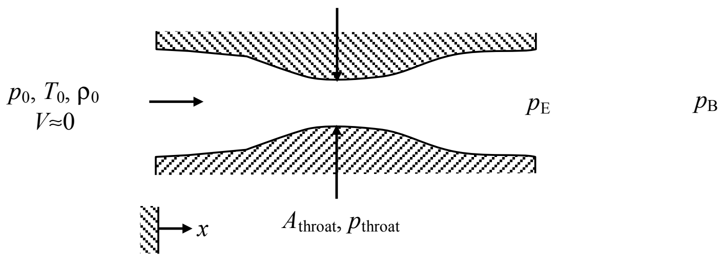
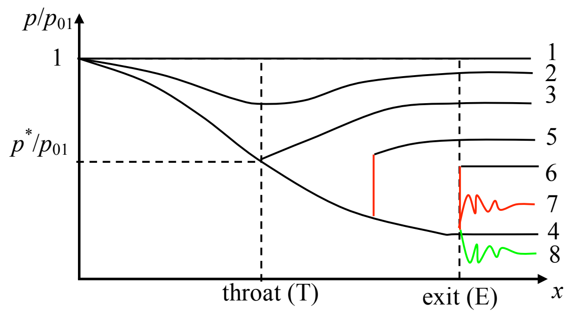
10. Flow in Converging-Diverging Nozzles

Consider flow through a converging-diverging nozzle (aka a deLaval nozzle) as shown below.



Let's hold the stagnation pressure, p_0, fixed and vary the back pressure, p_B. The plot below shows how the static pressure ratio, p/p_0, varies with location, x, for various values of the back pressure ratio, p_B/p_0:



Cases:

1. There is no flow through the device since $p_B = p_0$.
2. There is subsonic flow throughout the device and $p_E = p_B$. Also, $Ma_T < 1$, $A_T > A^*$, $Ma_E < 1$, $\dot{m} < \dot{m}_{choked}$. The flow everywhere is isentropic.
3. There is subsonic flow throughout the device except at the throat where $p_T = p^*$ ($Ma_T = 1$, $A_T = A^*$). The flow is now choked ($\dot{m} = \dot{m}_{choked}$); further decreases in p_B will not affect the flow upstream of the throat and the mass flow rate will remain at the choked mass flow rate value. The exit pressure will equal the back pressure, $p_E = p_B$, since the flow is subsonic at the exit ($Ma_E < 1$). The flow everywhere is isentropic.
4. Subsonic flow will occur in the converging section, sonic flow will occur at the throat ($p_T = p^*$, Ma_T $=1$, $A_T = A^*$), and supersonic flow will occur in the diverging section ($Ma_E > 1$). This type of flow is called **correctly expanded flow** or flow at **design conditions** since no shock waves form anywhere in the device and $p_E = p_B$. Note that p_E does not equal p_B because the flow is subsonic at the exit, but because the flow is at design conditions (a special case).
5. Subsonic flow will occur in the converging section and sonic flow will occur at the throat ($p_T = p^*$, Ma_T $= 1$, $A_T = A^*_1$). A portion of the diverging section will be supersonic with a normal shock wave occurring at a location such that the subsonic flow downstream of the flow will have an exit pressure equal to the back pressure: $p_E = p_B$ ($Ma_E < 1$). As the back pressure decreases, the shock wave moves downstream of the throat and toward the exit. The pressure rise across the shock wave also increases as the back pressure decreases. There is isentropic flow upstream of the shock and downstream of it, but across the shock the flow is non-isentropic.
6. This case is similar to case 5 except that the shock wave is precisely at the nozzle exit. The pressure just downstream of the shock wave equals the back pressure since the flow is subsonic there ($p_{E2} = p_B$, $Ma_{E2} < 1$). The flow everywhere within the C-D nozzle is isentropic except right at the exit.
7. The flow within the C-D nozzle (and the exit) is isentropic ($Ma_E > 1$). The normal shock that was located at the exit for case 6 has moved outside the device to form a complicated sequence of oblique shock waves alternating with expansion fans (these are 2D phenomena to be discussed in a following

section of notes). This case is called the **overexpanded case** since the diverging section of the device has an area that overexpands the flow to a pressure that is lower than the back pressure ($p_E < p_B$). External shock waves are required to compress the flow to match the back pressure.

