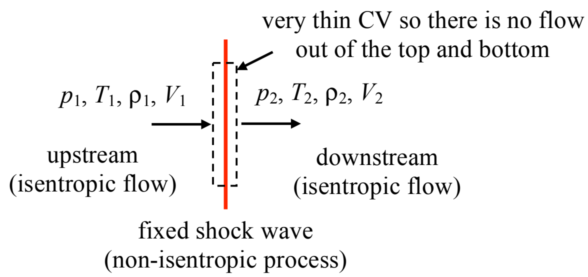
$c_2 > c_1$ since $T_1 > T_0$ and u_1 moves to the right



Notes:

1. The velocity of a shock wave is greater than the speed of sound since:
$$c^2 = \frac{\Delta p}{\Delta \rho}\left(1+\frac{\Delta \rho}{\rho}\right)$$
For a sound wave, $\Delta\rho \to d\rho \Rightarrow \Delta\rho/\rho \to 0$. For a shock wave, $\Delta\rho/\rho > 0$, so that $c_{\text{shock wave}} > c_{\text{sound wave}}$.
2. A shock wave is a pressure wave across which there is a finite change in the flow properties.
3. Shock waves only occur in supersonic flows.
4. Shock waves are typically very thin with thicknesses on the order of 1 μm. Thus, we consider the changes in the flow properties across the wave to be discontinuous.
5. The sudden change in flow properties across the shock wave occurs non-isentropically since the thermal and velocity gradients are large *within* the shock wave itself.

To analyze a shock wave, we'll use an approach similar to that used to examine a sound wave. Let's consider a fixed shock wave across which flow properties change:



COM:

$$\rho_1 V_1 A = \rho_2 V_2 A$$
$$\boxed{\rho_1 V_1 = \rho_2 V_2} \quad (143)$$

LME (in x-direction):

$$\dot{m}V_2 - \dot{m}V_1 = p_1 A - p_2 A$$
$$\boxed{\rho_1 V_1 (V_2 - V_1) = \rho_2 V_2 (V_2 - V_1) = p_1 - p_2} \quad (144)$$

COE:

$$\boxed{h_1 + \tfrac{1}{2}V_1^2 = h_2 + \tfrac{1}{2}V_2^2} \quad (145)$$

Also, $h_{01} = h_{02}$. Note that no heat is added to the CV, i.e., the process is adiabatic.

2nd Law:

$\boxed{s_2 > s_1}$ (since the process is adiabatic but irreversible) (146)

Thermal Equation of State (ideal gas law):

$$\boxed{\frac{p_1}{\rho_1 T_1} = \frac{p_2}{\rho_2 T_2} = R} \quad (147)$$

Caloric Equation of State (for a perfect gas):

$$\boxed{h = c_p T} \quad (148)$$

Combining Eqs. (143) and (144):

$$\frac{p_1}{\rho_1 V_1} - \frac{p_2}{\rho_2 V_2} = V_2 - V_1 \tag{149}$$

Substituting Eq. (147):

$$\frac{RT_1}{V_1} - \frac{RT_2}{V_2} = V_2 - V_1$$

Substituting Eqs. (145) and (148) then re-arranging:

$$\frac{R}{V_1}\left(T_0 - \frac{V_1^2}{2c_p}\right) - \frac{R}{V_2}\left(T_0 - \frac{V_2^2}{2c_p}\right) = V_2 - V_1$$

$$RV_2T_0 - \frac{RV_2V_1^2}{2c_p} - RV_1T_0 - \frac{RV_1V_2^2}{2c_p} = V_1V_2\left(V_2 - V_1\right)$$

$$V_1V_2\left(V_2 - V_1\right) = R\left[\left(V_2 - V_1\right)T_0 + \left(V_2 - V_1\right)\frac{V_1V_2}{2c_p}\right]$$

$$V_1V_2\left(V_2 - V_1\right) = R\left(V_2 - V_1\right)\left[T_0 + \frac{V_1V_2}{2c_p}\right]$$

$$V_1V_2 = R\left[T_0 + \frac{V_1V_2}{2c_p}\right]$$

$$V_1V_2\left(1 - \frac{R}{2c_p}\right) = RT_0$$

$$V_1V_2 = \frac{RT_0}{\left(1 - \frac{R}{2c_p}\right)}$$

Finally, substituting the relation:

$$\frac{R}{c_p} = \frac{c_p - c_v}{c_p} = \frac{\gamma - 1}{\gamma}$$

and re-arranging we have:

$$\boxed{V_1V_2 = \frac{2\gamma RT_0}{\gamma + 1}} \qquad \textbf{Prandtl's Equation} \tag{150}$$

