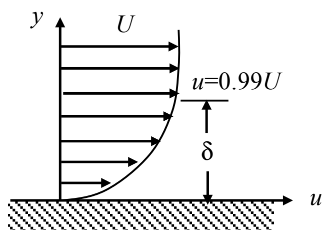
Boundary Layer Thickness Definitions

Before continuing further, we should define what we mean by the "thickness" of a boundary layer. There are three commonly used definitions:

1. **(99%) boundary layer thickness, δ**:

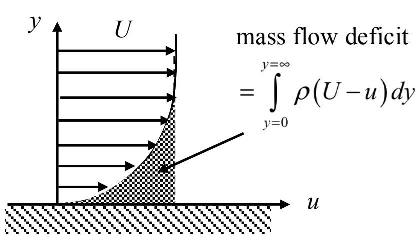
   This thickness definition is the most commonly used definition. The boundary layer thickness, δ, is defined as the distance from the boundary at which the fluid velocity, u, is 99% that of the outer velocity, U:

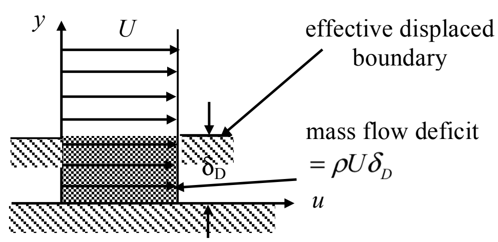
$$\boxed{u(y=\delta)=0.99U} \tag{1}$$



2. **displacement thickness, δ_D or δ^***:

   The displacement thickness, δ_D, is the distance at which the undisturbed outer flow is displaced from the boundary by a stagnant layer of fluid that removes the same mass flow as the actual boundary layer profile.



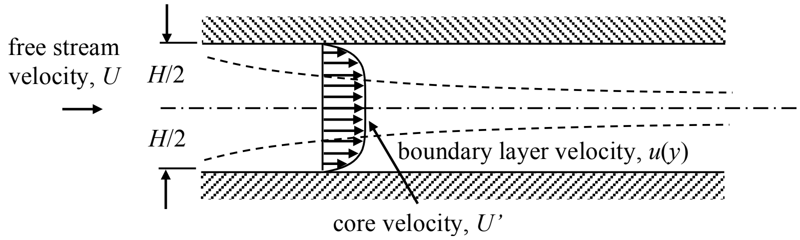


$$\int_0^\infty \rho(U-u)\,dy = \rho U \delta_D$$

$$\boxed{\delta_D = \int_0^\infty \left(1-\frac{u}{U}\right)dy \approx \int_0^\delta \left(1-\frac{u}{U}\right)dy} \tag{2}$$

E*xample*:

Consider the mass flow rate between two parallel plates in which a boundary layer has formed:



Let's just consider the lower half of the flow since the upper and lower halves are symmetric. The mass flow rate in the lower half is:

$$\dot{m}_{1/2} = \underbrace{\int_0^{\delta} \rho u\, dy}_{\text{mass flow rate in BL}} + \underbrace{\int_{\delta}^{H/2} \rho U'\, dy}_{\text{mass flow rate in outer flow}}$$

Note that the uniform, outer flow velocity at this cross-section is U' which is larger than U (to conserve mass). The second integral can be re-written as:

$$\int_{\delta}^{H/2} \rho U' dy = \int_0^{H/2} \rho U' dy - \int_0^{\delta} \rho U' dy$$

so that the mass flow rate is:

$$\dot{m}_{1/2} = \int_0^{\delta} \rho u\, dy + \int_0^{H/2} \rho U'\, dy - \int_0^{\delta} \rho U'\, dy = \int_0^{\delta} \rho \left(u - U'\right) dy + \int_0^{H/2} \rho U'\, dy$$