8. This case is similar to case 7 except that the flow outside of the device forms a sequence of expansion fans alternating with oblique shock waves (a sequence out of phase with the sequence mentioned in case 7). This case is called the **underexpanded case** since the diverging section of the device has an area that is not large enough to drop the exit pressure to the back pressure ($p_E > p_B$, $\text{Ma}_E > 1$). External expansion waves are required expand the flow to match the back pressure.

Notes:

1. The critical back pressure ratio corresponding to case 3 can be found from the isentropic relations (the flow throughout the entire device is isentropic). Assume that the geometry, and hence the exit-to-throat area ratio, A_e/A_t, is given. Since for case 5 the flow is choked we know that $A_t = A^*$. Furthermore, since the exit flow is subsonic we also know that $p_e = p_b$. From the area ratio we can determine the exit Mach number, Ma_e:

$$\frac{A_e}{A^*} = \frac{A_e}{A_t} = \frac{1}{\text{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \quad \text{(where the subsonic Ma}_e\text{ is chosen)}$$

The back pressure ratio, p_b/p_0, for case 5 can be determined given the exit Mach number.

$$\frac{p_b}{p_0} = \frac{p_e}{p_0} = \left(1+\frac{\gamma-1}{2}\text{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}}$$

2. The critical back pressure ratio corresponding to case 4 can be determined in a manner similar to that described above in Note 1. For case 4 however, the supersonic value for Ma_e should be used when determining the exit Mach number from the area ratio.

3. The critical back pressure ratio corresponding to case 6 can be found by combining the isentropic relations with the normal shock wave relations. When the shock wave occurs right at the exit of the device, the flow just upstream of the exit can be found from the isentropic relations:

$$\frac{A_e}{A_1^*} = \frac{A_e}{A_t} = \frac{1}{\text{Ma}_{e1}}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_{e1}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \quad \text{(where the supersonic Ma}_{e1}\text{ is chosen)}$$

$$\frac{p_{e1}}{p_{01}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_{e1}^2\right)^{\frac{\gamma}{1-\gamma}}$$

Note that the subscript '1' denotes the conditions just upstream of the shock wave. To determine the conditions just downstream of the shock we use the normal shock wave relations:

$$\frac{p_{e2}}{p_{e1}} = \frac{2\gamma}{\gamma+1}\text{Ma}_{e1}^2 - \frac{\gamma-1}{\gamma+1}$$

where p_{e2} is the pressure just downstream of the shock. Since the downstream flow is subsonic and because we're at the exit of the device, the downstream pressure, p_{e2}, must also equal the back pressure, p_b. Thus,

$$\frac{p_b}{p_{01}} = \frac{p_{e2}}{p_{01}} = \frac{p_{e2}}{p_{e1}}\frac{p_{e1}}{p_{01}} = \left[\frac{2\gamma}{\gamma+1}\text{Ma}_{e1}^2 - \frac{\gamma-1}{\gamma+1}\right]\left(1+\frac{\gamma-1}{2}\text{Ma}_{e1}^2\right)^{\frac{\gamma}{1-\gamma}}$$

4. The location of a shock wave for a back pressure in the range corresponding to case 3 and case 5 can be determined through iteration.
   a. Assume a location for the shock wave (e.g., pick a value for A/A_t since the geometry is known).
   b. Determine the Mach number and pressure just upstream of the shock, Ma_1 and p_1, using the isentropic relations as discussed in Note 2.

