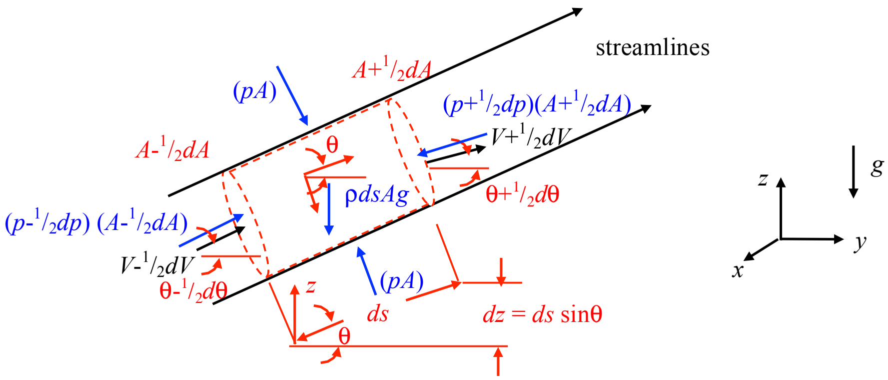
Another Approach to Deriving Bernoulli's Equation

We can also derive Bernoulli's equation by considering LME and COM applied to a differential control volume as shown below. In the following analysis, we'll make the following simplifying assumptions:

1. steady flow
2. inviscid fluid
3. incompressible fluid

Note that the control volume shown follows the streamlines.



First apply COM to the CV shown above:

$$\frac{d}{dt}\int_{CV}\rho dV+\int_{CS}\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)=0$$

where

$$\frac{d}{dt}\int_{CV}\rho dV=0 \qquad \text{(steady flow)}$$

$$\int_{CS}\left(\rho\mathbf{u}_{\text{rel}}\cdot d\mathbf{A}\right)=\rho\left(V+\tfrac{1}{2}dV\right)\left(A+\tfrac{1}{2}dA\right)$$

$$-\rho\left(V-\tfrac{1}{2}dV\right)\left(A-\tfrac{1}{2}dA\right)$$

$$=\rho VdA+\rho AdV+H.O.T.$$

(Note that there is no flow across a streamline.)

$$\therefore VdA=-AdV$$

Now apply the LME in the *s*-direction to the same CV:

$$\frac{d}{dt}\int_{\text{CV}} u_s \rho dV + \int_{\text{CS}} u_s \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) = F_{B,s} + F_{S,s}$$

where

$$\frac{d}{dt}\int_{\text{CV}} u_s \rho dV = 0 \qquad \text{(steady flow)}$$

$$\begin{aligned}\int_{\text{CS}} u_s \left(\rho \mathbf{u}_{\text{rel}} \cdot d\mathbf{A}\right) &= -\rho\left(V - \tfrac{1}{2}dV\right)^2\left(A - \tfrac{1}{2}dA\right) \\ &\quad + \rho\left(V + \tfrac{1}{2}dV\right)^2\left(A + \tfrac{1}{2}dA\right) \\ &= 2\rho V A dV + \rho V^2 dA + H.O.T.\end{aligned}$$

$$F_{B,s} = \rho ds A\left(-g\sin\theta\right) = -\rho A g \underbrace{ds\sin\theta}_{=dz} = -\rho A g dz$$

$$\begin{aligned}F_{S,s} &= \left(p - \tfrac{1}{2}dp\right)\left(A - \tfrac{1}{2}dA\right) - \left(p + \tfrac{1}{2}dp\right)\left(A + \tfrac{1}{2}dA\right) + p dA \\ &= -A dp + H.O.T.\end{aligned}$$

$$\therefore 2\rho V A dV + \rho V^2 dA = -\rho A g dz - A dp$$

Substitute the result from COM into the result from the LME and simplify:

$$2\rho V A dV + \underbrace{\rho V^2 dA}_{=-\rho V A dV} = -\rho A g dz - A dp$$

$$\frac{dp}{\rho} + V dV + g dz = 0$$

We can integrate this equation to get:

$$\frac{p}{\rho} + \tfrac{1}{2}V^2 + gz = \text{constant} \tag{150}$$

Again, it's important to review the assumptions built into the derivation for Eq. (150):

1. steady flow
2. inviscid fluid
3. incompressible fluid
4. flow along a streamline