6. Turbulent Boundary Layer over a Flat Plate (no pressure gradient)

This material isn't covered in ME309. You don't need to worry about it.

To analyze a turbulent boundary layer we must use the momentum integral approach coupled with experimental data since no exact solutions are known. To approximate the velocity profile in a turbulent boundary layer, recall the Law of the Wall (from a previous set of notes on turbulence):

$$\bar{u}/u^* = \frac{yu^*}{\nu} \qquad \text{for } yu^*/\nu \le 5 \qquad (8.64)$$

$$\bar{u}/u^* = \frac{1}{K'}\ln\left(\frac{yu^*}{\nu}\right)+c \qquad \text{for } yu^*/\nu > 5 \qquad (8.65)$$

where $u^* = (\tau_w/\rho)^{1/2}$ is the "friction velocity."

We could substitute this velocity profile into the momentum thickness integral:

$$\delta_M = \int_{y=0}^{y=\infty} \frac{u}{U_\infty}\left(1-\frac{u}{U_\infty}\right)dy \qquad (66)$$

and solve. Using the stated velocity profile, however, can get cumbersome. Instead, Prandtl suggested approximating the logarithmic turbulent velocity profile using a 1/7th power-law curve fit:

$$\frac{u}{U_\infty} = \left(\frac{y}{\delta}\right)^{1/7} \qquad \text{for } y \le \delta \qquad (8.67)$$

$$\frac{u}{U_\infty} = 1 \qquad \text{for } y \ge \delta \qquad (8.68)$$

Using this velocity profile, the momentum thickness becomes:

$$\delta_M = \int_{y=0}^{y=\infty} \frac{u}{U_\infty}\left(1-\frac{u}{U_\infty}\right)dy = \int_{y=0}^{y=\delta}\left(\frac{y}{\delta}\right)^{1/7}\left[1-\left(\frac{y}{\delta}\right)^{1/7}\right]dy$$

$$\delta_M = \frac{7}{72}\delta$$

To determine the shear stress, recall that from the Karman momentum integral equation:

$$\tau_w = \rho U_\infty^2 \frac{d\delta_M}{dx} = \frac{7}{72}\rho U_\infty^2\frac{d\delta}{dx} \qquad (69)$$

so that the friction coefficient becomes:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2} = \frac{7}{36}\frac{d\delta}{dx} \qquad (70)$$

Experimental wall friction data for turbulent boundary layers can be fit using:

$$C_f \approx 0.020\,\mathrm{Re}_\delta^{-1/6} \qquad (71)$$

where $\mathrm{Re}_\delta = (U_\infty\delta/\nu)$. Equating the two friction coefficients gives:

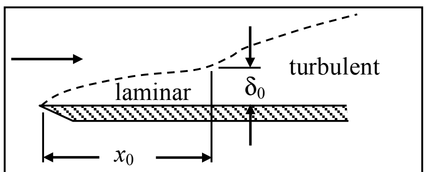
$$\frac{7}{36}\frac{d\delta}{dx} = 0.020\left(\frac{U_\infty\delta}{\nu}\right)^{-1/6}$$

$$\int_{\delta=\delta_0}^{\delta=\delta} \delta^{1/6}\,d\delta = 0.103\left(\frac{U_\infty}{\nu}\right)^{-1/6}\int_{x=x_0}^{x=x} dx$$

$$\delta^{7/6} - \delta_0^{7/6} = 0.120\left(\frac{U_\infty}{\nu}\right)^{-1/6}(x-x_0)$$

Assuming $\delta_0 = 0$ at $x_0 = 0$ gives:

$$\left(\frac{\delta}{x}\right)^{7/6} = 0.120\left(\frac{\nu}{U_\infty x}\right)^{1/6}$$



Note: The shear stress relationship holds strictly for the part of the BL that is turbulent. If the laminar BL thickness and distance downstream are small in comparison to the current thickness and location, then we may assume that the turbulent BL starts approximately at the leading edge, i.e., $\delta_0 = 0$ at $x_0 = 0$.

$$\boxed{\frac{\delta}{x} \approx \frac{0.163}{\mathrm{Re}_x^{1/7}}} \qquad (72)$$

From this relation we can also determine the displacement thickness, momentum thickness, friction factor, and drag coefficient. These relations are summarized below.

$$\boxed{\frac{\delta}{x} \approx \frac{0.16}{\mathrm{Re}_x^{1/7}}} \quad (73) \qquad \boxed{\frac{\delta_D}{x} \approx \frac{0.02}{\mathrm{Re}_x^{1/7}}} \quad (74) \qquad \boxed{\frac{\delta_M}{x} \approx \frac{0.016}{\mathrm{Re}_x^{1/7}}} \quad (75)$$

$$\boxed{C_f \approx \frac{0.027}{\mathrm{Re}_x^{1/7}}} \quad (76) \qquad \boxed{C_D \approx \frac{0.031}{\mathrm{Re}_L^{1/7}}} \quad (77) \qquad \mathrm{Re}_x = \frac{U_\infty x}{\nu}$$

Notes:

1. The boundary layer thickness grows as $\delta \sim x^{6/7}$ for a turbulent boundary layer whereas it grows as $\delta \sim x^{1/2}$ for a laminar boundary layer. Hence, a boundary layer grows more rapidly with distance for turbulent flow than for a laminar flow. The momentum and displacement thicknesses also increase more rapidly for turbulent boundary layers.
2. The shear stress decreases more rapidly for laminar flow than for a turbulent flow. The drag does not increase as rapidly in a laminar flow as compared to a turbulent flow.
3. Another experimental friction curve fit that is commonly used is:

$$C_f \approx \frac{0.0466}{\mathrm{Re}_\delta^{1/4}} \qquad (78)$$

which gives:

$$\boxed{\frac{\delta}{x} \approx \frac{0.382}{\mathrm{Re}_x^{1/5}}} \quad (79) \qquad \boxed{\frac{\delta_D}{x} \approx \frac{0.0478}{\mathrm{Re}_x^{1/5}}} \quad (80) \qquad \boxed{\frac{\delta_M}{x} \approx \frac{0.0371}{\mathrm{Re}_x^{1/5}}} \quad (81)$$

$$\boxed{C_f \approx \frac{0.0594}{\mathrm{Re}_x^{1/5}}} \quad (82) \qquad \boxed{C_D \approx \frac{0.0742}{\mathrm{Re}_L^{1/5}}} \quad (83) \qquad \mathrm{Re}_x = \frac{U_\infty x}{\nu}$$

White (in White, F.M., *Viscous Fluid Flow*, 2nd ed., McGraw-Hill) states that the experimental curve fit given by Eq. (78) is based on limited data and is not as accurate as the curve fit given by Eq. (71). This argument is supported by the plot shown below (plot from White, F.M., *Viscous Fluid Flow*, 2nd ed., McGraw-Hill).