$$\frac{A}{A_1^*} = \frac{A}{A_t} = \frac{1}{\text{Ma}_1}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_1^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \quad \text{(where the supersonic Ma}_1\text{ is chosen)}$$

$$\frac{p_1}{p_{01}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_1^2\right)^{\frac{\gamma}{1-\gamma}}$$

   c. Calculate the stagnation pressure ratio and sonic area ratio across the shock using the normal shock relations:

$$\frac{p_{02}}{p_{01}} = \frac{A_1^*}{A_2^*} = \left[\frac{\frac{\gamma+1}{2}\text{Ma}_1^2}{1+\frac{\gamma-1}{2}\text{Ma}_1^2}\right]^{\gamma/\gamma-1}\left[\frac{2\gamma}{\gamma+1}\text{Ma}_1^2 - \frac{\gamma-1}{\gamma+1}\right]^{1/1-\gamma}$$

   d. Determine the exit Mach number and exit pressure ratio using the isentropic relations and the downstream sonic area and stagnation pressure:

$$\frac{A_e}{A_2^*} = \frac{A_e}{A_t}\frac{A_1^*}{A_2^*} = \frac{1}{\text{Ma}_e}\left(\frac{1+\frac{\gamma-1}{2}\text{Ma}_e^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \quad \text{(where the subsonic Ma}_e\text{ is chosen)}$$

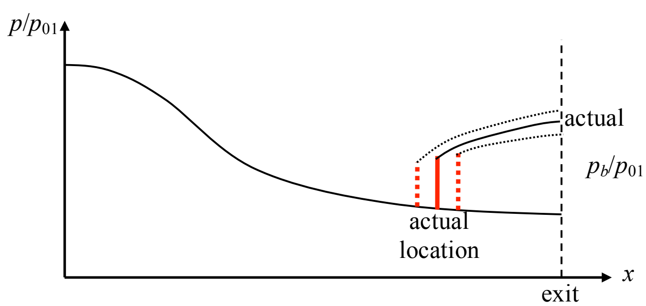
   Note that since the flow is choked, the throat area is equal to the upstream sonic area, i.e., $A_t = A_1^*$.

$$\frac{p_e}{p_{02}} = \left(1+\frac{\gamma-1}{2}\text{Ma}_e^2\right)^{\frac{\gamma}{1-\gamma}}$$

   Note that since the exit Mach number is subsonic, the exit pressure will equal the back pressure, i.e., $p_e = p_b$.
   e. Calculate the ratio of the back pressure to the upstream stagnation pressure:

$$\frac{p_b}{p_{01}} = \frac{p_e}{p_{02}}\frac{p_{02}}{p_{01}}$$

   f. Check to see if the back pressure ratio calculated in step (e) matches with the given back pressure ratio. If so, then the assumed location of the shock is correct. If not, then the go back to step (a) and repeat. If the back pressure ratio calculated in part (e) is less than the given back pressure ratio, then the assumed shock location is too far downstream. If the back pressure ratio calculated in part (e) is greater than the given back pressure ratio, then the assumed shock location is too far upstream.