Dividing both sides of the equation by the sound speed on either side of the shock wave and utilizing the definition for the Mach # for a perfect gas:

$$\frac{V_1}{\sqrt{\gamma R T_1}}\frac{V_2}{\sqrt{\gamma R T_2}} = \frac{2}{\gamma+1}\frac{\sqrt{\gamma R T_0}}{\sqrt{\gamma R T_1}}\frac{\sqrt{\gamma R T_0}}{\sqrt{\gamma R T_2}}$$

$$\mathrm{Ma}_1\mathrm{Ma}_2 = \frac{2}{\gamma+1}\sqrt{\frac{T_0}{T_1}}\sqrt{\frac{T_0}{T_2}}$$

Recall that for the adiabatic flow of a perfect gas:

$$\frac{T}{T_0} = \left(1+\frac{\gamma-1}{2}\mathrm{Ma}^2\right)^{-1}$$

So that:

$$\begin{aligned}\mathrm{Ma}_1\mathrm{Ma}_2 &= \frac{2}{\gamma+1}\sqrt{\frac{T_0}{T_1}}\sqrt{\frac{T_0}{T_2}} \\ &= \frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{1/2}\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2\right)^{1/2}\end{aligned}$$

After additional algebra we can reduce this equation to the following:

$$\boxed{\mathrm{Ma}_2^2 = \frac{(\gamma-1)\mathrm{Ma}_1^2+2}{2\gamma\mathrm{Ma}_1^2-(\gamma-1)}} \tag{151}$$

Relation between upstream Mach # (Ma_1) and downstream Mach # (Ma_2) across a normal shock wave.

Notes:

1. When $\mathrm{Ma}_1>1$, then $\mathrm{Ma}_2<1$ (supersonic to subsonic flow) and when $\mathrm{Ma}_1<1$, then $\mathrm{Ma}_2>1$ (subsonic to supersonic flow).
2. From experiments, we observe that shock waves never form in subsonic flows ($\mathrm{Ma}_1<1$) even though Eq. (151) does not give any indication of this. We'll use the 2nd law in a moment to show that shock waves can only form in supersonic flows ($\mathrm{Ma}_1>1$).

The temperature ratio across the shock wave can be determined using the adiabatic stagnation temperature relation for a perfect gas and noting that the stagnation temperature remains constant across a shock:

$$\frac{\left(T_2/T_0\right)}{\left(T_1/T_0\right)} = \frac{\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2\right)^{-1}}{\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)^{-1}}$$

$$\boxed{\frac{T_2}{T_1} = \frac{\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_1^2\right)}{\left(1+\frac{\gamma-1}{2}\mathrm{Ma}_2^2\right)}} \tag{152}$$

The pressure ratio across the shock can be determined by combining Eqs. (152), (147), and (143) along with the definition of the Mach number for a perfect gas:

$$\frac{p_2}{p_1} = \frac{\rho_2 T_2}{\rho_1 T_1} = \frac{V_1 T_2}{V_2 T_1} = \frac{\left(\sqrt{\gamma R T_1}\,\text{Ma}_1\right) T_2}{\left(\sqrt{\gamma R T_2}\,\text{Ma}_2\right) T_1}$$

$$= \frac{\text{Ma}_1}{\text{Ma}_2}\sqrt{\frac{T_2}{T_1}}$$

$$\boxed{\frac{p_2}{p_1} = \frac{\text{Ma}_1}{\text{Ma}_2}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_1^2}{1+\frac{\gamma-1}{2}\text{Ma}_2^2}\right)^{1/2}} \tag{153}$$

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{p_2 T_1}{p_1 T_2} = \frac{\text{Ma}_1}{\text{Ma}_2}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_1^2}{1+\frac{\gamma-1}{2}\text{Ma}_2^2}\right)^{1/2}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_1^2}{1+\frac{\gamma-1}{2}\text{Ma}_2^2}\right)$$

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{\text{Ma}_1}{\text{Ma}_2}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_1^2}{1+\frac{\gamma-1}{2}\text{Ma}_2^2}\right)^{3/2}} \tag{154}$$