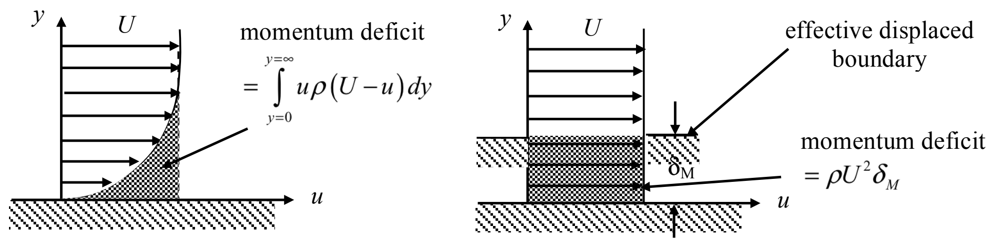
$$= -\rho U' \underbrace{\int_0^{\delta} \left(1 - \frac{u}{U'}\right) dy}_{=\delta_D} + \rho U' \left(H/2\right)$$

$$\boxed{\therefore \dot{m}_{1/2} = \rho\left(H/2 - \delta_D\right) U'}$$

Thus, if we replace the real velocity profile by a uniform velocity profile at that cross-section (which has velocity U'), we must decrease the effective cross-sectional area (per unit depth) of each half of the pipe at that particular location by the displacement thickness, δ_D, to maintain the same mass flow rate. Note that we would need to have another relation for δ_D in order to solve for U' and visa-versa.

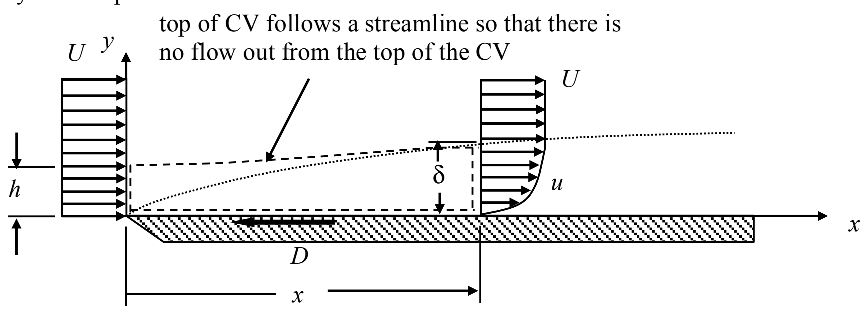
3. **momentum thickness, δ_M or Θ**:

The momentum thickness, δ_M, is the thickness of a stagnant layer that has the same momentum deficit, relative to the outer flow, as the actual boundary layer profile. The concept is similar to that for the displacement thickness except instead of a mass deficit, the momentum thickness considers the momentum deficit.



$$\int_0^\infty u\rho(U-u)\,dy = \rho U^2 \delta_M \quad \Rightarrow \quad \boxed{\delta_M = \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy \approx \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy} \tag{3}$$

Example:
Consider the boundary layer flow over a flat plate. We'll assume a steady flow with the pressure everywhere equal.



Let's apply the LME in the *x*-direction to the CV shown to determine the drag acting on the fluid (or the plate) over the distance *x*.

$$-D = \frac{d}{dt}\int_{CV} u_x \rho\, dV + \int_{CS} u_x \rho\left(\mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) \quad \Rightarrow \quad -D = \int_0^\delta \rho u^2 dy - \rho U^2 h$$

where the height, *h*, is found via COM on the same CV to be:

$$\rho U h = \int_0^\delta \rho u\, dy \quad \Rightarrow \quad h = \int_0^\delta \frac{u}{U}\, dy$$

so that the drag is:

$$-D = \int_0^\delta \rho u^2\, dy - \rho U^2 \int_0^\delta \frac{u}{U}\, dy \quad \Rightarrow \quad D = \rho U^2 \underbrace{\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy}_{=\delta_M}$$

$$\boxed{\therefore D = \rho U^2 \delta_M} \tag{4}$$

Thus we see that the drag is related to the momentum thickness. Note that this particular example is for a situation with no pressure gradients (the pressure is a constant). The drag expression will be different if the pressure varies for the flow but the resulting drag expression will still be in terms of the momentum thickness.

Notes:
1. Usually: $\delta > \delta_D > \delta_M$