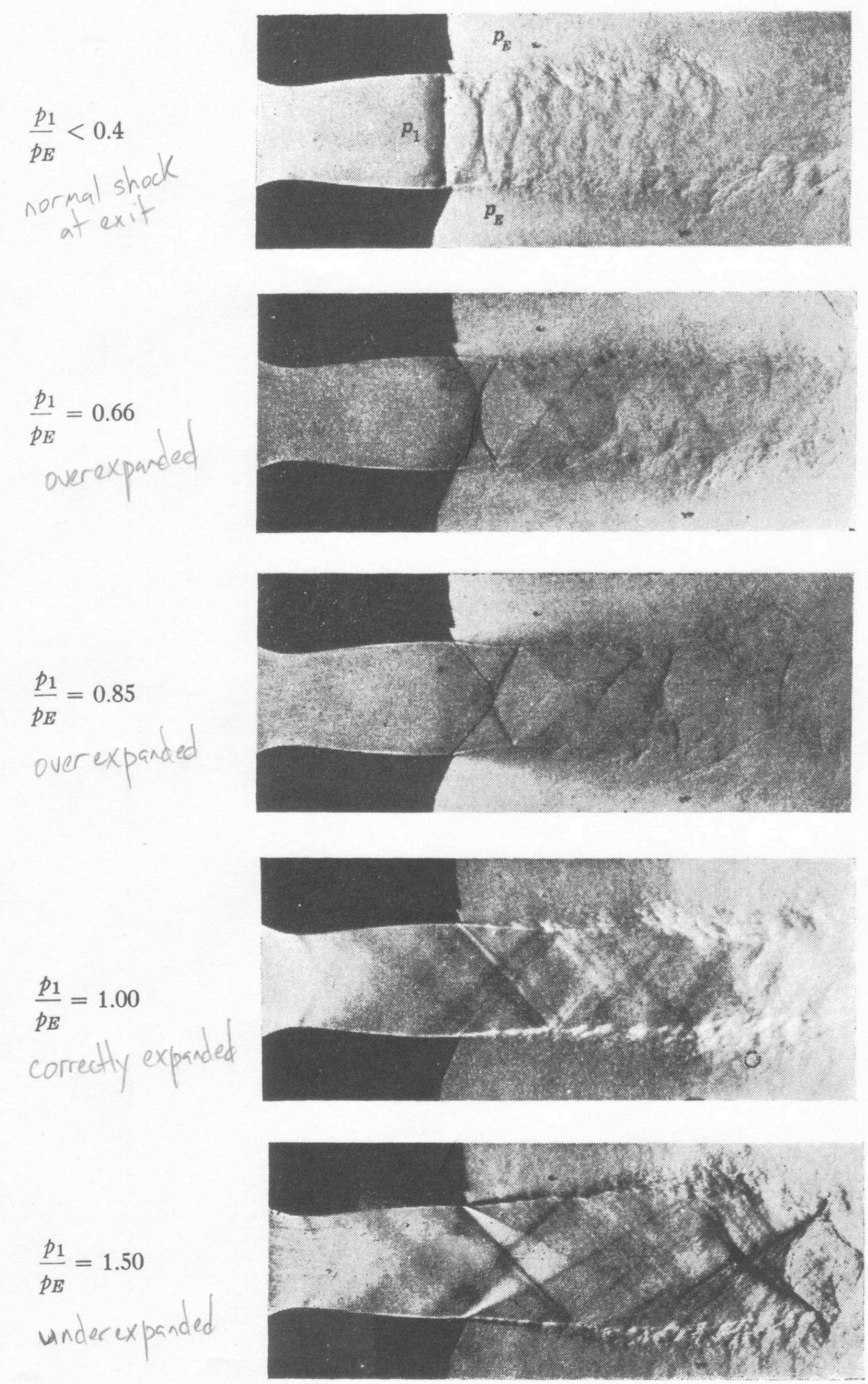


FIG. 5·4 Schlieren photographs of flow from a supersonic nozzle at different back pressures. (Figure from: Liepmann, H.W. and Roshko, A., *Elements of Gasdynamics*, Wiley.)

5. In real nozzles flows, the flow will typically separate from the nozzle walls as a result of the large adverse pressure gradient occurring across a shock wave. Interaction of the shock with the separated boundary layer results in a more gradual pressure rise than what is expected for the ideal, normal shock analysis.

   It is also possible that downstream pressure information can propagate upstream in the diverging section even when the core flow is supersonic. In a real flow, a boundary layer will form along the wall with the flow in part of this boundary layer being subsonic. Thus, pressure information can propagate upstream within the subsonic part of the boundary layer and affect the flow in the diverging section. When the back pressure is in the range corresponding to Case 7 (back pressure less than the exit pressure when a shock stands at the exit, and greater than the isentropic case corresponding to supersonic diverging section flow), oblique shocks will typically form within the diverging section and flow separation occurs as shown in the figures below. The exact pressure and location of the separation point are dependent on the boundary layer flow.