We can also determine the ratio of the isentropic stagnation pressures and densities across the shock wave:

$$\frac{p}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}^2\right)^{\gamma/(1-\gamma)}$$

$$\frac{\left(p_1/p_{01}\right)}{\left(p_2/p_{02}\right)} = \left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_1^2}{1+\frac{\gamma-1}{2}\text{Ma}_2^2}\right)^{\gamma/(1-\gamma)}$$

$$\frac{p_{02}}{p_{01}} = \left(\frac{p_2}{p_1}\right)\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_1^2}{1+\frac{\gamma-1}{2}\text{Ma}_2^2}\right)^{\gamma/(1-\gamma)} = \frac{\text{Ma}_1}{\text{Ma}_2}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_1^2}{1+\frac{\gamma-1}{2}\text{Ma}_2^2}\right)^{1/2}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_1^2}{1+\frac{\gamma-1}{2}\text{Ma}_2^2}\right)^{\gamma/(1-\gamma)}$$

$$\boxed{\frac{p_{02}}{p_{01}} = \frac{\text{Ma}_1}{\text{Ma}_2}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_1^2}{1+\frac{\gamma-1}{2}\text{Ma}_2^2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}} \tag{155}$$

$$\frac{\rho_{02}}{\rho_{01}} = \frac{p_{02}}{p_{01}}\frac{T_{01}}{T_{02}}$$

but since $T_{01} = T_{02}$

$$\boxed{\frac{\rho_{02}}{\rho_{01}} = \frac{p_{02}}{p_{01}} = \frac{\text{Ma}_1}{\text{Ma}_2}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_1^2}{1+\frac{\gamma-1}{2}\text{Ma}_2^2}\right)^{\frac{(1+\gamma)}{2(1-\gamma)}}} \tag{156}$$

The sonic area ratio across the shock can be determined from the fact that the mass flow rate across the shock must remain constant (COM):

$$\dot{m}_1 = \dot{m}_2$$

$$\rho_1^* V_1^* A_1^* = \rho_2^* V_2^* A_2^*$$

$$\frac{A_2^*}{A_1^*} = \frac{\rho_1^*}{\rho_2^*}\frac{V_1^*}{V_2^*}$$

The sonic ratios can be determined from the following:

$$\frac{\rho_1^*}{\rho_{01}} = \frac{\rho_2^*}{\rho_{02}} = \left(1+\frac{\gamma-1}{2}\right)^{1/(1-\gamma)}$$

$$\frac{\rho_1^*}{\rho_2^*} = \frac{\rho_{01}}{\rho_{02}} = \frac{\text{Ma}_1}{\text{Ma}_2}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_1^2}{1+\frac{\gamma-1}{2}\text{Ma}_2^2}\right)^{\frac{(\gamma+1)}{2(\gamma-1)}}$$

and

$$\frac{V_1^*}{V_2^*} = \frac{c_1^*}{c_2^*} = \sqrt{\frac{T_1^*}{T_2^*}} = \sqrt{\frac{\left(T_1^*/T_0\right)}{\left(T_2^*/T_0\right)}} = 1$$

Note that $T_{01} = T_{02}$ has been used in the previous equation. Substituting these two sonic ratios and simplifying:

$$\boxed{\frac{A_2^*}{A_1^*} = \frac{\text{Ma}_2}{\text{Ma}_1}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_1^2}{1+\frac{\gamma-1}{2}\text{Ma}_2^2}\right)^{\frac{(\gamma+1)}{2(\gamma-1)}}} \tag{157}$$

Note that we could have also used the isentropic area ratios on either side of the shock wave:

$$\frac{A_1}{A_1^*}=\frac{1}{\text{Ma}_1}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_1^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{(\gamma+1)}{2(\gamma-1)}} \quad \text{and} \quad \frac{A_2}{A_2^*}=\frac{1}{\text{Ma}_2}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_2^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{(\gamma+1)}{2(\gamma-1)}}$$

so that:

$$\frac{A_2^*}{A_1^*}=\frac{\left(A_1/A_1^*\right)}{\left(A_2/A_2^*\right)}=\frac{\text{Ma}_2}{\text{Ma}_1}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_1^2}{1+\frac{\gamma-1}{2}\text{Ma}_2^2}\right)^{\frac{(\gamma+1)}{2(\gamma-1)}} \qquad \text{where } A_1=A_2$$

Notes:

