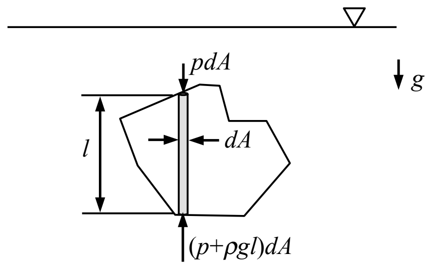
Buoyant Force and Center of Buoyancy
When an object is submerged in a fluid, the pressure acting on the object deeper in the fluid (i.e., in the direction of gravity) will be larger than the pressure acting on the object shallower in the fluid. As a result, there will be a net pressure force acting on the object. This net pressure force is known as the buoyant force.

To derive the value of the buoyant force, consider the vertical pressure forces acting on a narrow cylinder with cross-sectional area dA within a fully submerged object as shown in the following figure.



The net pressure force in the vertical direction on the narrow cylinder, assuming an incompressible fluid, is,

$$dF_{p,\text{net}} = \left(p + \rho_{\text{fluid}} g l\right) dA - p dA = \rho_{\text{fluid}} g l dA \quad \text{(acting opposite to gravity).} \tag{2.65}$$

The total net pressure force acting on the object is found by integrating these small bits of pressure force over the entire cross-sectional area of the object,

$$F_{p,\text{net}} = \int_A dF_{p,\text{net}} = \int_A \rho_{\text{fluid}} g l dA \quad \text{(acting opposite to gravity).} \tag{2.66}$$

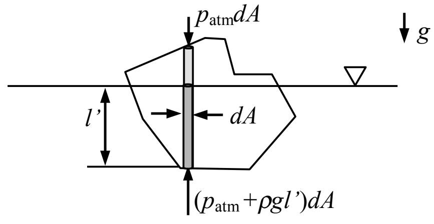
Since the density and gravity are assumed constant here, they may be pulled outside the integral,

$$F_{p,\text{net}} = \rho_{\text{fluid}} g \int_A l dA \quad \text{(acting opposite to gravity),} \tag{2.67}$$

$$\boxed{F_{p,\text{net}} = \rho_{\text{fluid}} g V_{\substack{\text{submerged}\\ \text{object}}} \quad \text{(acting opposite to gravity),}} \tag{2.68}$$

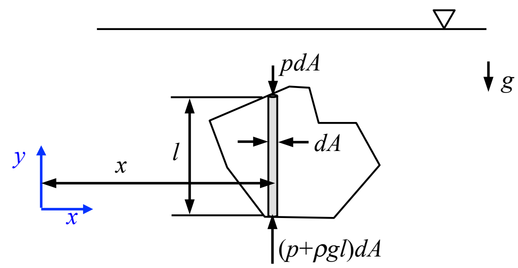
where the integral is simply the volume of the submerged object. This net pressure force acting on the object is referred to as the buoyant force.

Notes:
1. Equation (2.68) states that the buoyant force is equal to the weight of the fluid that's been displaced by the submerged object. This relationship is also known as **Archimede's Principle.**
2. The same analysis can be used for partially submerged objects. In that case, the pressure acting on the top of the object is atmospheric pressure while the pressure at the bottom is $p_{\text{atm}} + \rho_{\text{fluid}} g l'$ where l' is the length of the narrow cylinder that's *submerged in the fluid*. After integrating over the objects cross-sectional area (similar to Eq. (2.67)), we would arrive at exactly the same relation as in Eq. (2.68) except that the $V_{\text{submerged object}}$ refers to just that volume that is submerged in the fluid.



3. There is no net pressure force on the object in the directions perpendicular to gravity since the pressure only varies parallel to the gravitational vector.

The resultant buoyant force acts at the **center of buoyancy**. The center of buoyancy is found equating the moment caused by the resultant buoyant force acting at the center of buoyancy equal to the distributed moment caused by the distributed pressure forces. Consider moments about the *z* axis in figure below



$$x_{CB}\hat{\mathbf{i}}\times \underbrace{\rho g V\hat{\mathbf{j}}}_{\text{buoyant force}} = \int_A x\hat{\mathbf{i}}\times \underbrace{\rho g l dA\hat{\mathbf{j}}}_{\substack{\text{net pressure force}\\ \text{on narrow cylinder}}} \quad (V \text{ is the displaced object volume, } \rho \text{ is the fluid density}), \tag{2.69}$$

$$x_{CB}\rho g V\hat{\mathbf{k}} = \rho g\hat{\mathbf{k}}\int_A xl\,dA\,, \tag{2.70}$$

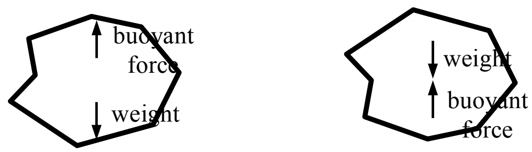
$$x_{CB}V = \int_V xV \quad (dV = l dA), \tag{2.71}$$

$$x_{CB} = \frac{1}{V}\int_V xV\,, \tag{2.72}$$

which is the center of displaced volume. Performing similar analyses about the *x* and *z* axes produces similar results. Thus, the **center of buoyancy is located at the center of the displaced volume**. This is true for both fully submerged and partially submerged objects.

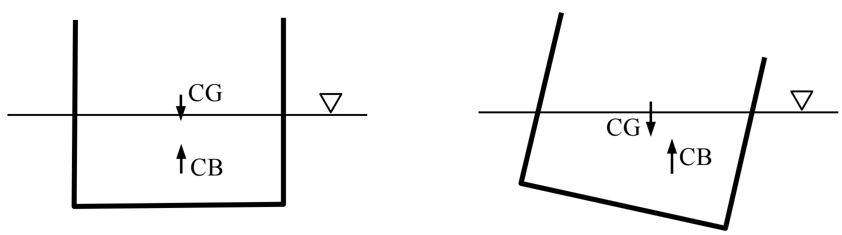
Stable Orientation of a Submerged Object

Submerged objects will be in an **equilibrium orientation** when the forces acting on the object are such that there is no net moment on the object. Considering only the object weight and a buoyant force, an equilibrium orientation will only occur when the two forces are co-linear, as shown in the figures below. Neither object experiences a net moment.



The object on the left is in a **stable equilibrium** while the object on the right is in an **unstable equilibrium**. The reason for the difference is that if each object is rotated slightly, the object on the left will experience a moment that restores it back to its original configuration. However, a small perturbation to the right-hand object will result in a moment that will cause the object to move away from its initial configuration.

The stability of partially submerged objects is a particularly important topic when considering the design of ships. The Swedish ship *Vasa* is a famous example of a ship that was unstable and "turtled" shortly after setting sail for the first time. Unfortunately, stability analysis of partially submerged objects can be complicated since the submerged volume and center of buoyancy changes as the orientation of the object rotates. For example, consider the stability of the simple shape shown in the figure below (CG is the center of gravity and CB is the center of buoyancy).



The initial configuration of the object (on the left) appears to be in unstable equilibrium with the center of gravity above the center of buoyancy. However, when the object is tilted (on the right), the center of buoyancy shifts to one side such that it acts to restore the object to its initial configuration. Hence, the object is actually initially in stable equilibrium.