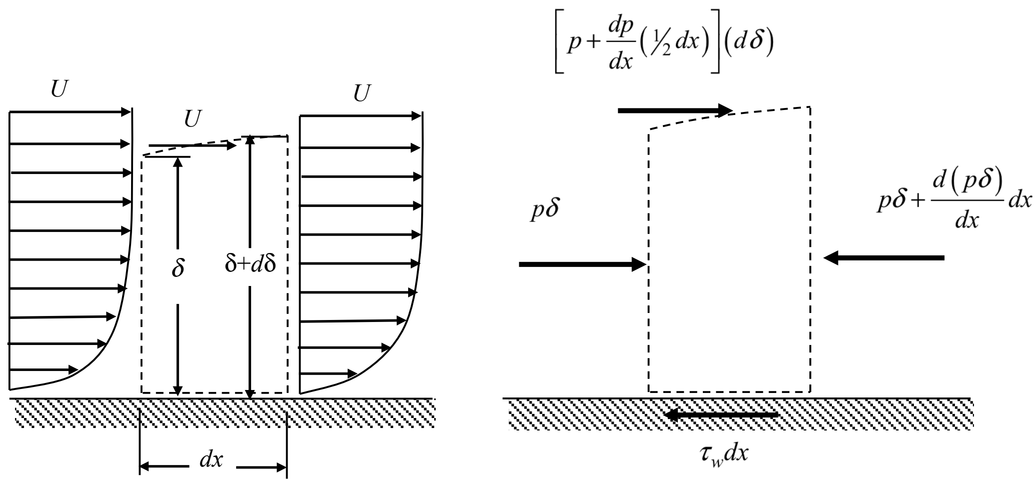
5. Approximate Methods: The Kármán Momentum Integral Equation

So far we've only examined boundary layer flows that lend themselves to similarity solutions. This, of course, is very restrictive. There are many non-similar boundary layer flows that we would also like to investigate. Since the majority of fluid mechanics problems are extremely complex, we often have to resort to empirical or semi-empirical methods for investigating the flows in greater detail. Here we'll discuss one such semi-empirical method used for investigating boundary layers called the Kármán Momentum Integral Equation (KMIE).

The idea is straightforward and relies on the Linear Momentum Equation. Consider a differential control volume as shown in the figure below. The top of the control volume is defined by the line separating the boundary layer region from the outer flow region (this is <u>not</u> a streamline).



Let's apply the linear momentum equation in the x-direction to the control volume:

$$F_{x,\text{on CV}} = \frac{d}{dt}\int_{\text{CV}} u_x \rho dV + \int_{\text{CS}} u_x \rho \left(\mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) \tag{40}$$

where

$$F_{x,\text{on CV}} = \underbrace{p\delta}_{\text{left}} - \underbrace{\left[p\delta + \frac{d}{dx}\left(p\delta\right)\left(dx\right)\right]}_{\text{right}} + \underbrace{\left[p + \frac{dp}{dx}\left(\tfrac{1}{2}dx\right)\right]\left(d\delta\right)}_{\text{top}} \underbrace{-\tau_w dx}_{\text{bottom}} \quad \text{(assuming unit depth)}$$

$$\frac{d}{dt}\int_{\text{CV}} u_x \rho dV = 0 \quad \text{(steady flow)}$$

$$\int_{\text{CS}} u_x \rho \left(\mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = \underbrace{-\int_{y=0}^{y=\delta} \rho u^2\, dy}_{\text{left}} + \underbrace{\left[\int_{y=0}^{y=\delta} \rho u^2\, dy + \frac{d}{dx}\left(\int_{y=0}^{y=\delta} \rho u^2\, dy\right) dx\right]}_{\text{right}} \underbrace{-U\frac{d}{dx}\left[\int_{y=0}^{y=\delta} \rho u\, dy\right] dx}_{\text{top}}$$

Note that the mass flow rate through the top is found via conservation of mass on the same control volume:

$$\underbrace{-\dot{m}_{\text{top}}}_{\text{in through top}} \underbrace{-\int_0^\delta \rho u\,dy}_{\text{in at left}} + \underbrace{\left[\int_0^\delta \rho u\,dy + \frac{d}{dx}\left(\int_0^\delta \rho u\,dy\right)dx\right]}_{\text{out through right}} = 0$$
$$\therefore \dot{m}_{\text{top}} = \frac{d}{dx}\left(\int_0^\delta \rho u\,dy\right)dx \tag{41}$$