1. Equations (152)-(157) may be written only in terms of Ma_1 by substituting Eq. (151). The resulting equations are:

$$\boxed{\text{Ma}_2^2=\frac{(\gamma-1)\text{Ma}_1^2+2}{2\gamma\text{Ma}_1^2-(\gamma-1)}} \tag{158}$$

$$\boxed{\frac{T_2}{T_1}=\left[2+(\gamma-1)\text{Ma}_1^2\right]\left[\frac{2\gamma\text{Ma}_1^2-(\gamma-1)}{(\gamma+1)^2\text{Ma}_1^2}\right]} \tag{159}$$

$$\boxed{\frac{\rho_2}{\rho_1}=\frac{V_1}{V_2}=\frac{(\gamma+1)\text{Ma}_1^2}{(\gamma-1)\text{Ma}_1^2+2}} \tag{160}$$

$$\boxed{\frac{p_2}{p_1}=\frac{2\gamma}{\gamma+1}\text{Ma}_1^2-\frac{\gamma-1}{\gamma+1}} \tag{161}$$

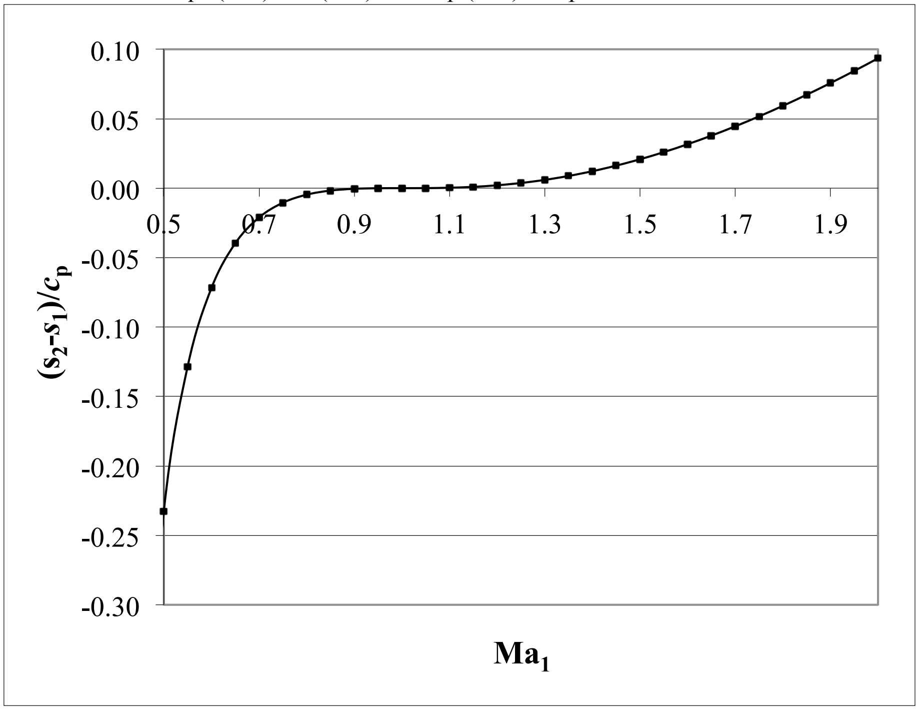
$$\boxed{\frac{T_{02}}{T_{01}}=1} \tag{162}$$

$$\boxed{\frac{p_{02}}{p_{01}}=\frac{\rho_{02}}{\rho_{01}}=\frac{A_1^*}{A_2^*}=\left[\frac{\frac{\gamma+1}{2}\text{Ma}_1^2}{1+\frac{\gamma-1}{2}\text{Ma}_1^2}\right]^{\gamma/\gamma-1}\left[\frac{2\gamma}{\gamma+1}\text{Ma}_1^2-\frac{\gamma-1}{\gamma+1}\right]^{1/1-\gamma}} \tag{163}$$