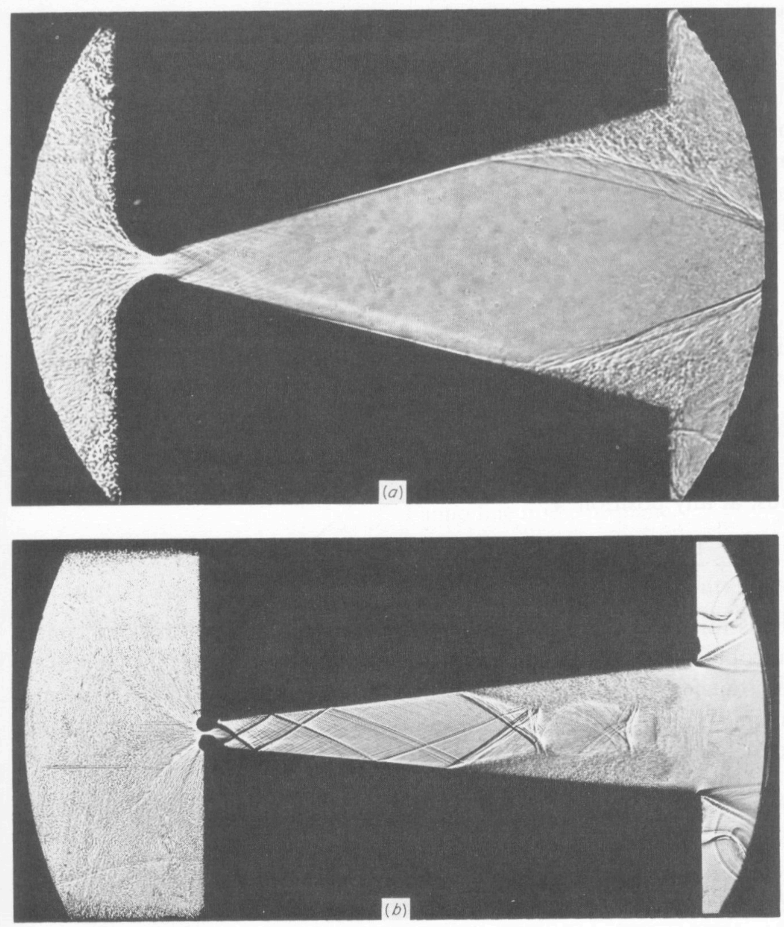
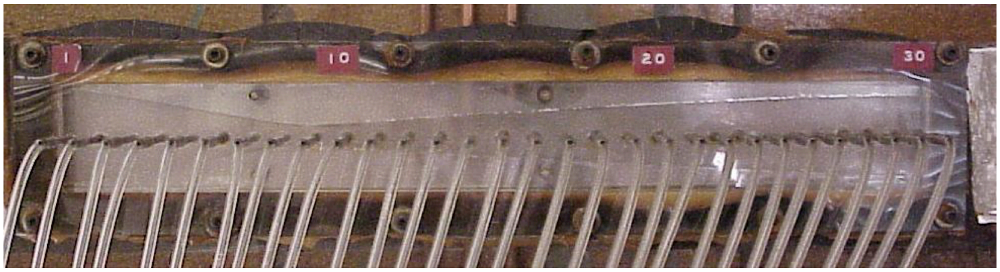


Figure 6.16
Shadowgraphs showing flow separation in supersonic nozzles. (*Courtesy of H. O. Amann, Ernst Mach Institut.*)

A converging-diverging nozzle with pressure taps along the length of the device. The flow is from left to right.



The pressure ratio as a function of the axial distance in the CD nozzle for various back pressures.