Substituting and simplifying (and neglecting higher order terms):

$$-\frac{dp}{dx}\delta dx - \tau_w dx = \frac{d}{dx}\left[\int_0^\delta \rho u^2 dy\right]dx - U\frac{d}{dx}\left[\int_0^\delta \rho u dy\right]dx$$
$$\Rightarrow -\frac{dp}{dx}\delta - \tau_w = \frac{d}{dx}\left[\int_0^\delta \rho u^2 dy\right] - U\frac{d}{dx}\left[\int_0^\delta \rho u dy\right] \tag{42}$$

Recall that the pressure at a given x location remains constant with y position so that we can find dp/dx in terms of the outer (potential) flow velocity using Bernoulli's equation outside of the boundary layer:

$$p + \tfrac{1}{2}\rho U^2 = \text{constant} \quad \Rightarrow \quad \frac{dp}{dx} + \rho U\frac{dU}{dx} = 0$$
$$\therefore \frac{dp}{dx} = -\rho U\frac{dU}{dx} \tag{43}$$

In addition, we'll re-write the boundary layer thickness in terms of an integral so that:

$$-\frac{dp}{dx}\delta = \left(\rho U\frac{dU}{dx}\right)\int_0^\delta dy = \frac{dU}{dx}\int_0^\delta \rho U dy \tag{44}$$

Additional re-arranging gives:

$$U\frac{d}{dx}\left[\int_0^\delta \rho u dy\right] = \frac{d}{dx}\left[\int_0^\delta \rho u U dy\right] - \frac{dU}{dx}\int_0^\delta \rho u dy \tag{45}$$

Substituting Eqs. (44) and (45) into Eq. (42) gives:

$$\frac{dU}{dx}\int_0^\delta \rho U dy - \tau_w = \frac{d}{dx}\left[\int_0^\delta \rho u^2 dy\right] - \frac{d}{dx}\left[\int_0^\delta \rho u U dy\right] + \frac{dU}{dx}\left[\int_0^\delta \rho u dy\right] \tag{46}$$

Additional re-arranging and simplifying gives:

$$\tau_w = \frac{d}{dx}\left[\int_0^\delta \rho u(U-u)\,dy\right] + \frac{dU}{dx}\int_0^\delta \rho(U-u)\,dy$$
$$= \frac{d}{dx}\left[\rho U^2 \underbrace{\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy}_{=\delta_M}\right] + \frac{dU}{dx}\rho U \underbrace{\int_0^\delta \rho\left(1-\frac{u}{U}\right)dy}_{=\delta_D} \tag{47}$$

Thus, if the fluid has constant density:

$$\boxed{\frac{\tau_w}{\rho} = \frac{d}{dx}\left[U^2\delta_M\right] + \delta_D U\frac{dU}{dx}} \tag{48}$$

This is known as the **Kármán Momentum Integral Equation (KMIE)**.

Notes:

1. If the pressure remains constant, then dU/dx=0 and

$$\tau_w = \rho U^2 \frac{d\delta_M}{dx} \tag{49}$$

2. The typical methodology for using the KMIE is as follows:
   a. Obtain an approximate expression for U=U(x) from inviscid flow theory (e.g., potential flow theory). Recall that Bernoulli's equation can be used to relate the pressure and U.
   b. Assume a velocity profile in the boundary layer subject to the appropriate boundary conditions, i.e., assume a form for:

$$\frac{u}{U} = f\left(\frac{y}{\delta}\right) \tag{50}$$

   subject to the boundary conditions:

$$\frac{u}{U}\left(\frac{y}{\delta}=0\right)=0 \quad \text{and} \quad \frac{u}{U}\left(\frac{y}{\delta}=1\right)=1 \tag{51}$$

   The form of the approximate velocity profile is typically found based on curve fits to experimental measurements of the boundary layer velocity profile.

   Note that higher order profiles will have additional boundary conditions. For example, a cubic curve fit will also have a boundary condition that matches the slope of the velocity profile at the freestream boundary.

   c. The shear stress at the wall for a laminar flow can also be determined from the Newtonian stress-strain rate constitutive relations to be:

$$\tau_w = \mu\left(\frac{U}{\delta}\right)\left.\frac{d\left(\frac{u}{U}\right)}{d\left(\frac{y}{\delta}\right)}\right|_{\frac{y}{\delta}=0} \tag{52}$$

   (For a turbulent flow, experimental data for the wall shear stress are used. This will be discussed in greater detail later in these notes.) This laminar wall shear stress must be the same shear stress as that found using the KMIE (Eq. (48)). Thus, we can equate the two shear stress expressions. The resulting differential equation can then be solved for the boundary layer thickness, δ, as a function of x.

3. This approximate technique can be used for either laminar or turbulent flows. In fact, this method is especially useful for analyzing turbulent boundary layer profiles (to be discussed in later notes).