2. Now let's examine the change in entropy across the shock using:

$$\boxed{s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}} \tag{164}$$

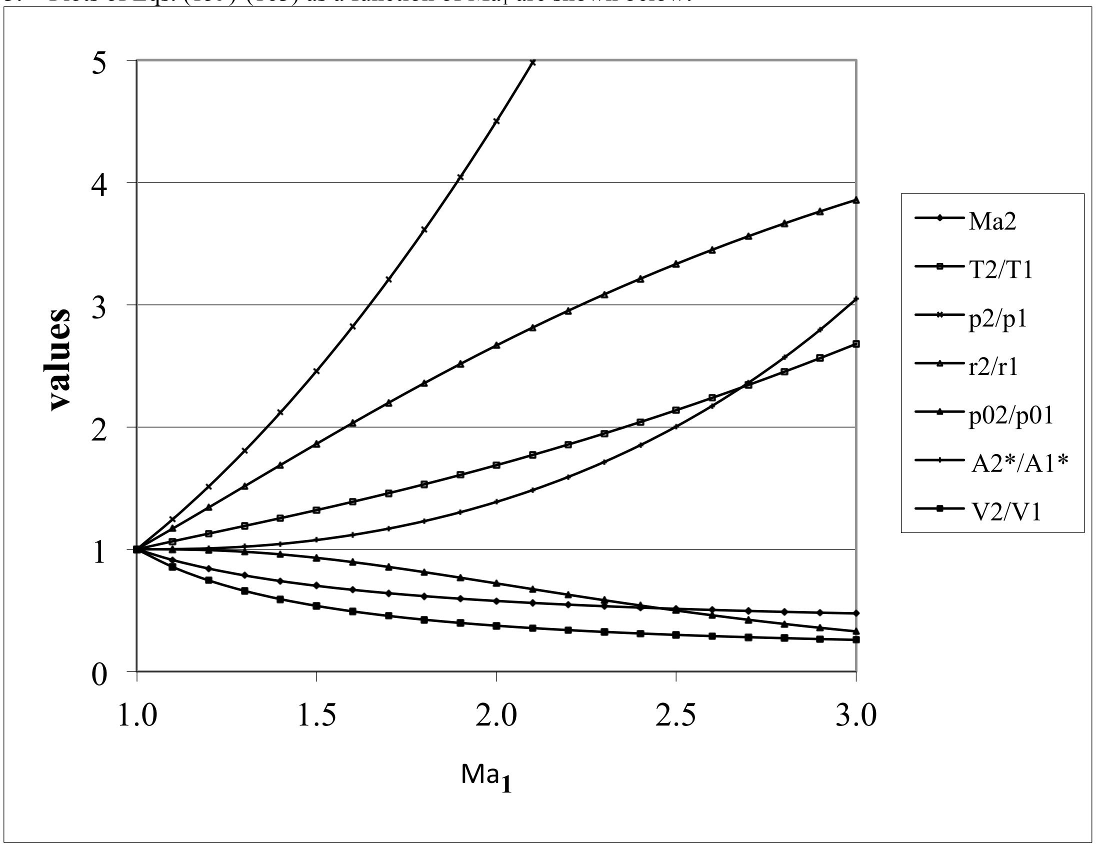
If we substitute Eqs. (159) and (161) into Eq. (164) and plot:



We observe that for $Ma_1 < 1$ the entropy decreases across the shock. The 2nd law, however, states that the entropy must increase across the shock (refer to Eq. (146)). Thus, **shock waves can only form when $Ma_1>1$**.

Also note that as the upstream Mach number approaches one ($Ma_1 \rightarrow 1$), the flow through the shock approaches an isentropic process. An infinitesimally weak shock wave, one occurring when $Ma_1=1$, results in an isentropic process. This type of shock is, in fact, just a sound wave.

3. Plots of Eqs. (159)-(163) as a function of Ma_1 are shown below:



The plot shows the following relations:

$T_2 > T_1$ and $T_2/T_1 \uparrow$ as $Ma_1 \uparrow$
$p_2 > p_1$ and $p_2/p_1 \uparrow$ as $Ma_1 \uparrow$
$\rho_2 > \rho_1$ and $\rho_2/\rho_1 \uparrow$ as $Ma_1 \uparrow$
$V_2 < V_1$ and $V_2/V_1 \downarrow$ as $Ma_1 \uparrow$
$T_{02} = T_{01}$
$p_{02} < p_{01}$ and $p_{02}/p_{01} \downarrow$ as $Ma_1 \uparrow$
$\rho_{02} < \rho_{01}$ and $\rho_{02}/\rho_{01} \downarrow$ as $Ma_1 \uparrow$
$A_2^* > A_1^*$ and $A_2^*/A_1^* \uparrow$ as $Ma_1 \uparrow$
$Ma_2 \downarrow$ as $Ma_1 \uparrow$ Furthermore, $\lim_{Ma_1 \to \infty} \left(Ma_2\right) = \left(\frac{\gamma - 1}{2\gamma}\right)^{1/2}$

4. The **shock strength** is defined as the change in pressure across the shock wave relative to the upstream pressure: $\Delta p/p_1 = p_2/p_1 - 1$. Viewing the trends shown in the previous plot, the larger the incoming Mach number the stronger the shock wave.